Filomat 34:6 (2020), 2101–2107
https://doi.org/10.2298/FIL2006101X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Strongly EP Elements in a Ring With Involution

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Abstract. In this paper, we introduce a new class of *EP* elements which is called strongly *EP* element and give some characterizations of strongly *EP* elements.

1. Introduction

Let R be an associative ring with 1, and let $a \in R$. a is said to be group invertible if there exists $a^\# \in R$ such that

$$aa^\# a = a, \qquad a^\# aa^\# = a^\#, \qquad aa^\# = a^\# a.$$

The element $a^\#$ is called a group inverse of a, which is uniquely determined by the above equations [3]. We denote the set of all group invertible elements of R by $R^\#$.

An involution in R is an anti-isomorphism $* : R \to R, a \mapsto a^*$ of degree 2, that is,

$$(a^*)^* = a, \qquad (a+b)^* = a^* + b^*, \qquad (ab)^* = b^* a^*.$$

If $a^* a = aa^*$, the element a is called normal [12].

An element a^+ is called the Moore-Penrose inverse (or MP-inverse) [7] of a, if

$$aa^+ a = a, \quad a^+ aa^+ = a^+, \quad (aa^+)^* = aa^+, \quad (a^+ a)^* = a^+ a.$$

If a^+ exists, then it is unique [12–14]. Denote by R^+ the set of all MP-invertible elements of R. If $a^* = a^+$, the element a is called partial isometry. An element $a \in R^\# \cap R^+$ satisfying $a^\# = a^+$ is said to be EP. We denote the set of all EP elements of R by R^{EP}. If $a \in R^{EP}$ and $a^* = a^+$, we say a is a strongly EP element. Denote by R^{PEP} the set of all strongly EP elements of R.

In [1], Baksalary, Styan and Trenkler explored various classes of matrices, such as partial isometries and EP elements, by using the representation of complex matrices and the matrix rank described in [12]. Recent researches on partial isometries have produced some interesting findings [6, 10].

At the same time, various characterizations of EP elements were investigated in [2, 4, 5, 7]. In general, EP elements are considered in the contexts of semigroups, rings and C^*-algebras.

Motivated by the above results, this work is intended to provide some equivalent conditions for an element to be an EP element and partial isometry by using solutions of some equations. Let $a \in R^\# \cap R^+$ and $\chi_a = \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*\}$. We show that $a \in R^{PEP}$ if and only if the equation $x = a^+ x (a^+)^*$ has at least one solution in χ_a. Also, we show that $a \in R^{PEP}$ if and only if the equations $xya^* = xya^\#$ has at least a solution in χ_a^2.

2010 *Mathematics Subject Classification*. 15A09; 15A27; 20M99
Keywords. Moore Penrose inverse, Group inverse, *EP* element, Partial isometry element, Strongly *EP* element
Received: 23 October 2019; Revised: 09 February 2020; Accepted: 12 February 2020
Communicated by Dijana Mosić
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2. Main Results

Lemma 2.1. *([6, Lemma 1.1 and Theorem 1.2])Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{EP}$;
2) $a^{+}a = aa^{+}$;
3) $a^{+}a = a^{\#}a$;
4) $aa^{+} = aa^{\#}$.

Observing the conditions 2) and 4) of Lemma 2.1, we obtain the following lemma.

Lemma 2.2. *[11]Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{EP}$;
2) $a^{+}a^{m+1} = a^{m}$ for some $m \geq 1$;
3) $a^{m} = a^{m+1}a^{+}$ for some $m \geq 1$.

Lemma 2.3. *[15, Corollary 2.14]Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{EP}$;
2) $aa^{+}a^{+} = a^{+}$;
3) $a^{+}a^{+}a = a^{+}$.

Lemma 2.4. *([6, Theorem 1.1]; [8]; [9]) (1) If $a \in R^{+}$, then $a^{+}aa^{*} = a^{*} = a^{*}aa^{+}$.*
(2) If $a \in R^{\#} \cap R^{+}$, then $a^{\#}a^{+}a = a^{\#} = aa^{+}a^{\#}$.

Lemma 2.5. *Let $a \in R^{\#} \cap R^{+}$. If $a^{*} = a^{+}aa^{\#}$, then $a \in R^{EP}$ and $a^{+} = a^{*}$.*

Proof. Since $a^{*} = a^{+}aa^{\#}$, we have $a^{*}a = a^{+}aa^{\#}a = a^{+}a$. Hence $a^{*} = a^{+}$ by [10, Theorem 2.1]. Consequently, $a^{+} = a^{*} = a^{+}aa^{\#}$, one obtains $a \in R^{EP}$ by [7, Theorem 2.1(xxii)]. □

Let $a \in R^{\#} \cap R^{+}$. Then $a^{*} = a^{+}aa^{\#}$ if and only if $aa^{*} = aa^{\#}$. Hence, Lemma 2.5 leads to the following corollary which conditions 2-3 were proved in [10].

Corollary 2.6. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{EP}$ and $a^{+} = a^{*}$;
2) $aa^{*} = aa^{\#}$;
3) $a^{*}a = a^{\#}a$;
4) $a^{*} = a^{\#}aa^{+}$;
5) $a^{*} = a^{+}aa^{\#}$.

Applying the involution $*$ on the condition 4) of Corollary 2.1, we have $a = aa^{+}(a^{\#})^{*}$. In this case, we have $a^{\#} = a^{+}$. Hence, $a = a^{+}a(a^{+})^{*}$, which implies that we can construct the following equation

$$x = a^{+}x(a^{+})^{*}. \tag{1}$$

Let $a \in R^{\#} \cap R^{+}$. If $a^{\#} = a^{+} = a^{*}$, then a is called a strongly *EP* element of R. We write by R^{PEP} to denote the set of all strongly *EP* elements of R. Using the equation (1), we can characterize strongly *EP* elements as follows.

Theorem 2.7. *Suppose $a \in R^{\#} \cap R^{+}$, then $a \in R^{PEP}$ if and only if the equation* (1) *has at least one solution in $\chi_a = \{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}, (a^{+})^{*}\}$.*

Proof. $\Longrightarrow$ Assume that $a \in R^{PEP}$. Then $a^{\#} = a^{+} = a^{*}$. It follows that $x = a$ is a solution of Equation (1) in χ_a.
$\Longleftarrow$ 1) If $x = a$ is a solution, then $a = a^{+}a(a^{+})^{*}$. Multiplying the equality on the left by $1 - a^{+}a$, we have $a = a^{+}a^{2}$, it follows that $a \in R^{EP}$ by Lemma 2.2. Hence $a = a^{+}a(a^{+})^{*} = aa^{+}(a^{+})^{*} = (a^{+})^{*}$, which implies $a \in R^{PEP}$;

2) If $x = a^{\#}$ is a solution, then $a^{\#} = a^{+}a^{\#}(a^{+})^{*}$. Multiplying the equality on the left by a, we have $aa^{\#} = a^{\#}(a^{+})^{*}$ by Lemma 2.4(2). Noting that $(1-a^{+}a)a^{\#} = (1-a^{+}a)a^{+}a^{\#}(a^{+})^{*} = 0$. Then $a^{\#} = a^{+}aa^{\#}$, one has $a \in R^{EP}$ by [7, Theorem 2.1(xix)], it follows that $a = a^2a^{\#} = aa^{\#}(a^{+})^{*} = a^{\#}a(a^{+})^{*} = a^{+}a(a^{+})^{*}$. Hence $a \in R^{PEP}$ by 1);

3) If $x = a^{+}$ is a solution, then $a^{+} = a^{+}a^{+}(a^{+})^{*}$. Multiplying the equality on the right by $1-a^{+}a$, we have $a^{+} = a^{+}a^{+}a$, it follows that $a \in R^{EP}$ by Lemma 2.3. Hence $a^{\#} = a^{+} = a^{+}a^{+}(a^{+})^{*} = a^{+}a^{\#}(a^{+})^{*}$, which gives $a \in R^{PEP}$ by 2);

4) If $x = a^{*}$ is a solution, then $a^{*} = a^{+}a^{*}(a^{+})^{*}$, that is $a^{*} = a^{+}a^{+}a$. Applying the involution on the equality, we have $a = a^{+}a(a^{+})^{*}$, which leads to $a \in R^{PEP}$ by 1);

5) If $x = (a^{\#})^{*}$ is a solution, then $(a^{\#})^{*} = a^{+}(a^{\#})^{*}(a^{+})^{*}$. Applying the involution on the equality, we have $a^{\#} = a^{+}a^{\#}(a^{+})^{*}$. Hence $a \in R^{PEP}$ by 2);

6) If $x = (a^{+})^{*}$ is a solution, then $(a^{+})^{*} = a^{+}(a^{+})^{*}(a^{+})^{*}$, which gives $a^{+} = a^{+}a^{+}(a^{+})^{*}$ by applying the involution. Hence $a \in R^{PEP}$ by 3); □

By the symmetricity of equation (1), we have the following equation

$$x = (a^{+})^{*}xa^{+}. \tag{2}$$

Similarly, we have the following theorem.

Theorem 2.8. *Suppose $a \in R^{\#} \cap R^{+}$, then $a \in R^{PEP}$ if and only if the equation* (2) *has at least one solution in χ_a.*

Corollary 2.9. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*

1) $a \in R^{PEP}$;

2) $a = a^{+}a(a^{\#})^{*}$;

3) $a^{\#} = a^{+}a^{\#}(a^{\#})^{*}$;

4) $(a^{+})^{*} = a^{+}(a^{+})^{*}(a^{\#})^{*}$.

Proof. 1) $\Longrightarrow$ i), ($i = 2, 3, 4$) It is routine.

2) $\Longrightarrow$ 1) Assume that $a = a^{+}a(a^{\#})^{*}$. Then $(1-a^{+}a)a = (1-a^{+}a)a^{+}a(a^{\#})^{*} = 0$, one has $a \in R^{EP}$ by Lemma 2.2. Hence $a = a^{+}a(a^{+})^{*}$ because $a^{+} = a^{\#}$. By the case 1) of proof of Theorem 2.1, we have $a \in R^{PEP}$.

3) $\Longrightarrow$ 2) Suppose that $a^{\#} = a^{+}a^{\#}(a^{\#})^{*}$. Then $a = aa^{\#}(a^{\#})^{*}$ by multiplying a^2 on the left. Noting that $(a^{\#})^{*}(1-aa^{+}) = 0$. Then we have $a(1-aa^{+}) = 0$, it follows that $a \in R^{EP}$ by Lemma 2.2. Hence $a = aa^{\#}(a^{\#})^{*} = a^{\#}a(a^{\#})^{*} = a^{+}a(a^{\#})^{*}$.

4) $\Longrightarrow$ 1) Assume that $(a^{+})^{*} = a^{+}(a^{+})^{*}(a^{\#})^{*}$. Then $a^{+} = a^{\#}a^{+}(a^{+})^{*}$ by applying involution on the equality, so one has $(1-aa^{+})a^{+} = (1-aa^{+})a^{\#}a^{+}(a^{+})^{*} = 0$ by Lemma 2.4(2), which gives $a \in R^{EP}$ by Lemma 2.3. Hence $a^{+} = a^{\#}a^{+}(a^{+})^{*} = a^{+}a^{\#}(a^{+})^{*} = a^{+}a^{+}(a^{+})^{*}$, by the case 3) of the proof of Theorem 2.1, we have $a \in R^{PEP}$. □

Corollary 2.10. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*

1) $a \in R^{PEP}$;

2) $a^{+} = a^{\#}a^{+}(a^{+})^{*}$;

3) $a^{*} = a^{\#}a^{*}(a^{+})^{*}$;

4) $(a^{\#})^{*} = a^{\#}(a^{\#})^{*}(a^{+})^{*}$.

Proof. 1) $\Longrightarrow$ i), ($i = 2, 3, 4$) It is evident.

2) $\Longrightarrow$ 1) Assume that $a^{+} = a^{\#}a^{+}(a^{+})^{*}$. Then $(1-aa^{+})a^{+} = (1-aa^{+})a^{\#}a^{+}(a^{+})^{*} = 0$ by Lemma 2.4(2), one has $a \in R^{EP}$ by Lemma 2.3, which gives $a^{+} = a^{\#}$. Hence $a^{+} = a^{+}a^{+}(a^{+})^{*}$. By the case 3) of proof of Theorem 2.1, we have $a \in R^{PEP}$.

3) $\Longrightarrow$ 1) Suppose that $a^{*} = a^{\#}a^{*}(a^{+})^{*}$. Then $(1-aa^{+})a^{*} = (1-aa^{+})a^{\#}a^{*}(a^{+})^{*} = 0$. Applying the involution on the equality, one has $a = a^2a^{+}$, so $a \in R^{EP}$ by Lemma 2.2. Hence $a^{*} = a^{\#}a^{*}(a^{+})^{*} = a^{+}a^{*}(a^{+})^{*}$, by the case 4) of proof of Theorem 2.1, we have $a \in R^{PEP}$.

4) $\Longrightarrow$ 1) Assume that $(a^{\#})^{*} = a^{\#}(a^{\#})^{*}(a^{+})^{*}$. Then $a^{\#} = a^{+}a^{\#}(a^{\#})^{*}$ by applying involution on the equality. Hence $a \in R^{PEP}$ by Corollary 2.2. □

Observing Corollary 2.3, we can easy obtain the following equation

$$x = a^{\#}x(a^{+})^{*}. \tag{3}$$

Modifying this equation as follows

$$x = xa^{\#}(a^{+})^{*}. \tag{4}$$

Theorem 2.11. *Suppose $a \in R^{\#} \cap R^{+}$, then $a \in R^{PEP}$ if and only if the equation* (4) *has at least one solution in $\chi_a = \{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}, (a^{+})^{*}\}$.*

Proof. $\Longrightarrow$ Assume that a a partial isometry, then $a^{+} = a^{*}$. It follows that $x = a$ is a solution of Equation (4) in χ_a.

$\Longleftarrow$ 1) If $x = a$ is a solution, then $a = aa^{\#}(a^{+})^{*}$. Multiplying the equality on the right by a^{*}, we have $aa^{*} = aa^{\#}aa^{+} = aa^{+}$, it follows that a is a partial isometry by [6, Theorem 2.1];

2) If $x = a^{\#}$ is a solution, then $a^{\#} = a^{\#}a^{\#}(a^{+})^{*}$. Multiplying the equality on the left by a^{2}, we have $a = aa^{\#}(a^{+})^{*}$. Hence, by 1), we have a is a partial isometry;

3) If $x = a^{+}$ is a solution, then $a^{+} = a^{+}a^{\#}(a^{+})^{*}$. Multiplying the equality on the left by a, we have $aa^{+} = a^{\#}(a^{+})^{*}$. Noting that $(a^{+})^{*}(1 - a^{+}a) = 0$. Then one has $aa^{+}(1 - a^{+}a) = 0$. Applying the involution on the last equality, we have $aa^{+} = a^{+}a^{2}a^{+}$, which gives $a = a^{+}a^{2}$. Hence $a \in R^{EP}$ by Lemma 2.2, which leads to $a = a^{2}a^{+} = a^{2}a^{+}a^{\#}(a^{+})^{*} = aa^{\#}(a^{+})^{*}$. Therefore a is a partial isometry by 1);

4) If $x = a^{*}$ is a solution, then $a^{*} = a^{*}a^{\#}(a^{+})^{*}$. Multiplying the equality on the left by $(a^{+})^{*}$, one obtains $aa^{+} = aa^{+}a^{\#}(a^{+})^{*} = a^{\#}(a^{+})^{*}$ by Lemma 2.4. Multiplying the last equality on the right by $1 - a^{+}a$, one has $aa^{+}(1 - a^{+}a) = 0$, applying the involution, we have $aa^{+} = a^{+}a^{2}a^{+}$, which implies $a = aa^{+}a = a^{+}a^{2}a^{+}a = a^{+}a^{2}$. Hence $a \in R^{EP}$ by Lemma 2.3, it follows that $a^{+} = a^{+}a^{+}a = a^{+}a^{\#}(a^{+})^{*} = a^{\#}a^{+}(a^{+})^{*}$, one obtains $a \in R^{PEP}$ by Corollary 2.3;

5) If $x = (a^{+})^{*}$ is a solution, then $(a^{+})^{*} = (a^{+})^{*}a^{\#}(a^{+})^{*}$. Applying the involution on the equality, we have $a^{+} = a^{+}(a^{\#})^{*}a^{+}$. It follows that $a = aa^{+}a = aa^{+}(a^{\#})^{*}a^{+}a = (a^{+}aa^{\#}aa^{+})^{*} = (a^{+})^{*}$. Hence $a = (a^{+})^{*} = (a^{+})^{*}a^{\#}(a^{+})^{*} = aa^{\#}(a^{+})^{*}$, one obtains a is a partial isometry by 1);

6)If $x = (a^{\#})^{*}$ is a solution, then $(a^{\#})^{*} = (a^{\#})^{*}a^{\#}(a^{+})^{*}$, it follows that $(a^{\#})^{*}(1-a^{+}aa) = (a^{\#})^{*}a^{\#}(a^{+})^{*}(1-a^{+}aa) = 0$. Applying the involution on the equality, we have $a^{\#} = a^{+}aa^{\#}$, which gives $a \in R^{EP}$. Hence $(a^{+})^{*} = (a^{\#})^{*} = (a^{\#})^{*}a^{\#}(a^{+})^{*} = (a^{+})^{*}a^{\#}(a^{+})^{*}$, which implies $a \in R^{PEP}$ by 5). □

Multiplying the equation (4) on the right by a^{*}, we have the following equation

$$xa^{*} = xa^{\#}aa^{+}. \tag{5}$$

In Equation (5), exchange a with a^{+}, or a^{+} with $a^{\#}$, we can obtain the following equation

$$xa^{*} = xa^{\#}. \tag{6}$$

Theorem 2.12. *Suppose $a \in R^{\#} \cap R^{+}$, then $a \in R^{PEP}$ if and only if the equation* (6) *has at least one solution in $\chi_a = \{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}, (a^{+})^{*}\}$.*

Proof. $\Longrightarrow$ Assume that $a \in R^{PEP}$, then $a^{+} = a^{*} = a^{\#}$. It follows that $x = a$ is a solution of Equation (6) in χ_a.

$\Longleftarrow$ 1) If $x = a$ is a solution, then $aa^{*} = aa^{\#}$. It follows that $a \in R^{PEP}$ by [6, Theorem 2.2(iv)];

2) If $x = a^{\#}$ is a solution, then $a^{\#}a^{*} = a^{\#}a^{\#}$. Multiplying the equality on the left by a^{2}, we have $aa^{*} = aa^{\#}$. Hence, by 1), we have $a \in R^{PEP}$;

3) If $x = a^{+}$ is a solution, then $a^{+}a^{*} = a^{+}a^{\#}$. It follows from [10, Theorem 2.3] that $a \in R^{PEP}$;

4) If $x = a^{*}$ is a solution, then $a^{*}a^{*} = a^{*}a^{\#}$. Multiplying the equality on the left by $(a^{+})^{*}$, one obtains $aa^{+}a^{*} = aa^{+}a^{\#} = a^{\#}$ by Lemma 2.4. By the proof of 3), one obtains $a \in R^{PEP}$;

5) If $x = (a^{\#})^{*}$ is a solution, then $(a^{\#})^{*}a^{*} = (a^{\#})^{*}a^{\#}$. It follows that $aa^{\#} = (a^{\#})^{*}a^{\#}$ by applying the involution on the two-sided. Noting that $(1 - a^{+}a)(a^{\#})^{*} = 0$. Then we have $(1 - a^{+}a)aa^{\#} = 0$, which gives $a \in R^{EP}$. So $a = aa^{\#}a = (a^{\#})^{*}a^{\#}a = (a^{+})^{*}a^{+}a = (a^{+})^{*}$, which implies $a \in R^{PEP}$.

6) If $x = (a^{+})^{*}$ is a solution, then $(a^{+})^{*}a^{*} = (a^{+})^{*}a^{\#}$, that is, $aa^{+} = (a^{+})^{*}a^{\#}$. It follows that $a^{*} = a^{*}aa^{+} = a^{*}(a^{+})^{*}a^{\#} = a^{+}aa^{\#}$. Hence $a \in R^{PEP}$ by Corollary 2.1. □

Corollary 2.13. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equavilent:*
1) $a \in R^{PEP}$;
2) $aa^{+}a^{*} = a^{\#}$;
3) $a^{*}a^{+}a = a^{\#}$;
4) $aa^{\#} = (a^{\#})^{*}a^{\#}$;
5) $aa^{\#} = a^{\#}(a^{\#})^{*}$.

Modifying the equation (5) as follows

$$xa^{*} = a^{\#}xaa^{+}. \tag{7}$$

Theorem 2.14. *Suppose $a \in R^{\#} \cap R^{+}$, then a is a partial isometry if and only if the equation (7) has at least one solution in $\rho_a = \{a, a^{\#}, a^{+}, a^{*}, (a^{\#})^{*}\}$.*

Proof. $\Longrightarrow$ Assume that a is partial isometry, then $a^{+} = a^{*}$. It follows that $x = a$ is a solution of Equation (7) in χ_a.

$\Longleftarrow$ 1) If $x = a$ is a solution, then $aa^{*} = a^{\#}a^{2}a^{+}$. It follows that $aa^{*} = aa^{+}$. Hence a partial isometry by [6, Theorem 2.1];

2) If $x = a^{\#}$ is a solution, then $a^{\#}a^{*} = a^{\#}a^{\#}aa^{+} = a^{\#}a^{+}$. By [10], a is partial isometry;

3) If $x = a^{+}$ is a solution, then $a^{+}a^{*} = a^{\#}a^{+}aa^{+} = a^{\#}a^{+}$. It follows that a is partial isometry from [10];

4) If $x = a^{*}$ is a solution, then $a^{*}a^{*} = a^{\#}a^{*}aa^{+} = a^{\#}a^{*}$. Multiplying the equality on the right by $(a^{+})^{*}$, one obtains $a^{*}a^{+}a = a^{\#}a^{+}a = a^{\#}$ by Lemma 2.4. Thus $a \in R^{PEP}$ by Corollary 2.4;

5) If $x = (a^{\#})^{*}$ is a solution, then $(a^{\#})^{*}a^{*} = a^{\#}(a^{\#})^{*}aa^{+} = a^{\#}(a^{\#})^{*}$. It follows that $aa^{\#} = a^{\#}(a^{\#})^{*}$ by applying the involution on the two-sided. Hence $a \in R^{PEP}$ by Corollary 2.4. □

Remark 2.15. *In Equation (7), choose $x = (a^{+})^{*}$, then we have $aa^{+} = a^{\#}(a^{+})^{*}aa^{+}$, so $a = a^{\#}(a^{+})^{*}a$, which leads to the following problem.*

Problem 2.16. *Let $a \in R^{\#} \cap R^{+}$. If $a = a^{\#}(a^{+})^{*}a$, is a partial isometry*?

However, we have the following proposition.

Proposition 2.17. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equivalent:*
(1) a is partial isometry;
(2) $a = a^{\#}(a^{+})^{*}a$ and $a^{*}a^{+} = a^{+}a^{*}$;
(3) $a = a(a^{+})^{*}a^{\#}$ and $a^{*}a^{+} = a^{+}a^{*}$.

Proof. (1) $\Longrightarrow$ (2) It is clear.

(2) $\Longrightarrow$ (1) Assume that $a = a^{\#}(a^{+})^{*}a$ and $a^{*}a^{+} = a^{+}a^{*}$. Then $a^{*} = a^{*}a^{+}(a^{\#})^{*} = a^{+}a^{*}(a^{\#})^{*} = a^{+}(a^{\#}a)^{*}$, it follows that $aa^{*} = aa^{+}(a^{\#}a)^{*} = (a^{\#}aaa^{+})^{*} = (aa^{+})^{*} = aa^{+}$. Hence a partial isometry by [6, Theorem 2.1]

Similarly, we can show that (1) $\Longleftrightarrow$ (3). □

Proposition 2.18. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equivalent:*
(1) a is partial isometry;
(2) $a = a^{\#}(a^{+})^{*}a$ and $(a^{+})^{*} \in comm(aa^{\#})$;
(3) $a = a(a^{+})^{*}a^{\#}$ and $(a^{+})^{*} \in comm(aa^{\#})$.

Proof. (1) $\Longrightarrow$ (2) It is clear.

(2) $\Longrightarrow$ (1) Assume that $a = a^{\#}(a^{+})^{*}a$ and $(a^{+})^{*} \in comm(aa^{\#})$. Then $aa^{\#} = a^{\#}(a^{+})^{*}aa^{\#} = a^{\#}aa^{\#}(a^{+})^{*} = a^{\#}(a^{+})^{*}$. It follows that $a = a^{2}a^{\#} = aa^{\#}(a^{+})^{*}$, so $aa^{*} = aa^{\#}(a^{+})^{*}a^{*} = aa^{\#}aa^{+} = aa^{+}$. Hence a partial isometry by [6, Theorem 2.1]

Similarly, we can show that (1) $\Longleftrightarrow$ (3). □

Proposition 2.19. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equivalent:*
(1) a is partial isometry;
(2) $a = a^{\#}(a^{+})^{*}a$ and $(a^{+})^{*} \in comm(a^{n})$ for some $n > 1$;
(3) $a = a(a^{+})^{*}a^{\#}$ and $(a^{+})^{*} \in comm(a^{n})$ for some $n > 1$.

Proof. (1) $\Longrightarrow$ (2) It is clear.

(2) $\Longrightarrow$ (1) Assume that $a = a^{\#}(a^{+})^{*}a$ and $(a^{+})^{*} \in comm(a^{n})$ for some $n > 1$. Then $a = a^{\#}(a^{+})^{*}a^{n}(a^{\#})^{n-1} = a^{\#}a^{n}(a^{+})^{*}(a^{\#})^{n-1} = a^{n-1}(a^{+})^{*}(a^{\#})^{n-1}$, which gives $a^{2} = a^{n}(a^{+})^{*}(a^{\#})^{n-1} = (a^{+})^{*}a^{n}(a^{\#})^{n-1} = (a^{+})^{*}a$. It follows that $a^{*}a^{2} = a^{*}(a^{+})^{*}a = a^{+}a^{2}$. Multiplying the above equation on the right by $a^{\#}$, we have $a^{*}a = a^{+}a$. So $aa^{*} = aa^{\#}(a^{+})^{*}a^{*} = aa^{\#}aa^{+} = aa^{+}$. Hence a partial isometry by [6, Theorem 2.1]

Similarly, we can show that (1) $\Longleftrightarrow$ (3).

□

Obversing the equation (6), we can obtain the following equation

$$yxa^{*} = yxa^{\#}. \tag{8}$$

Theorem 2.20. *Suppose $a \in R^{\#} \cap R^{+}$, then $a \in R^{PEP}$ if and only if the equation* (8) *has at least one solution in $\chi_a^2 = \{(c,d)|c,d \in \chi_a\}$.*

Proof. $\Longrightarrow$ It is an immediate corollary of Theorem 2.4.

$\Longleftarrow$ (1) If $y = a$, then $axa^{*} = axa^{\#}$.

(a) If $x = a$, then $a^{2}a^{*} = a^{2}a^{\#}$, it follows that $aa^{*} = a^{\#}a^{2}a^{*} = a^{\#}a^{2}a^{\#} = aa^{\#}$. Hence $a \in R^{PEP}$ by [6, Theorem 2.2(iv)];

(b) If $x = a^{\#}$, then $aa^{\#}a^{*} = aa^{\#}a^{\#}$, Multiplying the equality on the left by a, we have $aa^{*} = aa^{\#}$. Hence $a \in R^{PEP}$;

(c) If $x = a^{+}$, then $aa^{+}a^{*} = aa^{+}a^{\#}$. That is, $aa^{+}a^{*} = a^{\#}$, which gives $a^{+}a^{*} = a^{+}a^{\#}$ by multiplying a^{+} on the left. Hence $a \in R^{PEP}$ by the proof of case (3) of Theorem 2.4;

(d) If $x = a^{*}$, then $aa^{*}a^{*} = aa^{*}a^{\#}$. One has $a^{*}a^{*} = a^{+}aa^{*}a^{*} = a^{+}aa^{*}a^{\#} = a^{*}a^{\#}$. Hence $a \in R^{PEP}$ by the proof of case (4) of Theorem 2.4;

(e) If $x = (a^{\#})^{*}$, then $a(a^{\#})^{*}a^{*} = a(a^{\#})^{*}a^{\#}$. Multiplying the equality on the left by a^{+}, one has $(a^{\#})^{*}a^{*} = (a^{\#})^{*}a^{\#}$. Hence $a \in R^{PEP}$ by the proof of case (5) of Theorem 2.4;

(f) If $x = (a^{+})^{*}$, then $a(a^{+})^{*}a^{*} = a(a^{+})^{*}a^{\#}$. That is, $a^{2}a^{+} = a(a^{+})^{*}a^{\#}$, so $a^{2}a^{+}(1 - a^{+}a) = a(a^{+})^{*}a^{\#}(1 - a^{+}a) = 0$. Multiply the last equality on the left by $a^{+}a^{\#}$, one has $a^{+}(1 - a^{+}a) = 0$, it follows that $a \in R^{EP}$ by Lemma 2.3. Hence $a = a^{2}a^{+} = a(a^{+})^{*}a^{\#}$ and $a^{2} = a(a^{+})^{*}a^{\#}a = a(a^{+})^{*}a^{+}a = a(a^{+})^{*}$, it follows that $a^{*}a^{*} = a^{+}a^{*} = a^{\#}a^{*}$ by applying the involution on the last equality. Similar to the proof of Case (4) of Theorem 2.4, we have $a \in R^{PEP}$;

(2) If $y = a^{\#}$, then $a^{\#}xa^{*} = a^{\#}xa^{\#}$. Multiply the equation on the left by a^{2}, we have $axa^{*} = axa^{\#}$. Hence $a \in R^{PEP}$ by (1);

(3) If $y = a^{+}$, then $a^{+}xa^{*} = a^{+}xa^{\#}$.

(a) If $x = a$, then $a^{+}aa^{*} = a^{+}aa^{\#}$. It follows that $aa^{*} = aa^{+}aa^{*} = aa^{+}aa^{\#} = aa^{\#}$. Hence $a \in R^{PEP}$ by [6, Theorem 2.2(iv)];

(b) If $x = a^{\#}$, then $a^{+}a^{\#}a^{*} = a^{+}a^{\#}a^{\#}$. Multiplying the equality on the left by a, we have $a^{\#}a^{*} = a^{\#}a^{\#}$. Hence $a \in R^{PEP}$ by the proof of case (2) of Theorem 2.4;

(c) If $x = a^{+}$, then $a^{+}a^{+}a^{*} = a^{+}a^{+}a^{\#}$, which gives $a^{+}a^{+}a^{\#}(1 - aa^{+}) = 0$. Multiply the equality on the left by $a^{*}a$, one has $a^{*}a^{+}a^{\#}(1 - aa^{+}) = 0$, applying the involution on the last equality, we have $(1 - aa^{+})(a^{\#})^{*}(a^{+})^{*}a = 0$. Now we claim that $(a^{+})^{*}aR = aR$. (In fact, $a^{+}R = a^{*}R$ and $a^{+}a^{2}R = a^{+}R$ implies $(a^{+})^{*}aR = (a^{+}aa^{+})^{*}aR = (a^{+})^{*}a^{+}a^{2}R = (a^{+})^{*}a^{+}R = (a^{+})^{*}a^{*}R = aa^{+}R = aR$). Hence $0 = (1 - aa^{+})(a^{\#})^{*}(a^{+})^{*}aR = (1 - aa^{+})(a^{\#})^{*}aR = (1 - aa^{+})(a^{\#})^{*}aa^{+}R = (1 - aa^{+})(aa^{+}a^{\#})^{*}R = (1 - aa^{+})(a^{\#})^{*}R = (1 - aa^{+})a^{*}R$, which implies $a \in R^{EP}$. Hence $a^{+}a^{*} = aa^{+}a^{+}a^{*} = aa^{+}a^{+}a^{\#} = a^{+}a^{\#}$ and so $a \in R^{PEP}$ by the proof of case (3) of Theorem 2.4;

(d) If $x = a^{*}$, then $a^{+}a^{*}a^{*} = a^{+}a^{*}a^{\#}$. Noting that $Ra^{+} = Ra^{+}(a^{+})^{*}a^{*} \subseteq Ra^{*} = Ra^{*}aa^{+} \subseteq Ra^{+} = Ra^{+}(a^{+})^{*}(a^{\#})^{*}a^{*}a^{*} \subseteq Ra^{*}a^{*} \subseteq Ra^{*}$. Then $Ra^{+} = Ra^{*} = Ra^{*}a^{*}$, $Ra^{*}(1 - a^{+}a) = Ra^{*}a^{*}a^{*}(1 - a^{+}a) = Ra^{+}a^{*}a^{*}(1 - a^{+}a) = Ra^{+}a^{*}a^{\#}(1 - a^{+}a) = 0$, one has $a \in R^{EP}$. Hence $a^*a^* = a^+aa^*a^* = aa^+a^*a^* = aa^+a^*a^\# = a^*a^\#$, which implies $a \in R^{PEP}$ by the proof of case (4) of Theorem 2.4;

(e) If $x = (a^\#)^*$, then $a^+(a^\#)^*a^* = a^+(a^\#)^*a^\#$. Multiplying the equality on the right by $1 - aa^+$, one has $a^+(a^\#)^*a^\#(1 - aa^+) = 0$. Multiplying the last equality on the left by a^*a^*a, one obtains $a^*a^\#(1 - aa^+) = 0$. Hence $a^\#(1 - aa^+) = aa^+a^\#(1 - aa^+) = (a^+)^*a^*a^\#(1 - aa^+) = 0$, this gives $a \in R^{EP}$, it follows that $a^+ = a^+(a^+)^*a^* = a^+(a^\#)^*a^\# = a^+(a^+)^*a^+$ and $a = aa^+a = aa^+(a^+)^*a^+a = (a^+)^*a^+a = (a^+)^*$. Therefore $a \in R^{PEP}$.

(f) If $x = (a^+)^*$, then $a^+(a^+)^*a^* = a^+(a^+)^*a^\#$, that is, $a^+ = a^+(a^+)^*a^\#$, so $a^+a^+a = a^+(a^+)^*a^\#a^+a = a^+(a^+)^*a^\# = a^+$, which implies $a \in R^{EP}$. Hence $x = a^+ = a^\#$ is a solution, by (3)(e), we have $a \in R^{PEP}$;

(4) If $y = a^*$, then $a^*xa^* = a^*xa^\#$. Multiplying the equation on the left by $a^+(a^+)^*$, we have $a^+xa^* = a^+xa^\#$. Hence $a \in R^{PEP}$ by the case (3);

(5) If $y = (a^\#)^*$, then $(a^\#)^*xa^* = (a^\#)^*xa^\#$. Multiplying the equation on the left by $(a^*)^2$, one has $a^*xa^* = a^*xa^\#$, which implies $a \in R^{PEP}$ by the case (4);

(6) If $y = (a^+)^*$, then $(a^+)^*xa^* = (a^+)^*xa^\#$. Multiplying the equation on the left by aa^*, we have $axa^* = axa^\#$. Hence $a \in R^{PEP}$ by the case (1). □

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