Filomat 34:6 (2020), 2029–2035
https://doi.org/10.2298/FIL2006029A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A GT Generated by a Family of Maps

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Abstract. In this paper we define weak and strong generalized topologies. We show that a large range of generalized topological spaces can be characterized by the weak and strong generalized topologies. As a main result, we prove that a GTS is completely μ-regular if and only if μ is the weak GT generated by the family $C^*_{\mu,\nu}(X)$ of all bounded $(\mu, \nu_{\mathbb{R}})$-continuous functions.

1. Introduction

Generalized topological spaces [6] were defined and studied by Á. Császár in 2002. Let X be a nonempty set and $\mathcal{P}(X)$ denotes the power set of X. A subset μ of $\mathcal{P}(X)$ is called a *generalized topology* (GT as an acronym) on X if it contains $\emptyset$ and any union of elements of μ belongs to μ [6]. The pair (X, μ) is called a *generalized topological space* (GTS as an acronym). A subset A of X is called *μ-open* (or *μ-closed*) if $A \in \mu$ (or $X \setminus A \in \mu$). A GTS (X, μ) is called *strong* [5] if $X \in \mu$. For $A \subseteq X$, we show the union of all μ-open sets contained in A, by $i_\mu(A)$ and the intersection of all μ-closed sets containing A by $c_\mu(A)$. $i_\mu(A)$ and $c_\mu(A)$ are called *the interior and closure of A*, respectively [6]. The union of all elements of μ, i.e., $i_\mu(X)$ is denoted by M_μ [9]. In fact, M_μ is the largest μ-open subset of X. It is obvious that $M_\mu = X$ if and only if μ is strong. The family of all generalized topologies on a nonempty set X constructs a lattice which is denoted by $\mathcal{G}(X)$ [3]. Let $\mu, \nu \in \mathcal{G}(X)$. μ is called to be *finer* (*stronger*) than ν if $\nu \subseteq \mu$ [7]. In this case, we also say that ν is *coarse*r (*weaker*) than μ.

2. Preliminaries

Definition 2.1. ([7]) Let (X, μ) be a GTS. $\beta \subseteq \mu$ is called a base for μ if every $M \in \mu$ is a union of a subfamily of β.

As [9] if $\beta \subseteq \mathcal{P}(X)$, then the family $\mu(\beta)$ composed of $\emptyset$ and all sets $N \subseteq X$ of the form $N = \cup_{i \in I} B_i$, $B_i \in \beta$ and $I \neq \emptyset$ is arbitrary, is a GT on X and β is a base for $\mu(\beta)$ called the *generalized topology generated by β*.

Lemma 2.2. *Let X be a set and $\beta \subseteq \mathcal{P}(X)$. Then the following hold:*

(1) *$\mu(\beta)$ is the weakest GT on X containing β.*

(2) $\mu(\beta) = \cap\{\mu \in \mathcal{G}(X) | \beta \subseteq \mu\}$.

2010 *Mathematics Subject Classification*. Primary 54A05; Secondary 54A99

Keywords. Generalized topology, generalized continuity, weak (strong) generalized topologies, completely μ-regular, $C^*_{\mu,\nu}(X)$, GT generated by a family of maps

Received: 19 April 2019; Revised: 08 January 2020; Accepted: 10 January 2020

Communicated by Ljubiša D.R. Kočinac

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Proof. (1) If μ is a GT on X containing β, then it also contains all unions of some elements of β, i.e., $\mu(\beta) \subseteq \mu$.
(2) It is an immediate consequence of part (1). □

Proposition 2.3. *Let μ and μ' be two generalized topologies on X generated by bases β and β', respectively. Then μ' is finer than μ if and only if for every $x \in X$ and every $B \in \beta$ containing x, there exists $B' \in \beta'$ such that $x \in B' \subseteq B$.*

Proof. Necessity. Let μ' be finer than μ and $x \in B \in \beta$. Then B is a μ'-open set containing x and so by our assumption, there exists $B' \in \beta'$ such that $x \in B' \subseteq B$.

Sufficiency. Let for every $x \in X$ and every $B \in \beta$ containing x, there exists $B' \in \beta'$ such that $x \in B' \subseteq B$. Thus B is a union of some elements of β', i.e., $\beta \subseteq \mu(\beta') = \mu'$ and so by Lemma 2.2, μ' contains $\mu = \mu(\beta)$. □

Definition 2.4. ([6]) Let (X, μ) and (Y, ν) be two generalized topological spaces. A function $f : X \to Y$ is called (μ, ν)-continuous if $f^{-1}(U) \in \mu$, for every $U \in \nu$.

Lemma 2.5. ([4]) *If μ is a GT on X, $A \subseteq X$ and $x \in X$, then $x \in c_\mu(A)$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \emptyset$.*

The following two lemmas show that to specify the closure of a set in a GTS and to examine continuity of a function between two generalized topological spaces, it is just enough to consider sets of a base:

Lemma 2.6. ([14]) *Let μ be a GT on X, β a base for μ, $A \subseteq X$ and $x \in X$. Then $x \in c_\mu(A)$ if and only if $x \in B \in \beta$ implies $B \cap A \neq \emptyset$.*

Lemma 2.7. ([8]) *Let (X, μ) and (Y, ν) be two generalized topological spaces and β a base for ν. A function $f : X \to Y$ is (μ, ν)-continuous if and only if $f^{-1}(B) \in \mu$, for every $B \in \beta$.*

Definition 2.8. ([10]) Let (X, μ) be a GTS and $Y \subseteq X$. Also suppose that $\mu_Y = \{O \cap Y | O \in \mu\}$. Then μ_Y is a GT on Y which is called the relative GT on Y. In such case we say that Y is a μ-subspace of X.

Let (Z, ν) be a GTS, $f : X \to Z$ be a function and $Y \subseteq X$. Since $(f|Y)^{-1}(U) = f^{-1}(U) \cap Y$, for every ν-open set U, we have the following result:

Lemma 2.9. *Let (X, μ) and (Z, ν) be two generalized topological spaces and $Y \subseteq X$. If $f : (X, \mu) \to (Z, \nu)$ is (μ, ν)-continuous, then $f|Y$ is (μ_Y, ν)-continuous.*

3. A GT Generated by a Family of Maps

Definition 3.1. ([14]) Let $(X_\alpha, \mu_\alpha)_{\alpha \in A}$ be a family of generalized topological spaces and $\{f_\alpha\}_{\alpha \in A}$ be a family of of maps, where X is a set and f_α is a map of X to X_α. Then, $\mu_w = \left\{ \cup U | U \subseteq \{f_\alpha^{-1}(U_\alpha) : U_\alpha \in \mu_\alpha, \alpha \in A\} \right\}$ is called the GT on X generated by the family $\{f_\alpha\}_{\alpha \in A}$ of maps.

By Definition 2.1, we have the following result:

Proposition 3.2. *Let X be a nonempty set. If (X_α, μ_α) is a GTS, $f_\alpha : X \to X_\alpha$ is a function for each $\alpha \in A$ and μ_w is the GT on X generated by the family $\{f_\alpha\}_{\alpha \in A}$ of maps. Then the following statements hold:*

(1) $\{f_\alpha^{-1}(U_\alpha) | U_\alpha \in \mu_\alpha, \alpha \in A\}$ is a base for μ_w.

(2) If β_α is a base for μ_α for each $\alpha \in A$, then the set $\{f_\alpha^{-1}(U_\alpha) | U_\alpha \in \beta_\alpha, \alpha \in A\}$ forms a base for μ_w.

Let (X_α, μ_α) be a GTS for each $\alpha \in A$. We note that the GT on X generated by the family $\{f_\alpha : X \to (X_\alpha, \mu_\alpha)\}_{\alpha \in A}$ of maps is the smallest element between all GTs μ on X making each f_α (μ, μ_α)-continuous. The following result is needed.

Proposition 3.3. ([11]) *Let (X, μ) be a GTS and Y a μ-subspace of X with the relative GT μ_Y. Then μ_Y is the GT on Y generated by the inclusion map $\iota : Y \to X$.*

Lemma 3.4. *Let (X, μ), (Y, ν) and (Z, λ) be generalized topological spaces. If $f : X \to Y$ is (μ, ν)-continuous, and $g : Y \to Z$ is (ν, λ)-continuous, then $g \circ f$ is (μ, λ)-continuous.*

Theorem 3.5. *Let $\{(X_\alpha, \mu_\alpha)\}_{\alpha\in A}$ be a family of generalized topological spaces and $f_\alpha : X \to (X_\alpha, \mu_\alpha)$ be a function for each $\alpha \in A$. Let the GT on X generated by the family $\{f_\alpha\}_{\alpha\in A}$ of maps is equal to μ. A function f from a GTS (Y, ν) to (X, μ) is (ν, μ)-continuous if and only if $f_\alpha \circ f$ is (ν, μ_α)-continuous, for each $\alpha \in A$.*

Proof. Necessity. Let f be (ν, μ)-continuous. Then by Lemma 3.4, $f_\alpha \circ f$ is (ν, μ_α)-continuous, for each $\alpha \in A$.

Sufficiency. If $f_\alpha \circ f$ is (ν, μ_α)-continuous, for each $\alpha \in A$, then $f^{-1}(f_\alpha^{-1}(U_\alpha)) = (f_\alpha \circ f)^{-1}(U_\alpha)$ is ν-open for each $U_\alpha \in \mu_\alpha$. By Proposition 3.2, the set of all $f_\alpha^{-1}(U_\alpha)$, where $\alpha \in A$ and $U_\alpha \in \mu_\alpha$, form a base for μ and so by Lemma 2.7, f is (ν, μ)-continuous. □

Definition 3.6. Let $\{(X_\alpha, \mu_\alpha)\}_{\alpha\in A}$ be a family of generalized topological spaces, $X = \prod_{\alpha\in A} X_\alpha$ be its Cartesian Product and $\pi_\alpha : \prod_{\alpha\in A} X_\alpha \to X_\alpha$ be the projection map, i.e., $\pi_{\alpha 0}$ assigns to the point $x = \{x_\alpha\}_{\alpha\in A}$ its $\alpha 0$th coordinates $x_{\alpha 0}$.

(1) ([14]) The GT on X generated by the family $\{\pi_\alpha\}_{\alpha\in A}$ of maps is called the generalized product topology (briefly GPT) denoted by $\prod_{\alpha\in A} \mu_\alpha$.

(2) ([9]) $\mathcal{B} = \{\prod_{\alpha\in A} M_\alpha | M_\alpha \in \mu_\alpha$ *for all* $\alpha \in A$ and $M_\alpha = M_{\mu_\alpha}$ for all but a finite number of indices $\alpha\}$ is a base for a GT on X called the product (or Csaszar product in [11]) of the GT's μ_α and it is denoted by $\mathbf{P}_{\alpha\in A}\mu_\alpha$.

By Theorem 3.5 and Definition 3.6(1) we have the following result.

Corollary 3.7. *Let $\{(X_\alpha, \mu_\alpha)\}_{\alpha\in A}$ be a family of generalized topological spaces and $X = \prod_{\alpha\in A} X_\alpha$ be its Cartesian Product. If $\mu = \prod_{\alpha\in A} \mu_\alpha$ is the GPT on X, then a function f from a GTS (Y, ν) to (X, μ) is (ν, μ)-continuous if and only if $\pi_\alpha \circ f$ is (ν, μ_α)-continuous, for each $\alpha \in A$.*

Remark 3.8. It is well known that Definition 3.6(1) and Definition 3.6(2) are equivalent in the class of topological spaces, and so Corollary 3.7 holds if μ is the product of topologies $\{\mu_\alpha\}_{\alpha\in A}$, i.e., $\mu = \mathbf{P}_{\alpha\in A}\mu_\alpha$, where μ_α is a topology for all $\alpha \in A$. But, Corollary 3.7 does not hold if μ is the Csaszar product of generalized topologies $\{\mu_\alpha\}_{\alpha\in A}$ even if all μ_α's are strong as it is shown in Example 3.9. Thus, Definition 3.6(1) and Definition 3.6(2) are not equivalent in general.

Example 3.9. Let $X_1 = X_2 = \{a, b\}$ and $\mu_1 = \{\emptyset, \{a\}, X_1\}$, $\mu_2 = \{\emptyset, \{b\}, X_2\}$, $Y = \{a, b, c\}$, $\nu = \{\emptyset, \{a, b\}, \{b, c\}, Y\}$ and $\mu = \mathbf{P}_{i\in\{1,2\}}\mu_i$. Then the function $f : Y \to X_1 \times X_2$ with $f(a) = (a, a)$, $f(b) = (a, b)$ and $f(c) = (b, b)$ is not (ν, μ)-continuous since $f^{-1}(\{a\} \times \{b\}) = \{b\}$ is not ν-open. It is easy to check that the functions $\pi_1 \circ f$ and $\pi_2 \circ f$ are (ν, μ_1)-continuous and (ν, μ_2)-continuous, respectively.

Definition 3.10. Suppose we are given a GTS (X, μ), a family $\{(X_\alpha, \mu_\alpha)\}_{\alpha\in A}$ of generalized topological spaces and a family $\mathcal{F} = \{f_\alpha\}_{\alpha\in A}$ of maps, where $f_\alpha : (X, \mu) \to (X_\alpha, \mu_\alpha)$ is a (μ, μ_α)-continuous. If for every $x \in X$ and every μ-closed subset F of X such that $x \in X \setminus F$ there exists $\alpha \in A$ such that $f_\alpha(x) \notin c_{\mu_\alpha}(f_\alpha(F))$, then we say that the family $\mathcal{F}$ separates points and μ-closed sets.

Theorem 3.11. *Let (X, μ) be a GTS, $\{(X_\alpha, \mu_\alpha)\}_{\alpha\in A}$ be a family of generalized topological spaces and $\mathcal{F} = \{f_\alpha\}_{\alpha\in A}$ be a family of maps, where*

$$f_\alpha : (X, \mu) \to (X_\alpha, \mu_\alpha) \text{ is } (\mu, \mu_\alpha)\text{-continuous for all } \alpha \in A.$$

Then, μ is equal to the GT on X generated by the family $\mathcal{F}$ of maps if and only if the family $\mathcal{F}$ separates points and μ-closed sets.

Proof. Necessity. Let μ be equal to the GT on X generated by the family $\mathcal{F}$ of maps. If F is μ-closed and $x \in X \setminus F$, then there exists a μ_α-open set U_α for some $\alpha \in A$ such that $x \in f_\alpha^{-1}(U_\alpha) \subseteq X \setminus F$ since $\{f_\beta^{-1}(U_\beta) \mid U_\beta \in \mu_\beta, \beta \in A\}$ is a base for μ by Proposition 3.2. If $y = f_\alpha(x) \in U_\alpha \cap f_\alpha(F)$, then $y \in U_\alpha$ and there is $z \in F$ such that $y = f_\alpha(z)$, and so $z \in f_\alpha^{-1}(U_\alpha) \cap F$ which is a contradiction.

Sufficiency. Suppose that the family $\mathcal{F}$ separates points and μ-closed sets. If $x \in U$ for some μ-open set U, then $F = X \setminus U$ is μ-closed and $x \in X \setminus F$, and so by supposition there exists $\alpha \in A$ such that $f_\alpha(x) \notin c_{\mu_\alpha}(f_\alpha(F))$. Thus by Lemma 2.5, there is a μ_α-open set U_α such that $f_\alpha(x) \in U_\alpha$ and $U_\alpha \cap f_\alpha(F) = \emptyset$. If $y \in f_\alpha^{-1}(U_\alpha)$, then $f_\alpha(y) \in U_\alpha$ so $f_\alpha(y) \notin f_\alpha(F)$ which shows that $y \notin F = X \setminus U$ consequently $y \in U$, and so $x \in f_\alpha^{-1}(U_\alpha) \subseteq U$. Thus by our assumption, μ is the GT on X generated by the family $\mathcal{F}$ of maps. □

Theorem 3.12. *Let (X, μ) be a GTS and $\{(X_\alpha, \mu_\alpha)\}_{\alpha \in A}$ be a family of generalized topological spaces such that for each $\alpha \in A$ there exists a family $\{(Y_{\alpha_\lambda}, \mu_{\alpha_\lambda})\}_{\lambda \in \Lambda_\alpha}$ of generalized topological spaces. If μ is equal to the GT on X generated by the family $\{f_\alpha : X \to (X_\alpha, \mu_\alpha)\}_{\alpha \in A}$ of maps and each μ_α is equal to the GT on X_α generated by the family $\{g_{\alpha_\lambda} : X_\alpha \to (Y_{\alpha_\lambda}, \mu_{\alpha_\lambda})_{\lambda \in \Lambda_\alpha}$ of maps, then μ is equal to the GT on X generated by the family $\{g_{\alpha_\lambda} \circ f_\alpha : X \to (Y_{\alpha_\lambda}, \mu_{\alpha_\lambda})\}_{\alpha \in A, \lambda \in \Lambda_\alpha}$ of maps.*

Proof. Let U be a μ-open set and $x \in U$. Then by our assumption and Proposition 3.2, there exists a μ_α-open set U_α for some $\alpha \in A$ such that

$$x \in f_\alpha^{-1}(U_\alpha) \subseteq U. \tag{1}$$

Since μ_α is equal to the GT on X_α generated by the family $\{g_{\alpha_\lambda} : X_\alpha \to (Y_{\alpha_\lambda}, \mu_{\alpha_\lambda})_{\lambda \in \Lambda_\alpha}$ of maps and $f_\alpha(x) \in U_\alpha$, there is $\lambda \in \Lambda_\alpha$ and $U_{\alpha_\lambda} \in \mu_{\alpha_\lambda}$ such that $f_\alpha(x) \in g_{\alpha_\lambda}^{-1}(U_{\alpha_\lambda}) \subseteq U_\alpha$. Thus

$$x \in f_\alpha^{-1}(g_{\alpha\lambda}^{-1}(U_{\alpha\lambda})) = (g_{\alpha\lambda} \circ f_\alpha)^{-1}(U_{\alpha\lambda}) \subseteq f_\alpha^{-1}(U_\alpha), \tag{2}$$

and so (1) and Proposition 3.2 complete the proof. □

Theorem 3.13. *Let (X, μ) be a GTS, $\{(X_\alpha, \mu_\alpha)\}_{\alpha \in A}$ be a family of generalized topological spaces and $\mathcal{F} = \{f_\alpha : X \to X_\alpha\}_{\alpha \in A}$ be a family of maps. If μ is equal to the GT on X generated by the family $\mathcal{F}$ of maps and Y is a μ-subspace of X with the relative GT μ_Y. Then μ_Y is equal to the GT on Y generated by the the family $\{f_\alpha|Y\}_{\alpha \in A}$ of maps.*

Proof. By Proposition 3.3, μ_Y is the GT on Y generated by the inclusion map $\iota : Y \to X$ and $f_\alpha|Y = f_\alpha o \iota$ for every $\alpha \in A$. Now Theorem 3.12 shows that μ_Y is equal to the GT on Y generated by the family $\{f_\alpha \circ \iota : Y \to X_\alpha\}_{\alpha \in A}$ of maps and this completes the proof. □

Let $\beta = \{(-\infty, b) | b \in \mathbb{R}\} \cup \{(a, +\infty) | a \in \mathbb{R}\}$. Recall that the family $\nu_{\mathbb{R}}$ composed of $\emptyset$ and all sets $G \subseteq \mathbb{R}$ of the form $G = \cup_{i \in I} B_i$, where $B_i \in \beta$ and $I \neq \emptyset$ is arbitrary, is a GT on $\mathbb{R}$ [8]. It is called the standard GT on $\mathbb{R}$.

Notation 3.14. Let X be a nonempty set and (X, μ) be a GTS. We denote by $C_{\mu,\nu_{\mathbb{R}}}(X)$ the set of all $(\mu, \nu_{\mathbb{R}})$-continuous functions. A function $f \in C_{\mu,\nu_{\mathbb{R}}}(X)$ is called to be bounded if $|f(x)| < n$, for some $n \in \mathbb{N}$ and each $x \in X$. The subset of all bounded functions in $C_{\mu,\nu_{\mathbb{R}}}(X)$ is denoted by $C^*_{\mu,\nu_{\mathbb{R}}}(X)$. Let $c \in \mathbb{R}$ denote the constant function on X with image $\{c\}$. If $f, g \in C_{\mu,\nu_{\mathbb{R}}}(X)$, then the functions $f \pm g$, fg, $-f$ and cf are defined as usual.

Lemma 3.15. *Let f be a continuous real-valued function on $\mathbb{R}$. Then, $f \in C_{\nu_{\mathbb{R}},\nu_{\mathbb{R}}}(\mathbb{R})$ if and only if $x \leq y$ implies $f(x) \leq f(y)$ or $x \leq y$ implies $f(x) \geq f(y)$.*

Proof. The proof is straightforward. □

Let X be a nonempty set and (X, μ) be a GTS. Then, it is easy to see that μ is not strong if and only if $C_{\mu,\nu_{\mathbb{R}}}(X) = \emptyset$ if and only if $C^*_{\mu,\nu_{\mathbb{R}}}(X) = \emptyset$. Let (X, μ) be a GTS which is strong. It is easy to check that if f and g are in $C_{\mu,\nu_{\mathbb{R}}}(X)$ ($C^*_{\mu,\nu_{\mathbb{R}}}(X)$), then so are $-f$, cf and $f \pm c$. However, as the following example shows, $f \pm g$ and fg need not be in $C_{\mu,\nu_{\mathbb{R}}}(X)$ ($C^*_{\mu,\nu_{\mathbb{R}}}(X)$).

Example 3.16. Suppose that $f(x) = x^3$ and $g(x) = x$. Then by Lemma 3.15, f and g are $(\nu_{\mathbb{R}}, \nu_{\mathbb{R}})$-continuous, but $f - g = x^3 - x$ and $fg = x^4$ are not so, since $(f - g)^{-1}(-\infty, 0)$ and $(fg)^{-1}(-\infty, 1)$ are not $\nu_{\mathbb{R}}$-open.

Definition 3.17. ([11]) A GTS (X, μ) is called completely μ-regular if for every μ-closed set $F \subset X$ and for every $x \in X$ such that $x \notin F$, there exists a $(\mu, \nu_{\mathbb{R}})$-continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$, for every $y \in F$.

Let (X, μ) be a completely μ-regular GTS. If there is a proper μ-closed set, then it is an immediate consequence of the above definition that (X, μ) is strong.

Example 3.18. ([1]) Let $X = \{a, b, c, d\}$ and $\mu = \mathcal{P}(X) \setminus \{\{a\}, \{b\}, \{c\}, \{d\}\}$. Then,

$$\mathcal{B} = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, d\}\}$$

is a base for the GTS (X, μ) such that every member of $\mathcal{B}$ is μ-closed. Therefore, the GTS (X, μ) is completely μ-regular.

The following interesting result is needed in this paper.

Theorem 3.19. ([11]) *Every μ-subspace (Y, μ_Y) of a completely μ-regular GTS (X, μ) is completely μ-regular.*

Lemma 3.20. *A GTS (X, μ) is completely μ-regular if and only if for every μ-closed set $F \subset X$ and every $x \in X$ such that $x \notin F$, there exists $f \in C^*_{\mu,\nu_{\mathbb{R}}}(X)$ and $a \in \mathbb{R}$ such that one of the following conditions hold:*

(1) $f(x) > a$ and $f(y) \leq a$, for each $y \in F$;

(2) $f(x) < a$ and $f(y) \geq a$, for each $y \in F$.

Proof. Necessity. Obvious.

Sufficiency. Let for every μ-closed set $F \subset X$ and every $x \in X$ such that $x \notin F$, there exists $f \in C^*_{\mu,\nu_{\mathbb{R}}}(X)$ and $a \in \mathbb{R}$ such that one of the conditions (1) or (2) holds. If $F \subset X$ is a μ-closed set and $x \in X \setminus F$, then there exists a $(\mu, \nu_{\mathbb{R}})$-continuous function $f : X \to [c, d]$ and $a \in \mathbb{R}$ such that one of the conditions (1) or (2) holds. Without loss of generality we can assume that the condition (1) holds. Thus by Lemma 3.15, the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} 1 & t < a \\ \frac{t-f(x)}{a-f(x)} & a \leq t \leq f(x) \\ 0 & t > f(x) \end{cases}$$

is $(\nu_{\mathbb{R}}, \nu_{\mathbb{R}})$-continuous and therefore by Lemma 3.4, $h = g \circ f : \mathbb{R} \to [0, 1]$ is $(\mu, \nu_{\mathbb{R}})$-continuous such that $h(x) = 0$ and $h(y) = 1$, for every $y \in F$. □

Proposition 3.21. *Let (X, μ) be a GTS. Then, (X, μ) is completely μ-regular if and only if the family $C^*_{\mu,\nu_{\mathbb{R}}}(X)$ separates points and μ-closed sets.*

Proof. Let (X, μ) be completely μ-regular and $x \in X \setminus F$. Then by Lemma 3.20, there exists $f \in C^*_{\mu,\nu_{\mathbb{R}}}(X)$ such that one of the conditions in Lemma 3.20 holds. Without loss of generality we can assume that the condition (1) in Lemma 3.20 holds. Thus $f(x) \notin c_{\nu_{\mathbb{R}}}(f(F))$ since the $\nu_{\mathbb{R}}$-open set $U = (b, \infty)$ contains $f(x)$ but $U \cap f(F) = \emptyset$, where b is an arbitrary number in the open interval $(a, f(x))$. The converse is obvious. □

Theorem 3.22. *A GTS (X, μ) is completely μ-regular if and only if μ is equal to the GT on X generated by the family $C^*_{\mu,\nu_{\mathbb{R}}}(X)$ of maps.*

Proof. If (X,μ) is completely μ-regular, then by Proposition 3.21, $C^*_{\mu,\nu_{\mathbb{R}}}(X)$ separate points and μ-closed sets. Now Theorem 3.11 implies that μ is equal to the GT on X generated by the family $C^*_{\mu,\nu_{\mathbb{R}}}(X)$ of maps.

Conversely, let μ be equal to the GT on X generated by the family $C^*_{\mu,\nu_{\mathbb{R}}}(X)$ of maps. If F is a μ-closed set and $x \notin F$, then $U = F^c$ is a μ-open set containing x. Thus, there exists a $\nu_{\mathbb{R}}$-open subset V of $\mathbb{R}$ and a function $f \in C^*_{\mu,\nu_{\mathbb{R}}}(X)$ such that $x \in f^{-1}(V) \subseteq U$ since μ is the GT on X generated by $C^*_{\mu,\nu_{\mathbb{R}}}(X)$. By Proposition 3.2, we can suppose that V is of the form $(-\infty, b)$ or $(a, +\infty)$ in which $a, b \in \mathbb{R}$. As we mentioned before, $-f \in C^*_{\mu,\nu}(X)$, for each $f \in C^*_{\mu,\nu}(X)$. Furthermore, $f^{-1}(-\infty, a) = (-f)^{-1}(-a, +\infty)$. Thus replacing f by $-f$, enables us to assume that V has the form $(a, +\infty)$. Thus $x \in f^{-1}(a, +\infty) \subseteq U$. This is apparent that $f(x) > a$, and $f(y) \leq a$, for every $y \notin U$. Now Lemma 3.20 completes the proof. □

A space (X,μ) is called μ-T_1 [15] if for every pair of distinct points x and y of X, there exists a μ-open subset U of X such that $x \in U$ and $y \notin U$. μ-T_1 completely μ-regular spaces are called $T_{3.5}$ (or Tychonoff) in [11]. Makai et al. considered $[0,1]$ with the relative topology ν_R and showed that in the category of GenTop (GTS's with continuous functions), a GTS X is $T_{3.5}$ if and only if it is homeomorphic to a subspace Y of a power of the GTS [0,1] (see [11, Theorem 4.12]).

Definition 3.23. Let $\{\mu_\alpha\}_{\alpha\in A}$ be a family of generalized topologies on X. The supremum of μ_α's is defined to be $\mu(\cup_{\alpha\in A}\mu_\alpha)$, i.e., the GT on X generated by the base $\cup_{\alpha\in A}\mu_\alpha$ and it is denoted by $\sup_{\alpha\in A}\mu_\alpha$.

The following result is an immediate consequence of Lemma 2.2(1), Definition 3.2 and Definition 3.23:

Theorem 3.24. *Let $\{\mu_\alpha\}_{\alpha\in A}$ be a family of generalized topologies on X.*

(1) $\sup_{\alpha\in A}\mu_\alpha$ *is the weakest GT on X containing all μ_α's.*

(2) $\sup_{\alpha\in A}\mu_\alpha$ *is the weak GT generated by the family $\{I_\alpha : X \to (X,\mu_\alpha)\}_{\alpha\in A}$ of identity maps.*

Definition 3.25. Let $\{(X_\alpha,\mu_\alpha)\}_{\alpha\in A}$ be a family of generalized topological spaces and let f_α be a map from X_α to a set Y, for each $\alpha \in A$. The strong generalized topology (briefly, strong GT) induced by the family $\mathcal{F} = \{f_\alpha\}_{\alpha\in A}$ of maps on Y is the GT μ_s which consists of all sets U in Y such that $f_\alpha^{-1}(U)$ is μ_α-open, for each $\alpha \in A$.

Definition 3.26. Let $\{\mu_\alpha\}_{\alpha\in A}$ be a family of generalized topologies on a nonempty set X. The infimum of μ_α's is defined to be the GT $\bigcap_{\alpha\in A}\mu_\alpha$ on X denoted by $\inf_{\alpha\in A}\mu_\alpha$.

The following result is an immediate consequence of Definitions 3.25 and 3.26:

Theorem 3.27. *Let $\{\mu_\alpha\}_{\alpha\in A}$ be a family of generalized topologies on a nonempty set X. Then the following statements hold:*

(1) $\inf_{\alpha\in A}\mu_\alpha$ *is the strongest GT on X contained in all μ_α's.*

(2) $\inf_{\alpha\in A}\mu_\alpha$ *is the strong GT on X induced by the family $\{I_\alpha : (X,\mu_\alpha) \to X\}_{\alpha\in A}$ of identity maps.*

The following result is an immediate consequence of Definitions 3.25 and 2.4.

Theorem 3.28. *Let $\{(X_\alpha,\mu_\alpha)\}_{\alpha\in A}$ be a family of generalized topological spaces and let f_α be a map from X_α to a set Y, for each $\alpha \in A$. The strong generalized topology μ_s induced by the family $\mathcal{F} = \{f_\alpha\}_{\alpha\in A}$ of maps on Y is the finest GT μ on Y making each f_α (μ_α,μ)-continuous.*

Theorem 3.29. *Let Y have the strong GT μ generated by the family $\{f_\alpha : X_\alpha \to Y\}_{\alpha\in A}$ of maps. If (Z,ν) is a GTS, then a map $g : (Y,\mu) \to (Z,\nu)$ is (μ,ν)-continuous if and only if $g \circ f_\alpha$ is (μ_α,ν)-continuous, for each $\alpha \in A$.*

Proof. Necessity. It follows from Theorem 3.28 and Lemma 3.4.

Sufficiency. Suppose that $g \circ f_\alpha : X_\alpha \to Z$ is (μ_α,ν)-continuous, for each $\alpha \in A$, and $U \subseteq Z$ is a ν-open set. Thus $f_\alpha^{-1}(g^{-1}(U)) = (g \circ f_\alpha)^{-1}(U)$ is μ_α-open, for each $\alpha \in A$. Therefore by Definition 3.25, $g^{-1}(U)$ is μ-open which yields that g is (μ,ν)-continuous. □

Theorem 3.30. *Let $\{(X_\alpha, \mu_\alpha)\}_{\alpha\in A}$ be a family of generalized topological spaces and let $\mathcal{F} = \{f_\alpha : (X_\alpha, \mu_\alpha) \to Y\}_{\alpha\in A}$ be a family of maps such that for each $\alpha \in A$, there exists a family $\{(Z_{\alpha_\lambda}, \mu_{\alpha_\lambda})\}_{\lambda\in\Lambda_\alpha}$ of generalized topological spaces. If ν_s is the strong GT on Y induced by the family $\mathcal{F}$ of maps and for every $\alpha \in A$, μ_α is equal to the strong GT on X_α induced by the family $\mathcal{G}_\alpha = \{g_{\alpha\lambda} : (Z_{\alpha\lambda}, \mu_{\alpha\lambda}) \to X_\alpha\}_{\lambda\in\Lambda_\alpha}$ of maps, then ν_s is equal to the strong GT on Y induced by the family $\{f_\alpha \circ g_{\alpha\lambda} : (Z_{\alpha\lambda}, \mu_{\alpha\lambda}) \to Y\}_{\alpha\in A,\lambda\in\Lambda_\alpha}$ of maps.*

Proof. Let ν be the strong GT on Y generated by the family $\{f_\alpha \circ g_{\alpha\lambda} : (Z_{\alpha\lambda}, \mu_{\alpha\lambda}) \to Y\}_{\alpha\in A,\lambda\in\Lambda_\alpha}$ of maps. We claim that $\nu = \nu_s$. Let $U \subseteq Y$ be ν_s-open. Then by our assumption, $f_\alpha^{-1}(U) \in \mu_\alpha$ for each $\alpha \in A$ and so for each $\lambda \in \Lambda_\alpha$, $(f_\alpha \circ g_{\alpha\lambda})^{-1}(U) = g_{\alpha\lambda}^{-1}(f_\alpha^{-1}(U))$ is $\mu_{\alpha\lambda}$-open which implies that U is ν-open. Thus $\nu_s \subseteq \nu$.

Conversely, suppose that $U \subseteq Y$ be ν-open. Then by our assumption, $(f_\alpha \circ g_{\alpha\lambda})^{-1}(U)$ is $\mu_{\alpha\lambda}$-open for each $\alpha \in A$ and $\lambda \in \Lambda_\alpha$, and so $g_{\alpha\lambda}^{-1}(f_\alpha^{-1}(U))$ is $\mu_{\alpha\lambda}$-open, for each $\lambda \in \Lambda_\alpha$. Thus, $f_\alpha^{-1}(U)$ is μ_α-open since μ_α is the strong GT on X_α induced by the family $\mathcal{G}_\alpha$ of maps. Thus, U is ν_s-open since ν_s is the strong GT on Y induced by the family $\mathcal{F}$ of maps. Therefore, $\nu \subseteq \nu_s$ and this completes the proof. □

Acknowledgement

The authors thank referees for their careful reading and valuable comments.

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