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Approximating Common Fixed Points by a New Faster Iteration Process

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Abstract. In this paper, we propose a three-step iteration process and show that this process converges faster than a number of existing iteration processes. We give a numerical example followed by graphs to validate our claim. We prove strong and weak convergence results for approximating common fixed points for two nonexpansive mappings. Again we reconfirm our results by examples and tables. Further, we provide some applications of the our iteration process.

1. Introduction

A point which always remains invariant when subject to a transformation is referred to as a fixed point. The existence of such points has a significant role in various fields of mathematics which include: Topology, Algebraic Topology, Nonlinear Operators, Differential Equations (both ordinary and partial) and Functional Analysis. Many operator equations, by suitable manipulations can be written in the form $x = Px$ wherein the solution of the equation is represented by a fixed point. By using techniques of fixed point theory, one can effectively obtain appropriate and adequate solutions of operator equations representing phenomena occurring in different fields of nonlinear sciences. Thus, the aim of finding solutions to these equations is to locate the fixed point and its approximation.
Approximation of fixed points in different domains for nonlinear mappings using the different iterative processes is the thrust of fixed point theory. Owing to it's importance fixed point theory is attracting young researchers across the world and in the last few years many iterative processes have been obtained in different domains. To name a few, we have Mann [12], Ishikawa [8], Noor [13], Agarwal et al. [2], Abbas and Nazir [1], Thakur et al. [19, 20], Picard-Mann hybrid [9], RK [15] and M^* [21].
Following Khan [9], Ullah and Arshad [22] modified his Picard-Mann hybrid iteration process and named it M-iteration process and they, without giving any analytical proof, gave an example to show that it converges faster than Picard-S [6] and S iteration [2]. In order to define their iteration, let D be a nonempty closed convex subset of a uniformly convex Banach space A and $P : D \to D$ be a nonexpansive mapping.

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Then, a sequence $\{j_n\}$ is constructed from arbitrary $j_1 \in D$ by:

$$\begin{cases} h_n = (1-\varrho_n)j_n + \varrho_n P j_n \\ i_n = P h_n \\ j_{n+1} = P i_n \end{cases} \tag{1.1}$$

for each $n \in \mathbb{N}$ and $\{\varrho_n\}$ is a sequence in $(0,1)$.

Motivated and inspired by the research going on in this direction,we introduce a new modified iteration process for approximating common fixed points of two nonexpansive mappings to achieve a better rate of convergence. Let $P_1, P_2 : D \to D$ be two nonexpansive mappings, then the sequence $\{g_n\}$ is generated iteratively by $g_1 \in D$ and

$$\begin{cases} e_n = (1-\varrho_n)g_n + \varrho_n P_1 g_n \\ f_n = P_1((1-\varsigma_n)e_n + \varsigma_n P_2 e_n) \\ g_{n+1} = P_2 f_n \end{cases} \tag{1.2}$$

for each $n \in \mathbb{N}$ and $\{\varrho_n\}$ and $\{\varsigma_n\}$ are sequences in $(0,1)$.

Obviously, our process deals with the common fixed points and it is very well-known that a common fixed point problem has direct link with a minimization process.

Remark:

Note that our process is comparable with the iterative processes mentioned above. For example,

- It reduces to Ullah and Arshad [22] when $P_1 = P_2 = P$.
- It is independent of Ullah and Arshad [21] and Thakur et al. [19, 20].
- None of the above mentioned iterative processes deal with common fixed points whereas our process does.

The aim of this paper is to prove that the newly defined iteration process (1.2) converges faster than iteration process (1.1) for contractive-like mappings. Also, we prove some weak and strong convergence results involving the iteration process (1.2) for nonexpansive mappings. Further, we provide a numerical example to show that our process (1.2) converges faster than a number of existing iteration processes. At the end, we apply our iteration process to find solution of a variational inequality problem and a constrained minimization problem.

2. Preliminaries

We begin by recalling some known lemmas and definitions which will be frequently used throughout the text.

A mapping $P : D \to D$ is said to be nonexpansive if $\|Pq - Pr\|\| \le \|q - r\|$ for all $q, r \in D$. A point $k \in D$ is said to be a fixed point of P if $Pk = k$. We will denote the set of fixed points of P by $F(P)$.

Definition 2.1. A Banach space A is said to be uniformly convex if for each $\alpha \in (0,2]$ there is a $\beta > 0$ such that for $r, q \in A$ with $\|r\| \le 1$, $\|q\| \le 1$ and $\|r - q\| > \alpha$, we have

$$\left\|\frac{r+q}{2}\right\| < 1 - \beta.$$

Definition 2.2. A Banach space A is said to satisfy the Opial's condition if for any sequence $\{g_n\}$ in A which converges weakly to $g \in A$ i.e. $g_n \rightharpoonup g$ implies that

$$\limsup_{n\to\infty} \|g_n - g\| < \limsup_{n\to\infty} \|g_n - y\|$$

for all $y \in A$ with $y \neq g$.

A mapping $P : D \to A$ is demiclosed at $a \in A$ if for each sequence $\{g_n\}$ in D and each $b \in A$, $g_n \rightharpoonup b$ and

$Pg_n \to a$ imply that $b \in D$ and $Pb = a$.
The following definitions about the rate of convergence were given by Berinde [4].
Definition 2.3. Let $\{a_n\}$ and $\{b_n\}$ be two real sequences converging to a and b respectively. Then, $\{a_n\}$ converges faster than $\{b_n\}$ if $\lim\limits_{n\to\infty} \frac{\|a_n-a\|}{\|b_n-b\|} = 0$.
Definition 2.4. Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes converging to the same fixed point p. If $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero such that $\|u_n - p\| \leq a_n$ and $\|v_n - p\| \leq b_n$ for all $n \geq 1$, then we say that $\{u_n\}$ converges faster than $\{v_n\}$ to p if $\{a_n\}$ converges faster then $\{b_n\}$.
Next, we list two lemmas which will be useful in our subsequent discussion.
Lemma 2.1. ([18]) Let D be a nonempty closed convex subset of a uniformly convex Banach space A and P a nonexpansive mapping on D. Then, $I - P$ is demiclosed at zero.
Lemma 2.2. ([16]) Let A be a uniformly convex Banach space and $\{j_n\}$ be any sequence such that $0 < p \leq j_n \leq q < 1$ for some $p, q \in \mathbb{R}$ and for all $n \geq 1$. Let $\{u_n\}$ and $\{v_n\}$ be any two sequences of A such that $\limsup\limits_{n\to\infty} \|u_n\| \leq r$, $\limsup\limits_{n\to\infty} \|v_n\| \leq r$ and $\limsup\limits_{n\to\infty} \|j_n u_n + (1 - j_n)v_n\| = r$ for some $r \geq 0$. Then, $\lim\limits_{n\to\infty} \|u_n - v_n\| = 0$.
As a background of our exposition, we now mention some contractive mappings.
Suppose that for a mapping $P : D \to D$, there exist real numbers a, b, c satisfying $0 < a < 1$, $0 < b, c < \frac{1}{2}$ such that, for each pair $x, y \in D$, at least one of the following is true:

$$\begin{cases} (z_1) & \|Px - Py\| \leq a\|x - y\|, \\ (z_2) & \|Px - Py\| \leq b(\|x - Px\| + \|y - Py\|), \\ (z_3) & \|Px - Py\| \leq c(\|x - Py\| + \|y - Px\|). \end{cases} \tag{2.1}$$

A mapping P satisfying the contractive conditions (z_1), (z_2) and (z_3) in (2.1) is called a Zamfirescu operator [23]. An operator satisfying condition (z_2) is called a Kannan operator, while the mapping satisfying condition (z_3) is called a Chatterjea operator. As shown in [5], the contractive condition (2.1) leads to

$$\begin{cases} (b_1) & \|Px - Py\| \leq \delta\|x - y\| + 2\delta\|x - Px\| \quad if \quad one \quad uses \quad (z_2) \\ (b_2) & \|Px - Py\| \leq \delta\|x - y\| + 2\delta\|x - Py\| \quad if \quad one \quad uses \quad (z_3), \end{cases} \tag{2.2}$$

for all $x, y \in D$, where $\delta := max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$, $\delta \in [0, 1)$, and it was shown that this class of operators is wider than the class of Zamfirescu operators. Any mapping satisfying condition (b_1) or (b_2) is called a quasi-contractive operator.
Osilike and Udomene [14] considered operator P for which there exist real numbers $L \geq 0$ and $\delta \in [0, 1)$ such that for all $x, y \in D$,

$$\|Px - Py\| \leq \delta\|x - y\| + L(\|x - Px\|). \tag{2.3}$$

Imoru and Olantiwo [7] gave a more general definition: An operator P is called a contractive-like operator if there exists a constant $\delta \in [0, 1)$ and a strictly increasing and continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that for each $x, y \in D$,

$$\|Px - Py\| \leq \delta\|x - y\| + \varphi(\|x - Px\|). \tag{2.4}$$

3. Rate of Convergence Results

In this section, first we show that our iteration process (1.2) converges faster than the M-iteration process (1.1) for contractive-like mappings. It must be noted here that Ullah and Arshad [22] never gave the rate of convergence of their process analytically. They just gave an example. However, we not only give the proof analytically but also validate with a nontrivial example.
Theorem 3.1. Let D be a nonempty closed convex subset of a Banach space A and $P_1, P_2 : D \to D$ be two contractive-like mappings as in (2.4) with $F(P_1) \cap F(P_2) \neq \emptyset$. If $\{g_n\}$ is a sequence defined by (1.2), then $\{g_n\}$

converges faster than $\{j_n\}$ given by the iteration process (1.1).
Proof. From (1.2) and (2.4), for any $k \in F(P_1) \cap F(P_2)$,

$$\begin{aligned} \|e_n - k\| &= \|(1-\varrho_n)g_n + \varrho_n P_1 g_n - k\| \\ &\le (1-\varrho_n)\|g_n - k\| + \varrho_n\|P_1 g_n - k\| \\ &\le (1-\varrho_n)\|g_n - k\| + \varrho_n\delta\|g_n - k\| \\ &= (1-(1-\delta)\varrho_n)\|g_n - k\| \end{aligned}$$

and

$$\begin{aligned} \|f_n - k\| &= \|P_1((1-\varsigma_n)e_n + \varsigma_n P_2 e_n) - k\| \\ &\le \delta(\|(1-\varsigma_n)e_n + \varsigma_n P_2 e_n - k\|) \\ &\le \delta((1-\varsigma_n)\|e_n - k\| + \varsigma_n\delta\|e_n - k\|) \\ &= \delta(1-(1-\delta)\varsigma_n)\|e_n - k\| \\ &\le \delta(1-(1-\delta)\varsigma_n)(1-(1-\delta)\varrho_n)\|g_n - k\|. \end{aligned}$$

As, $\{\varrho_n\}, \{\varsigma_n\}$ are sequences in $(0,1)$, we can find $\varrho_n, \varsigma_n \in \mathbb{R}$ such that $\varrho_n \le \varrho < 1$ and $\varsigma_n \le \varsigma < 1$ for all $n \in \mathbb{N}$. So,

$$\begin{aligned} \|g_{n+1} - k\| &= \|P_2 f_n - k\| \\ &\le \delta\|f_n - k\| \\ &\le \delta^2(1-(1-\delta)\varsigma_n)(1-(1-\delta)\varrho_n)\|g_n - k\| \\ &\le \delta^2(1-(1-\delta)\varsigma)(1-(1-\delta)\varrho)\|g_n - k\| \\ &\vdots \\ &\le \delta^{2n}(1-(1-\delta)\varsigma)^n(1-(1-\delta)\varrho)^n\|g_1 - k\|. \end{aligned}$$

Now, for $k \in F(P)$ using (1.1), we have

$$\begin{aligned} \|h_n - k\| &= \|(1-\varrho_n)j_n + \varrho_n P j_n - k\| \\ &\le (1-\varrho_n)\|j_n - k\| + \varrho_n\|P j_n - k\| \\ &\le (1-\varrho_n)\|j_n - k\| + \varrho_n\delta\|j_n - k\| \\ &= (1-(1-\delta)\varrho_n)\|j_n - k\| \end{aligned}$$

and

$$\begin{aligned} \|i_n - k\| &= \|P h_n - k\| \\ &\le \delta\|h_n - k\| \\ &\le \delta(1-(1-\delta)\varrho_n)\|j_n - k\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|j_{n+1} - k\| &= \|P i_n - k\| \\ &\le \delta\|i_n - k\| \\ &\le \delta^2(1-(1-\delta)\varrho_n)\|j_n - k\| \\ &\le \delta^2(1-(1-\delta)\varrho)\|j_n - k\| \\ &\vdots \\ &\le \delta^{2n}(1-(1-\delta)\varrho)^n\|j_1 - k\|. \end{aligned}$$

Let

$$b_n = \delta^{2n}(1-(1-\delta)\varrho)^n\|j_1 - k\|$$

and

$$a_n = \delta^{2n}(1-(1-\delta)\varsigma)^n(1-(1-\delta)\varrho)^n\|g_1 - k\|.$$

Then,

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{\delta^{2n}(1-(1-\delta)\varsigma)^n(1-(1-\delta)\varrho)^n\|g_1-k\|}{\delta^{2n}(1-(1-\delta)\varrho)^n\|j_1-k\|} \\ &\to 0 \quad as \quad n\to\infty.\end{aligned}$$

Hence, $\{g_n\}$ converges faster than $\{j_n\}$.

Theorem 3.2. Let D be a nonempty closed convex subset of a Banach space A and $P_1, P_2 : D \to D$ be two contractive-like mappings with $F(P_1) \cap F(P_2) \neq \emptyset$. If $\{g_n\}$ is a sequence defined by (1.2), then $\{g_n\}$ converges to a common fixed point of P_1 and P_2.
Proof. From (1.2), for any $k \in F(P_1) \cap F(P_2)$,

$$\begin{aligned}\|e_n - k\| &= \|(1-\varrho_n)g_n + \varrho_n P_1 g_n - k\| \\ &\leq (1-\varrho_n)\|g_n - k\| + \varrho_n\|P_1 g_n - k\| \\ &\leq (1-\varrho_n)\|g_n - k\| + \varrho_n\delta\|g_n - k\| \\ &= (1-(1-\delta)\varrho_n)\|g_n - k\|\end{aligned}$$

and

$$\begin{aligned}\|f_n - k\| &= \|P_1((1-\varsigma_n)e_n + \varsigma_n P_2 e_n) - k\| \\ &\leq \delta(\|(1-\varsigma_n)e_n + \varsigma_n P_2 e_n - k\|) \\ &\leq \delta((1-\varsigma_n)\|e_n - k\| + \varsigma_n\delta\|e_n - k\|) \\ &= \delta(1-(1-\delta)\varsigma_n)\|e_n - k\| \\ &\leq \delta(1-(1-\delta)\varsigma_n)(1-(1-\delta)\varrho_n)\|g_n - k\|.\end{aligned}$$

As, $\{\varrho_n\}$, $\{\varsigma_n\}$ are sequences in $(0, 1)$ and $0 \leq \delta < 1$, we have $(1-(1-\delta)\varrho_n) < 1$ and $(1-(1-\delta)\varsigma_n) < 1$ for all $n \in \mathbb{N}$. So,

$$\begin{aligned}\|g_{n+1} - k\| &= \|P_2 f_n - k\| \\ &\leq \delta\|f_n - k\| \\ &\leq \delta^2(1-(1-\delta)\varsigma_n)(1-(1-\delta)\varrho_n)\|g_n - k\| \\ &\leq \delta^2\|g_n - k\| \\ &\vdots \\ &\leq \delta^{2n}\|g_1 - k\|.\end{aligned}$$

Since, $0 \leq \delta < 1$, we get

$$\lim_{n\to\infty} \delta^n = 0.$$

Thus,

$$\lim_{n\to\infty} \|g_{n+1} - k\| = 0.$$

Now, we present an example to show that our process (1.2) converges faster than many iterations namely: Thakur et. al, M, M^*, Abbas and Nazir, Agarwal, Noor, SP iteration processes for contractive-like mappings.
Example 1. Let $A = \mathbb{R}$ and $D = [0, 6]$. Let $P : D \to D$ be a mapping defined as

$$P(x) = \begin{cases} \frac{x}{4} & x \in [0, 3) \\ \frac{x}{8} & x \in [3, 6]. \end{cases}$$

for all $x \in D$.
Proof: Clearly $x = 0$ is the fixed point of P. First, we prove that P is a contractive-like mapping but not

a contraction. Since P is not continuous at $x = 3 \in [0, 6]$, so P is not a contraction. We show that P is a contractive-like mapping. For this, define $\varphi : [0, \infty) \to [0, \infty)$ as $\varphi(x) = \frac{x}{6}$. Then, φ is a strictly increasing as well as continuous function. Also, $\varphi(0) = 0$.
We need to show that

$$\|Px - Py\| \leq \delta\|x - y\| + \varphi(\|x - Px\|) \tag{A}$$

for all $x, y \in [0, 6]$ and δ is a constant in $[0, 1)$.
Before going ahead, let us note the following. When $x \in [0, 3)$,

$$\|x - Px\| = \left\|x - \frac{x}{4}\right\| = \frac{3x}{4}$$

and

$$\varphi\left(\frac{3x}{4}\right) = \frac{3x}{24} = \frac{x}{8}. \tag{3.1}$$

Similarly, when $x \in [3, 6]$,then

$$\|x - Px\| = \left\|x - \frac{x}{8}\right\| = \frac{7x}{8}$$

and

$$\varphi\left(\frac{7x}{8}\right) = \frac{7x}{48}. \tag{3.2}$$

Now, consider the following cases:
Case A: Let $x, y \in [0, 3)$. Using (3.1) we get

$$\begin{aligned} \|Px - Py\| &= \|\tfrac{x}{4} - \tfrac{y}{4}\| \\ &= \tfrac{1}{4}\|x - y\| + \varphi(\tfrac{3x}{4}) \\ &= \tfrac{1}{4}\|x - y\| + \varphi(\|x - Px\|). \end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{4}$.
Case B: Let $x \in [0, 3)$ and $y \in [3, 6]$. Using (3.1), we get

$$\begin{aligned} \|Px - Py\| &= \|\tfrac{x}{4} - \tfrac{y}{8}\| \\ &\leq \tfrac{1}{8}\|x - y\| + \left\|\tfrac{x}{8}\right\| \\ &\leq \tfrac{1}{4}\|x - y\| + \varphi(\tfrac{3x}{4}) \\ &= \tfrac{1}{4}\|x - y\| + \varphi(\|x - Px\|). \end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{4}$.
Case C: Let $x \in [3, 6]$ and $y \in [0, 3)$. Using (3.2), we get

$$\begin{aligned} \|Px - Py\| &= \|\tfrac{x}{8} - \tfrac{y}{4}\| \\ &\leq \tfrac{1}{4}\|x - y\| + \left\|\tfrac{7x}{48}\right\| \\ &= \tfrac{1}{4}\|x - y\| + \varphi(\|x - Px\|). \end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{4}$.
Case D: Let $x, y \in [3, 6]$. Using (3.2), we get

$$\begin{aligned} \|Px - Py\| &= \|\tfrac{x}{8} - \tfrac{y}{8}\| \\ &\leq \tfrac{1}{4}\|x - y\| + \left\|\tfrac{7x}{48}\right\| \\ &= \tfrac{1}{4}\|x - y\| + \varphi(\|x - Px\|). \end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{4}$.
Consequently, (A) is satisfied for $\delta = \frac{1}{4}$ and $\varphi(x) = \frac{x}{6}$ in all the possible cases. Thus, P is a contractive-like mapping.
Now, using P, we show that our iteration process converges at a better rate. Set $\varrho_n = \varsigma_n = \gamma_n = \frac{n}{n+1}$ for each $n \in \mathbb{N}$. Then, we get the following Table 1 and Table 2 with the initial value 4.5.

Step	Agarwal Iteration	Abbas Iteration	Thakur New	M Iteration	Our Iteration
1	4.5	4.5	4.5	4.5	4.5
2	0.59765625	0.3911132813	0.1098632813	0.158203125	0.09887695313
3	0.099609375	0.03802490234	0.004577636719	0.004943847656	0.001544952393
		⋮			
6	0.000224198103	0.00002489541683	$1.609878382 \times 10^{-7}$	$7.920898497 \times 10^{-8}$	$1.624403012 \times 10^{-9}$
7	0.00002516509319	$2.005060904 \times 10^{-6}$	$4.517515867 \times 10^{-9}$	1.7680577×10^{-9}	$1.294964136 \times 10^{-11}$

Table 1:

Step	Noor Iteration	Picard S Iteration	Thakur Iteration	M^* Iteration	Our Iteration
1	4.5	4.5	4.5	4.5	4.5
2	2.570800781	0.1494140625	0.4790039063	0.1604003906	0.09887695313
3	1.035461426	0.006225585938	0.05987548828	0.004177093506	0.001544952393
⋮		⋮			
8	0.0005774046657	$1.63495252 \times 10^{-10}$	$1.234039751 \times 10^{-7}$	$3.981751878 \times 10^{-12}$	$9.563638943 \times 10^{-14}$
9	0.00008791758285	$4.163073546 \times 10^{-12}$	$6.474899929 \times 10^{-9}$	$4.608509118 \times 10^{-14}$	$6.641415933 \times 10^{-16}$
10	0.00001221642288	$1.021253979 \times 10^{-13}$	$3.157525419 \times 10^{-10}$	$4.986550882 \times 10^{-16}$	$4.384372237 \times 10^{-18}$

Table 2:

Also, the Figure 1 and Figure 2 shows that our iteration process (1.2) converges faster to $x = 0$ which is a fixed point of P.

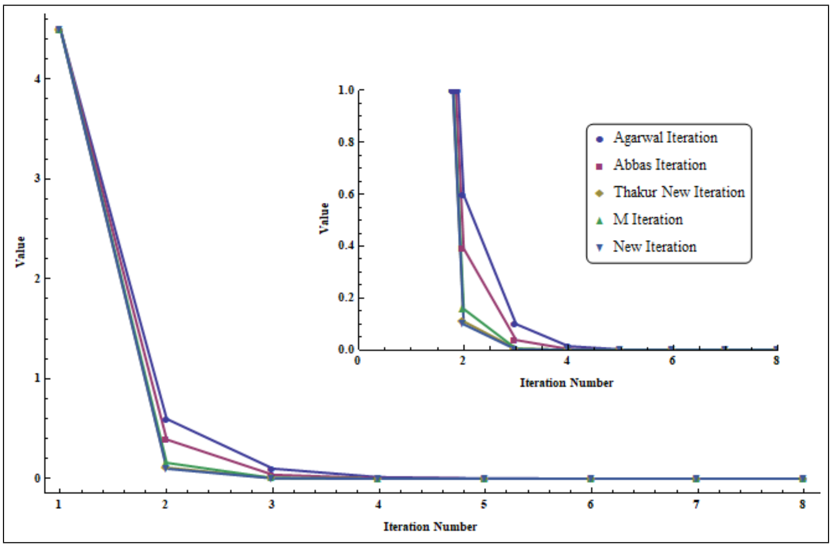

Figure 1:

Thus, it is evident from the tables as well as graphs that the newly introduced iteration process converges at a much faster rate than a number of the existing iteration processes.

4. Convergence Results

Our results in this section are just new of their kind. They are independent of, for example, [19, 20] and [21, 22].They are better in the sense of approximating common fixed points as opposed to [19, 20] and [21, 22].

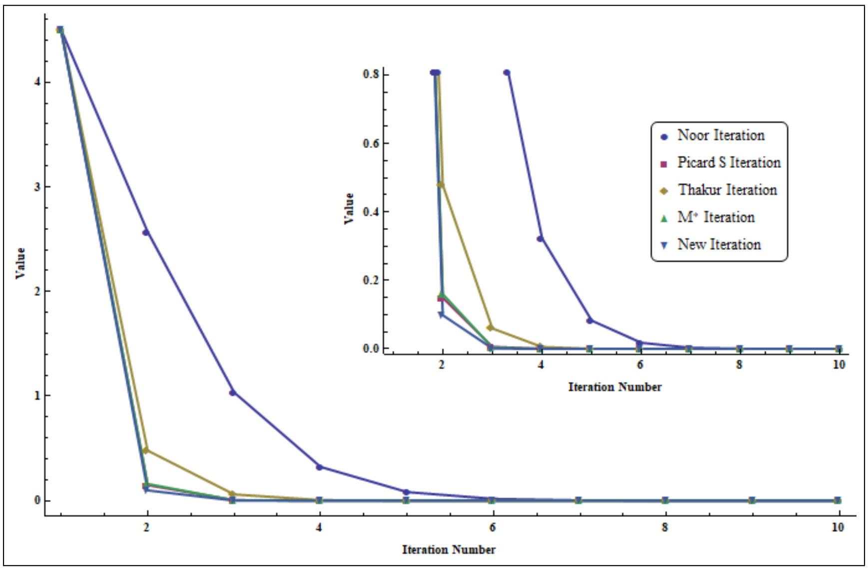

Figure 2:

Lemma 4.1. Let D be a nonempty closed convex subset of a Banach space A and $P_1, P_2 : D \to D$ be two nonexpansive mappings with $F(P_1) \cap F(P_2) \neq \emptyset$. Let $\{g_n\}$ be defined by the iteration process (1.2). Then
(i) $\lim_{n\to\infty} \|g_n - k\|$ exists for all $k \in F(P_1) \cap F(P_2)$,
(ii) $\lim_{n\to\infty} \|P_1 g_n - g_n\| = \lim_{n\to\infty} \|P_2 g_n - g_n\| = 0$.
Proof. Let $k \in F(P_1) \cap F(P_2)$. Then, using (1.2) we get

$$\begin{aligned} \|e_n - k\| &= \|(1-\varrho_n)g_n + \varrho_n P_1 g_n - k\| \\ &\leq (1-\varrho_n)\|g_n - k\| + \varrho_n\|P_1 g_n - k\| \\ &\leq (1-\varrho_n)\|g_n - k\| + \varrho_n\|g_n - k\| \\ &= \|g_n - k\| \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \|f_n - k\| &= \|P_1((1-\varsigma_n)e_n + \varsigma_n P_2 e_n) - k\| \\ &\leq (1-\varsigma_n)\|e_n - k\| + \varsigma_n\|P_2 e_n - k\| \\ &\leq (1-\varsigma_n)\|e_n - k\| + \varsigma_n\|e_n - k\| \\ &= \|e_n - k\| \\ &\leq \|g_n - k\|. \end{aligned} \tag{4.2}$$

Using (4.1) and (4.2) we obtain

$$\begin{aligned} \|g_{n+1} - k\| &= \|P_2 f_n - k\| \\ &\leq \|f_n - k\| \\ &\leq \|e_n - k\| \\ &\leq \|g_n - k\|. \end{aligned}$$

Thus, $\{\|g_n - k\|\}$ is bounded and non-increasing for all $k \in F(P_1) \cap F(P_2)$ which gives that $\lim_{n\to\infty} \|g_n - k\|$ exists for all $k \in F(P_1) \cap F(P_2)$.
(ii) Let $\lim_{n\to\infty} \|g_n - k\| = \chi$.
From (4.1) and (4.2), we get

$$\limsup_{n\to\infty} \|e_n - k\| \leq \chi \tag{4.3}$$

and

$$\limsup_{n\to\infty} \|f_n - k\| \le \chi. \tag{4.4}$$

Now,

$$\chi = \lim_{n\to\infty} \|g_{n+1} - k\| = \lim_{n\to\infty} \|P_2 f_n - k\|.$$

So,

$$\lim_{n\to\infty} \|P_2 f_n - k\| = \chi.$$

Also,

$$\|P_2 f_n - k\| \le \|f_n - k\|$$

which gives

$$\chi \le \liminf_{n\to\infty} \|f_n - k\|.$$

So, using (4.4), we have

$$\lim_{n\to\infty} \|f_n - k\| = \chi. \tag{4.5}$$

From (4.2), we obtain

$$\|f_n - k\| \le \|e_n - k\|,$$

which yields

$$\chi \le \liminf_{n\to\infty} \|e_n - k\|. \tag{4.6}$$

Owing to (4.3) and (4.6), we get

$$\lim_{n\to\infty} \|e_n - k\| = \chi, \tag{4.7}$$

and hence in view of Lemma 2.2.

$$\lim_{n\to\infty} \|P_1 g_n - g_n\| = 0. \tag{4.8}$$

Now,

$$\begin{aligned} \|g_n - e_n\| &= \|(1-\varrho_n)g_n + \varrho_n P_1 g_n - g_n\| \\ &= \|\varrho_n (P_1 g_n - g_n).\| \end{aligned}$$

So, by using (4.8), we obtain

$$\lim_{n\to\infty} \|g_n - e_n\| = 0. \tag{4.9}$$

Consider,

$$\begin{aligned} \|f_n - k\| &= \|P_1((1-\varsigma_n)e_n + \varsigma_n P_2 e_n) - k\| \\ &\le \|(1-\varsigma_n)e_n + \varsigma_n P_2 e_n - k\| \\ &\le \|e_n - k\|, \end{aligned}$$

which on using (4.5) and (4.7) gives

$$\lim_{n\to\infty} \|(1-\varsigma_n)e_n + \varsigma_n P_2 e_n - k\| = \chi. \tag{4.10}$$

Also, since P_2 is nonexpansive so we obtain

$$\limsup_{n\to\infty} \|P_2 e_n - k\| \le \limsup_{n\to\infty} \|e_n - k\| = \chi. \tag{4.11}$$

Owing to (4.7), (4.10), (4.11) and Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|P_2e_n - e_n\| = 0. \tag{4.12}$$

Now, consider

$$\begin{aligned} \|P_2g_n - g_n\| &\leq \|P_2g_n - P_2e_n\| + \|P_2e_n - e_n\| + \|e_n - g_n\| \\ &\leq \|g_n - e_n\| + \|P_2e_n - e_n\| + \|e_n - g_n\|. \end{aligned}$$

On using (4.9) and (4.12), we get

$$\lim_{n\to\infty} \|P_2g_n - g_n\| = 0.$$

Now, we prove the weak convergence of iteration process (1.2).

Theorem 4.1. Let D be a nonempty closed convex subset of a uniformly convex Banach space A which satisfies the Opial's condition and $P_1, P_2 : D \to D$ be two nonexpansive mapping with $F(P_1) \cap F(P_2) \neq \emptyset$. If $\{g_n\}$ is defined by the iteration process (1.2), then $\{g_n\}$ converges weakly to a common fixed point of P_1 and P_2.

Proof. Let $k \in F(P_1) \cap F(P_2)$. Then, from Lemma 4.1 $\lim_{n\to\infty} \|g_n - k\|$ exists. In order to show the weak convergence of the iteration process (1.2) to a common fixed point of P_1 and P_2, we will prove that $\{g_n\}$ has a unique weak subsequential limit in $F(P_1) \cap F(P_2)$. For this, let $\{g_{n_j}\}$ and $\{g_{n_k}\}$ be two subsequences of $\{g_n\}$ which converges weakly to w and y respectively. By Lemma 4.1, we have $\lim_{n\to\infty} \|P_1g_n - g_n\| = \lim_{n\to\infty} \|P_2g_n - g_n\| = 0$ and using the Lemma 2.1, we have $I - P_1$ and $I - P_2$ are demiclosed at zero. So $w, y \in F(P_1) \cap F(P_2)$.

Next, we show the uniqueness. Since $w, y \in F(P_1) \cap F(P_2)$, so $\lim_{n\to\infty} \|g_n - w\|$ and $\lim_{n\to\infty} \|g_n - y\|$ exists. Let $w \neq y$. Then, by Opial's condition, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|g_n - w\| &= \lim_{n_j\to\infty} \|g_{n_j} - w\| \\ &< \lim_{n_j\to\infty} \|g_{n_j} - y\| \\ &= \lim_{n\to\infty} \|g_n - y\| \\ &= \lim_{n_k\to\infty} \|g_{n_k} - y\| \\ &< \lim_{n_k\to\infty} \|g_{n_k} - w\| \\ &= \lim_{n\to\infty} \|g_n - w\| \end{aligned}$$

which is a contradiction, so $w = y$. Thus, $\{g_n\}$ converges weakly to a common fixed point of P_1 and P_2.

Next, we establish some strong convergence results for iteration process (1.2).

Theorem 4.2. Let D be a nonempty closed convex subset of a Uniformly convex Banach space A and $P_1, P_2 : D \to D$ be two nonexpansive mappings with $F(P_1) \cap F(P_2) \neq \emptyset$. If $\{g_n\}$ is defined by the iteration process (1.2), then $\{g_n\}$ converges to a point of $F(P_1) \cap F(P_2)$ if and only if $\liminf_{n\to\infty} d(g_n, F(P_1) \cap F(P_2)) = 0$.

Proof. If the sequence $\{g_n\}$ converges to a point $k \in F(P_1) \cap F(P_2)$, then it is obvious that $\liminf_{n\to\infty} d(g_n, F(P_1) \cap F(P_2)) = 0$.

For the converse part, assume that $\liminf_{n\to\infty} d(g_n, F(P_1) \cap F(P_2)) = 0$. From Lemma 4.1, we have $\lim_{n\to\infty} \|g_n - k\|$ exists for all $k \in F(P_1) \cap F(P_2)$, which gives

$$\|g_{n+1} - k\| \leq \|g_n - k\| \text{ for any } k \in F(P_1) \cap F(P_2)$$

which yields

$$d(g_{n+1}, F(P_1) \cap F(P_2)) \leq d(g_n, F(P_1) \cap F(P_2)). \tag{4.13}$$

Thus, $\{d(g_n, F(P_1) \cap F(P_2))\}$ forms a decreasing sequence which is bounded below by zero as well, so we get that $\lim_{n\to\infty} d(g_n, F(P_1) \cap F(P_2))$ exists. As, $\liminf_{n\to\infty} d(g_n, F(P_1) \cap F(P_2)) = 0$ so $\lim_{n\to\infty} d(g_n, F(P_1) \cap F(P_2)) = 0$.

Next, we prove that $\{g_n\}$ is a Cauchy sequence in D. Let $\epsilon > 0$ be arbitrarily chosen. Since $\liminf_{n\to\infty} d(g_n, F(P_1) \cap F(P_2)) = 0$, there exists n_0 such that for all $n \geq n_0$, we have

$$d(g_n, F(P_1) \cap F(P_2)) < \frac{\epsilon}{4}.$$

In particular,

$$\inf\{\|g_{n_0} - k\| : k \in F(P_1) \cap F(P_2)\} < \frac{\epsilon}{4},$$

so there must exist a $b \in F(P_1) \cap F(P_2)$ such that

$$\|g_{n_0} - b\| < \frac{\epsilon}{2}.$$

Thus, for $m, n \geq n_0$, we have

$$\|g_{n+m} - g_n\| \leq \|g_{n+m} - b\| + \|g_n - b\| < 2\|g_{n_0} - b\| < 2\frac{\epsilon}{2} = \epsilon$$

which shows that $\{g_n\}$ is a Cauchy sequence. Since D is a closed subset of a Banach space A, therefore $\{g_n\}$ must converge in D. Let, $\lim_{n\to\infty} g_n = s$ for some $s \in D$.

Now using $\lim_{n\to\infty} \|P_1 g_n - g_n\| = 0$, we get

$$\begin{aligned} \|s - P_1 s\| &\leq \|s - g_n\| + \|g_n - P_1 g_n\| + \|P_1 g_n - P_1 s\| \\ &\leq \|s - g_n\| + \|g_n - P_1 g_n\| + \|g_n - s\| \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

and hence $s = P_1 s$. Similarly, we can show that $s = P_2 s$, thus $s \in F(P_1) \cap F(P_2)$. This proves our result.

Two mappings $P_1, P_2 : D \to D$ are said to satisfy the Condition (A) ([17]) if there exists a nondecreasing function $t : [0, \infty) \to [0, \infty)$ with $t(0) = 0$ and $t(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|u - P_1 u\| \geq t(d(u, F(P_1) \cap F(P_2)))$$

or

$$\|u - P_2 u\| \geq t(d(u, F(P_1) \cap F(P_2)))$$

for all $u \in D$.

Theorem 4.3. Let D be a nonempty closed convex subset of a uniformly convex Banach space A. Let $P_1, P_2 : D \to D$ be two nonexpansive mapping such that $F(P_1) \cap F(P_2) \neq \emptyset$ and $\{g_n\}$ be the sequence defined by (1.2). If P_1 and P_2 satisfies Condition (A), then $\{g_n\}$ converges strongly to a point of $F(P_1) \cap F(P_2)$.

Proof. From (4.13) we get $\lim_{n\to\infty} d(g_n, F(P_1) \cap F(P_2))$ exists.
Also, by Lemma 4.1, we have $\lim_{n\to\infty} \|g_n - P_1 g_n\| = \lim_{n\to\infty} \|g_n - P_2 g_n\| = 0$.
It follows from condition (A) that

$$\lim_{n\to\infty} t(d(g_n, F(P_1) \cap F(P_2))) \leq \lim_{n\to\infty} \|g_n - P_1 g_n\| = 0$$

or

$$\lim_{n\to\infty} t(d(g_n, F(P_1) \cap F(P_2))) \leq \lim_{n\to\infty} \|g_n - P_2 g_n\| = 0$$

so that $\lim_{n\to\infty} t(d(g_n, F(P_1) \cap F(P_2))) = 0$.
Since t is a non-decreasing function satisfying $t(0) = 0$ and $t(r) > 0$ for all $r \in (0, \infty)$, therefore $\lim_{n\to\infty} d(g_n, F(P_1) \cap$

$F(P_2)) = 0$.
By Theorem 4.2., the sequence $\{g_n\}$ converges strongly to a point of $F(P_1) \cap F(P_2)$.

We now reconfirm our above convergence result with the help of the following example.

Example 2. Let $A = \mathbb{R}$ and $D = [1, 50]$. Let $P_1, P_2 : D \to D$ be two mappings defined as $P_1(k) = \sqrt{k^2 - 9k + 54}$ and $P_2(k) = \sqrt{k^2 - 7k + 42}$ for all $k \in D$. Clearly $k = 6$ is the common fixed point of P_1 and p_2. Set $\varrho_n = \frac{n}{n+1}$ and $\varsigma_n = 0.7$ for all $n \in \mathbb{N}$. Then, we get the Table 3 and Figure 3 for three different initial values.

Step	when $g_1 = 2$	when $g_1 = 8$	when $g_1 = 15$
1	2	8	15
2	5.93505362831552	6.12583922716451	8.1126797070467
3	5.99803076818424	6.00400176170888	6.11071817599684
		⋮	
10	6.000000000000000	6.00000000000001	6.00000000000034
11	6.000000000000000	6.000000000000000	6.00000000000001
12	6.000000000000000	6.000000000000000	6.000000000000000

Table 3:

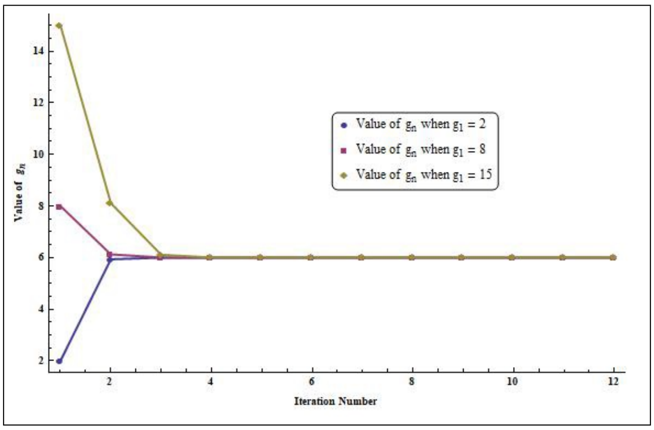

Figure 3:

5. Applications

Let A be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$, respectively. Let D be a nonempty closed convex subset of A and $T : D \to A$ a nonlinear operator. Then, T is said to be:
(i) monotone if $\langle Tu - Tv, u - v \rangle \geq 0$ for all $u, v \in D$,
(ii) λ- strongly monotone if there exist a constant $\lambda > 0$ such that $\langle Tu - Tv, u - v \rangle \geq \lambda \|u - v\|^2$ for all $u, v \in D$,
(iii) v- inverse strongly monotone (v-ism) if there exist a constant $v > 0$ such that $\langle Tu - Tv, u - v \rangle \geq v\|Tu - Tv\|^2$ for all $u, v \in D$.
The variational inequality problem denoted by $VI(D, T)$ is to find a point $z \in D$ such that $\langle Tz, z - u \rangle \geq 0$ for all $u \in D$. The set of solutions of variational inequality problem is denoted by $\Omega(D, T)$. The variational inequalities were initially studied by Stampachhia [10, 11]. Such a problem is connected with convex minimization problem, the complementarity problem,the problem of finding a point $u \in A$ satisfying $0 = Tu$ and so on. The existence and approximation of solutions are important aspects in the study of variational inequalities. The variational inequality problem is equivalent to finding the set of fixed points of the operator $P_D(I - \mu T)$, i.e., $F(P_D(I - \mu T)) = \Omega(D, T)$, where $\mu > 0$ is a constant and P_D is the metric projection from A onto D. If T is L-Lipschitzian and λ - strongly monotone, then the operator $F_{(}\mu)$ is a

contraction on D provided that $0 < \mu < 2\lambda/L^2$. Then, it follows from the Banach contraction principle that $VI(D,T)$ has a unique solution u^* and the sequence of the Picard iteration process, given by

$$u_{n+1} = P_D(I - \mu T)u_n, \qquad n \in \mathbb{N}$$

converges strongly to u^*.
Now, in view of Theorem 3.1 and Theorem 3.2, we have the following results:
Theorem 5.1. Let D be a nonempty closed convex subset of a Hilbert space A and $T : D \to A$ a L-Lipschitzian and λ- strongly monotone operator. Suppose $\{\varrho_n\}$ and $\{\varsigma_n\}$ are sequences in $(0,1)$. Then for $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{g_n\}$ constructed from $g_1 \in D$ and defined by

$$\begin{cases} e_n = (1-\varrho_n)g_n + \varrho_n P_D(I-\mu T)g_n \\ f_n = P_D(I-\mu T)((1-\varsigma_n)e_n + \varsigma_n P_D(I-\mu T)e_n) \quad , \quad n \in \mathbb{N} \\ g_{n+1} = P_D(I-\mu T)f_n \end{cases}$$

converges strongly to $g^* \in \Omega(D,T)$.
Theorem 5.2. Let D be a nonempty closed convex subset of a Hilbert space A and $T : D \to A$ a υ- inverse strongly monotone mapping, where $\upsilon > 0$ is a constant. Suppose $\Omega(D,T) \neq \phi$ and $\mu \in (0, 2\upsilon)$. Let $\{g_n\}$ be a sequence in D constructed from $g_1 \in D$ and defined by

$$\begin{cases} e_n = (1-\varrho_n)g_n + \varrho_n P_D(I-\mu T)g_n \\ f_n = P_D(I-\mu T)((1-\varsigma_n)e_n + \varsigma_n P_D(I-\mu T)e_n) \quad , \quad n \in \mathbb{N} \\ g_{n+1} = P_D(I-\mu T)f_n \end{cases}$$

where $\{\varrho_n\}$ and $\{\varsigma_n\}$ are sequences in $(0,1)$. Then, $\{g_n\}$ converges weakly to a solution of the variational inequality $VI(D,T.)$
Algorithms for signal and image processing are often iterative constrained optimization procedures designed to minimize a convex differentiable function $q(x)$ over a closed convex set D in A. It is well known that every L-Lipschitzian operator is 2/L-ism. Therefore, the following process converges to minimizer of q.
Theorem 5.3. Let D be a nonempty closed convex subset of a Hilbert space A and q a convex and differentiable function on an open set E containing the set D. Suppose that ∇q is a L-Lipschitz continuous operator on E, $\mu \in (0, 2/L)$ and minimizers of q relative to the set D exist. Let $\{g_n\}$ be a sequence in D constructed from $g_1 \in D$ and defined by

$$\begin{cases} e_n = (1-\varrho_n)g_n + \varrho_n P_D(I-\mu\nabla q)g_n \\ f_n = P_D(I-\mu\nabla q)((1-\varsigma_n)e_n + \varsigma_n P_D(I-\mu\nabla q)e_n) \quad , \quad n \in \mathbb{N} \\ g_{n+1} = P_D(I-\mu\nabla q)f_n \end{cases}$$

where $\{\varrho_n\}$ and $\{\varsigma_n\}$ are sequences in $(0,1)$. Then, $\{g_n\}$ converges weakly to a minimizer of q.

6. Conclusion

We conclude from the above sections that our iteration process (1.2) converges faster than the M-iteration process (1.1) for contractive-like mappings. It must be noted here that Ullah and Arshad [22] never gave the rate of convergence of their process analytically. They just gave an example. However, we not only give the proof analytically but also validate with an example. As far as convergence results are concerned, our results are just new of their kind. They are independent of, for example, [19, 20] and [21, 22]. Keeping in mind that common fixed point problem has a direct link with minimization problem, our results are better in the sense of approximating common fixed points as opposed to [19, 20] and [21, 22]. We have not only validated our resuts by example, but also applied to find the solution of variational inequality and optimization problem.

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