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Convergence Theorems for Nonspreading Mappings and Equilibrium Problems in Hadamard Spaces

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Abstract. In this paper, we introduce a new iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of nonspreading mappings and a finite family of nonexpansive multivalued mappings in Hadamard space. We state and prove strong and Δ convergence theorems of the proposed iterative process. The results obtained in this paper extend and improve some recent known results.

1. Introduction

Let (X, d) be a metric space. Berg and Nikolaev [4] introduced the concept of quasilinearization in metric spaces. A pair $(a, b) \in X \times X$ is denoted by $\overrightarrow{ab}$ and is called a vector. Let $\mathbb{R}$ be the set of real numbers. The quasilinearization is the map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad (a, b, c, d \in X). \tag{1}$$

It is obvious that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ and

$$d^2(a, b) = d^2(a, x) + d^2(b, x) - 2\langle \overrightarrow{ax}, \overrightarrow{bx} \rangle \tag{2}$$

for all $a, b, c, d, x \in X$.
A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. For other equivalent definitions and basic properties, we refer the reader to standard texts such as [3, 6]. Complete CAT(0) spaces are often called Hadamard spaces.

Equilibrium problems were originally studied in [5] as a unifying class of variational problems. Let C be a nonempty set and $\Phi : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem $EP(C, F)$ is to find $x \in C$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in C.$$

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Choosing different settings for Φ, one may include, e.g., minimization, minimax inequalities, variational inequalities, and fixed point problems as special subclass of equilibrium problems. The set of all solutions of $EP(C,F)$ is denoted by $\mathcal{E}(C,F)$. Almost all of attempts of researchers in past years to solve $EP(C,F)$ were in Banach or Hilbert spaces [5, 9, 16, 26, 28, 30, 34] and recently in Hadamard manifolds [7, 8, 29]. As Reich and Shafrir [32] have suggested, some kinds of hyperbolic spaces can be a suitable context for some notions in nonlinear analysis. Despite the lack of linear structure, some fundamental concepts of nonlinear analysis have been generalized from Hilbert spaces to Hadamard space; see, for instance, [2, 10, 19, 23] and references therein.

Very recently, Kumam in [25] studied the KKM principle in Hadamard spaces and used this principle to prove existence theorems for equilibrium problems in such spaces. Let X be a CAT(0) space, $x, y \in X$ and $t \in [0,1]$. We write $tx \oplus (1-t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z,x) = (1-t)d(x,y) \quad \text{and} \quad d(z,y) = td(x,y). \tag{3}$$

We also denote by $[x,y]$ the geodesic segment joining from x to y, that is, $[x,y] = \{tx \oplus (1-t)y : t \in [0,1]\}$. A subset C of X is convex if $[x,y] \subseteq C$ for all $x, y \in C$. A function $f : C \to \mathbb{R}$, with C being convex, is called convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

for any $x, y, \in C$ and each $t \in [0,1]$.
In the literature, the following conditions on the bifunction $\Phi : C \times C \to \mathbb{R}$ are natural for solving the equilibrium problem:

$(A1)$ $\Phi(x,x) = 0$ for all $x \in C$,

$(A2)$ Φ is monotone, i.e., $\Phi(x,y) + \Phi(y,x) \leq 0$, for any $x, y \in C$,

$(A3)$ Φ is upper-hemicontinuous, i.e. for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Phi(tz \oplus (1-t)x, y) \leq \Phi(x,y),$$

$(A4)$ $\Phi(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$.

For a subset C of a CAT(0) space X, we denote by $\mathcal{CB}(C)$, $\mathcal{K}(C)$ and $\mathcal{P}(C)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of C, respectively. The Hausdorff metric H on $\mathcal{CB}(C)$ is defined by

$$H(A,B) := max\left\{\sup_{x\in A} dist(y,A),\ \sup_{y\in B} dist(x,B)\right\}$$

for all $A, B \in \mathcal{CB}(C)$, where $dist(y,A) = inf\{d(y,x) : x \in A\}$. Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of all fixed points of T will be denote by $F(T)$. A multivalued mapping $T : C \to \mathcal{CB}(C)$ is called nonexpansive if

$$H(Tx,Ty) \leq d(x,y) \quad \forall x, y \in C.$$

Approximating fixed points (and common fixed points) of nonexpansive multivalued mappings using iterative sequences have been investigated by various authors (see; e.g., [11, 15, 33]).

The class of nonspreading mappings as an important class of mappings in Banach spaces was introduced and studied in [17, 24]. This definition can be rewritten in metric space setting. Let C be a subset of a CAT(0) space, a mapping $T : C \to C$ is said to be nonspreading if

$$2d^2(Tx,Ty) \leq d^2(x,Ty) + d^2(Tx,y) \tag{4}$$

for all $x, y \in C$. By (1), this is equivalent to

$$d^2(Tx,Ty) \leq d^2(x,y) + 2\langle\overrightarrow{xTx}, \overrightarrow{yTy}\rangle. \tag{5}$$

Note that for nonspreading mapping T if $F(T) \neq \emptyset$, then T is quasi-nonexpansive, i.e., $d(Tx, p) \leq d(x, p)$ for all $x \in C$ and $p \in F(T)$.

Inspired by [15] and [17] we introduce an iterative process for finding a common element of the set of solutions of equilibrium problem and the set of common fixed points of a finite family of nonexpansive multivalued mappings and a finite family of nonspreading mappings in the setting of Hadamard spaces. Also, we establish the strong and Δ convergence of the proposed iterative process.

2. Preliminaries

As we mentioned in the preceding section, many concepts of nonlinear analysis have been generalized to CAT(0) metric spaces. Now, we recall some of them which are needed in the next section. The metric space X is said to satisfy the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leqslant d(a, b)d(c, d),$$

for all $a, b, c, d \in X$. It is known [4, Corollary 3] that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

The concept of Δ-convergence introduced by Lim [27] in 1976 was shown by Kirk and Panyanak [22] in CAT(0) spaces to be very similar to the weak convergence in Hilbert space setting. Next, we give the concept of Δ-convergence and collect some basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [12] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.
A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. We say that a subset C of X is Δ-closed if for every sequence $\{x_n\} \subset C$ that Δ-converges to x we have $x \in C$. Uniqueness of asymptotic center implies that CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y). \tag{6}$$

Let $\{v_1, v_2, \ldots, v_n\} \subset X$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset (0, 1)$ with $\sum_{i=1}^{n} \lambda_i = 1$. Dhompongsa et al. in [11] introduced the following concept by induction,

$$\bigoplus_{i=1}^{n} \lambda_i v_i := (1 - \lambda_n)\left(\frac{\lambda_1}{1 - \lambda_n} v_1 \oplus \frac{\lambda_2}{1 - \lambda_n} v_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} v_{n-1}\right) \oplus \lambda_n v_n. \tag{7}$$

Note that the definition of $\oplus$ in (7) is an ordered one in the sense that it depends on the order of points $v_1, v_2, \ldots, v_n$.

We need following lemmas in the sequel.

Lemma 2.1. [22] *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.2. [13] *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C. In the other words, every closed convex subset of a complete CAT(0) space is Δ-closed.*

Lemma 2.3. [18] *Let X be a complete CAT(0) space, $\{x_n\}$ be a bounded sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if* $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ *for all $y \in X$.*

Lemma 2.4. *[14, Lemma 2.4] Let X be a CAT(0) space. Then*

$$d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z) \tag{8}$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Utilizing (8) we can see that

$$d\left(\bigoplus_{i=1}^{n} \lambda_i v_i, z\right) \leq \sum_{i=1}^{n} \lambda_i d(v_i, z) \tag{9}$$

for each $z \in X$.

Lemma 2.5. [14, Lemma 2.5] *A geodesic space X is a CAT(0) space if and only if the following inequality*

$$d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y), \tag{10}$$

is satisfied for all $x, y, z \in X$ and $t \in [0,1]$.

Considering (7) and twice using (10) we have the following lemma.

Lemma 2.6. *Let X be a CAT(0) space, $x, y, z, q \in X$ and $\alpha, \beta, \gamma \in [0,1]$ be such that $\alpha + \beta + \gamma = 1$. Then*

$$\begin{aligned} d^2(\alpha x \oplus \beta y \oplus \gamma z, q) \leq & \alpha d^2(x,q) + \beta d^2(y,q) + \gamma d^2(z,q) \\ & - \frac{\alpha\beta}{1-\gamma} d^2(x,y) - \gamma(1-\gamma) d^2\left(\frac{\alpha}{1-\gamma} x \oplus \frac{\beta}{1-\gamma} y, z\right). \end{aligned} \tag{11}$$

Let C be a nonempty closed convex subset of a complete CAT(0) space X. It is known that for any $x \in X$ there exists a unique point $u \in C$ such that

$$d(x,u) = \inf_{y \in C} d(x,y).$$

The mapping $P_C : X \to C$ defined by $P_C x = u$ is called the metric projection from X onto C. Dehghan and Rooin [10] obtained the following characterization of metric projection in CAT(0) metric spaces.

Theorem 2.7. [10, Theorem 2.2] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then*

$$u = P_C x \quad \text{if and only if} \quad \langle \overrightarrow{ux}, \overrightarrow{yu}\rangle \geq 0, \quad \text{for all } y \in C.$$

Lemma 2.8. *[1, Lemma 4.3] Let C be a nonempty closed convex subset of a complete CAT(0) space X and $\{z_n\}$ be a sequence in X such that*

$$d(z_{n+1}, z) \leq d(z_n, z)$$

for all $z \in C$ and $n \geq 0$. Then, $\{P_C z_n\}$ converges to some $u \in C$.

Lemma 2.9. *[2, Lemma 3.2.3] Let X be an Hadamard space and $\varphi : X \to \mathbb{R}$ be a lower semicontinuous and convex function. If the sequence $\{x_n\}$ in X, Δ-converges to x_0, then*

$$\varphi(x_0) \leq \liminf_{n\to\infty} \varphi(x_n). \tag{12}$$

Definition 2.10. *[20] Suppose that $C \subset X$ is closed and convex, and $\Phi : C \times C \to \mathbb{R}$. The resolvent of Φ is the mapping $J_\Phi : X \rightrightarrows C$ defined by*

$$J_\Phi(x) = \{z \in C : \Phi(z, y) - \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \geq 0, \ \ \forall y \in C\}, \ \ \forall x \in X.$$

Theorem 2.11. *[25, Theorem 5.2] Suppose that Φ has the properties $(A1) - (A4)$. Then $\mathrm{dom}(J_\Phi) = X$ and J_Φ is single-valued.*

The following proposition gives several valuable properties for a resolvent of a monotone bifunction.

Proposition 2.12. *[25, Proposition 5.4] Suppose that Φ is monotone and $\mathrm{dom}(J_\Phi) \neq \emptyset$. Then, the following properties hold:*
(i) J_Φ is single-valued,
(ii) if $\mathrm{dom}(J_\Phi) \supset C$, then J_Φ is nonexpansive restricted to C,
(iii) if $\mathrm{dom}(J_{\mu f}) \supset C$ for any $\mu > 0$, then $F(J_\Phi) = \mathcal{E}(C, \Phi)$.

Although not mentioned in [25, Proposition 5.4], its proof shows that J_Φ is firmly nonexpansive, i.e.,

$$d^2(J_\Phi x, J_\Phi y) \leq \langle \overrightarrow{J_\Phi x J_\Phi y}, \overrightarrow{xy} \rangle, \quad \forall x, y \in C. \tag{13}$$

3. Main results

We begin with an example of nonspreading mapping which is not nonexpansive.

Example 3.1. *Consider $\mathbb{R}^2$ with the usual Euclidean norm $\|\cdot\|$. Let $X = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial metric d_r where $d_r(x, \ y) = d(x, \ y) = \|x - y\|$ if x and y are situated on a Euclidean straight line passing through the origin $\mathbf{0} = (0, 0)$ and $d_r(x, \ y) = d(x, \ \mathbf{0}) + d(y, \ \mathbf{0}) := \|x\| + \|y\|$ otherwise (see [21] and [31] page 65). Let $A = \{x \in X \ : \ \|x\| \leq 1\}$, $B = \{x \in X : \|x\| \leq 2\}$, and define the mapping $T : X \to X$ as follows:*

$$Tx = \begin{cases} 0 & (x \in B)\ ; \\ P_A(x) = \frac{x}{\|x\|} & (x \in X \backslash B)\ . \end{cases}$$

We show that T is nonspreading mapping. We write the inequality (4) as

$$d_r^2(x, \ Ty) + d_r^2(Tx, \ y) - 2d_r^2(Tx, \ Ty) \geq 0. \tag{14}$$

(i) In the case that $x, y \in B$ we have $d_r(Tx, \ Ty) = 0$ and so (14) clearly holds.

(ii) (a) In the case that $x, y \in X \backslash B$ are on a straight ray initiating from the origin, again we have $d_r(Tx, \ Ty) = 0$ and so (14) holds.

(b) If $x, y \in X \backslash B$ are not on a straight ray initiating from the origin, then

$$\begin{aligned} d_r^2(x, \ Ty) + d_r^2(Tx, \ y) - 2d_r^2(Tx, \ Ty) &= (\|x\| + 1)^2 + (\|y\| + 1)^2 - 4 \\ &\geq \ 3^2 + 3^2 - 4 \geq 0. \end{aligned}$$

(iii) (a) Let $x \in B$ and $y \in X \backslash B$ be on a straight ray initiating from the origin. Then,

$$\begin{aligned} d_r^2(x, \ Ty) + d_r^2(Tx, \ y) - 2d_r^2(Tx, \ Ty) &= \|x - Ty\|^2 + \|y\|^2 - 2 \\ &\geq \ \|x - Ty\|^2 \geq 0. \end{aligned}$$

(b) If $x \in B$ and $y \in X \backslash B$ are not on a straight ray initiating from the origin, then

$$\begin{aligned} d_r^2(x, \ Ty) + d_r^2(Tx, \ y) - 2d_r^2(Tx, \ Ty) &= (\|x\| + 1)^2 + \|y\|^2 - 2 \\ &\geq \ (\|x\| + 1)^2 \geq 0. \end{aligned}$$

Note that T is not nonexpansive mapping. In fact, if $x = (2 - 1/4, 0)$ *and* $y = (2 + 1/4, 0)$*, then we have*

$$d_r(Tx, Ty) = 1 > \frac{1}{2} = d_r(x,\ y).$$

Next, we show the demiclosedness of nonspreading mapping which is essentially used in the proof of our main theorem.

Lemma 3.2. *Let X be an Hadamard space, C be a nonempty closed convex subset of X and $T : C \to X$ be a nonspreading mapping. Then, $I - T$ is demiclosed, i.e., $\{x_n\}\Delta$- converges to z and $d(x_n,\ Tx_n) \to 0$ imply $z \in F(T)$.*

Proof. Since $T : C \to X$ is a nonspreading mapping, we have

$$d^2\ (Tx,\ Ty) \leq d^2(x,\ y) + 2\langle\overrightarrow{xTx}, \overrightarrow{yTy}\rangle \tag{15}$$

for all $x, y \in C$. Suppose $\{x_n\}\Delta$-converges to z and $d(x_n,\ Tx_n) \to 0$. Since $\{x_n\}$ is bounded, then $\{Tx_n\}$ is too. Also, using the Cauchy-Schwarz inequality, we obtain

$$\lim_{n\to\infty}\langle\overrightarrow{x_nTx_n}, \overrightarrow{ab}\rangle = 0, \tag{16}$$

$$\lim_{n\to\infty}\langle\overrightarrow{x_nTx_n}, \overrightarrow{Tx_nb}\rangle = 0, \tag{17}$$

for all $a, b \in X$. Replacing x and y respectively by x_n and z in (15) we get

$$d^2\ (Tx_n,\ Tz) \leq d^2(x_n,\ z) + 2\langle\overrightarrow{x_nTx_n}, \overrightarrow{zTz}\rangle. \tag{18}$$

Suppose that $Tz \neq z$. From (6), (2) and (16)-(18), we have

$$\begin{aligned}
\limsup_{n\to\infty} d^2(x_n,\ z) &< \limsup_{n\to\infty} d^2(x_n, Tz)\\
&= \limsup_{n\to\infty}(d^2(x_n,\ Tx_n) + d^2(Tz,\ Tx_n) + 2\langle\overrightarrow{x_nTx_n}, \overrightarrow{Tx_nTz}\rangle)\\
&= \limsup_{n\to\infty}\left(d^2(x_n,\ Tx_n) + d^2(z,\ x_n) + 2\langle\overrightarrow{x_nTx_n}, \overrightarrow{zTz}\rangle + 2\langle\overrightarrow{x_nTx_n}, \overrightarrow{Tx_nTz}\rangle\right)\\
&= \limsup_{n\to\infty} d^2(x_n,\ z).
\end{aligned}$$

From this contradiction we get the conclusion. □

We also need demiclosedness of multivalued nonexpansive mapping which can be found in [11, Lemma 3.2].

Lemma 3.3. *Let X be an Hadamard space, C be a nonempty closed convex subset of X and $T : C \to \mathcal{K}(X)$ be a nonexpansive mapping. If $\{x_n\}$ is regular, Δ-converges to z and* $\mathrm{dist}(x_n,\ Tx_n) \to 0$*, then $z \in F(T)$.*

Note that very bounded sequence in Hadamard spaces has a regular subsequence (see [22, p. 3690]) and so we can omit the regularity assumption on $\{x_n\}$ from original Lemma [11, Lemma 3.2]. We are now in a position to prove our main theorem for finding common element of the set of solutions of an equilibrium problem and fixed points of nonexpansive mappings and nonspreading mappings in an Hadamard space.

Theorem 3.4. *Let C be a nonempty closed convex subset of an Hadamard space X and $N \geq 1$ be an integer, Φ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let, for $i = 1, 2, \ldots, N$, $f_i : C \to C$ be a finite family of nonspreading mappings and $T_i : C \to \mathcal{K}(C)$ be a finite family of nonexpansive multivalued mappings. Assume that*

$\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(f_i) \cap \mathcal{E}(C, \Phi) \neq \emptyset$ and $T_i(p) = \{p\}$, $(i = 1, 2, \ldots, N)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in C$ and then by

$$\begin{cases} \Phi(u_n,\ y) - \langle \overrightarrow{u_n x_n}, \overrightarrow{u_n y} \rangle \geq 0,\ \forall y \in C \\ x_{n+1} = \alpha_n u_n \oplus \beta_n f_n u_n \oplus \gamma_n z_n,\ \forall n \geq 1 \end{cases}$$

where $z_n \in T_n u_n$, $T_n = T_{n(\mathrm{mod}(N))}$, $f_n = f_{n(\mathrm{mod}(N))}$ and $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ satisfy the condition $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a,\ b] \subset (0, 1)$. Then, the sequences $\{x_n\}$ and $\{u_n\}\Delta$- converge to an element of $q \in \mathcal{F}$, where $q = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. We present the proof in five steps.

Step 1. We first prove that $\lim\limits_{n\to\infty} d(x_n,\ q)$ exist for all $q \in \mathcal{F}$. Indeed, for each $q \in \mathcal{F}$, from the definition of J_Φ in Definition 2.10, we have $u_n = J_\Phi x_n$. This together with Proposition 2.12 implies that

$$d(u_n,\ q) = d(J_\Phi x_n,\ J_\Phi q) \leq d(x_n,\ q), \tag{19}$$

for all $n \geq 1$. Also, since every nonspearding mapping is quasi-nonexpansive, then

$$d(f_n u_n,\ q) = d(f_n u_n,\ f_n q) \leq d(u_n,\ q). \tag{20}$$

Moreover, nonexpansiveness of T_n implies that

$$d(z_n,\ q) = \mathrm{dist}(z_n,\ Tq) \leq \sup_{y \in T_n u_n} \mathrm{dist}(y,\ Tq) \leq H(T_n u_n,\ T_n q) \leq d(u_n,\ q). \tag{21}$$

The inequalities (19)-(21) together with (9) imply that

$$\begin{aligned} d(x_{n+1},\ q) &= d(\alpha_n u_n \oplus \beta_n f_n u_n \oplus \gamma_n z_n,\ q) \\ &\leq \alpha_n d(u_n,\ q) + \beta_n d(f_n u_n,\ q) + \gamma_n d(z_n,\ q) \\ &\leq d(u_n,\ q) \\ &\leq d(x_n,\ q). \end{aligned} \tag{22}$$

Thus, $\lim\limits_{n\to\infty} d(x_n,\ q)$ exists for all $q \in \mathcal{F}$.

Step 2. We claim that $\lim\limits_{n\to\infty} d(x_{n+i},\ x_n) = \lim\limits_{n\to\infty} d(u_{n+i},\ u_n) = 0$ for all $i \in \{1, 2, \ldots,\ N\}$. Using (11) and inequalities (19)-(21) we get

$$\begin{aligned} d^2(x_{n+1},\ q) =& d^2\ (\alpha_n \oplus \beta_n f_n u_n \oplus \gamma_n z_n,\ q) \\ \leq& \alpha_n d^2(u_n,\ q) + \beta_n d^2(f_n u_n,\ q) + \gamma_n d^2(z_n,\ q) \\ &- \frac{\alpha_n \beta_n}{1-\gamma_n} d^2(u_n,\ f_n u_n) - \gamma_n(1-\gamma_n) d^2\left(\frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n,\ z_n\right) \\ &\leq d^2(x_n,\ q) - \frac{\alpha_n \beta_n}{1-\gamma_n} d^2(u_n,\ f_n u_n) \\ &- \frac{\alpha_n \beta_n}{1-\gamma_n} d^2(u_n,\ f_n u_n) - \gamma_n(1-\gamma_n) d^2\left(\frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n,\ z_n\right). \end{aligned} \tag{23}$$

Therefore

$$\frac{a^2}{1-b} d^2(u_n,\ f_n u_n) \leq \frac{\alpha_n \beta_n}{1-\gamma_n} d^2(u_n,\ f_n u_n) \leq d^2(x_n,\ q) - d^2(x_{n+1},\ q) \tag{24}$$

and

$$\begin{aligned} a(1-b) d^2\left(\frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n,\ z_n\right) &\leq \gamma_n(1-\gamma_n) d^2\left(\frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n,\ z_n\right) \\ &\leq d^2(x_n,\ q) - d^2(x_{n+1},\ q). \end{aligned} \tag{25}$$

Using Step 1 and taking the limit as $n \to \infty$ yields that

$$\lim_{n\to\infty} d(u_n,\ f_n u_n) = 0, \tag{26}$$

$$\lim_{n\to\infty} d\left(\frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n,\ z_n\right) = 0. \tag{27}$$

Considering (3) we obtain

$$\begin{aligned} d(u_n,\ z_n) &\le d\left(u_n,\ \frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n\right) + d\left(\frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n,\ z_n\right) \\ &= \frac{\beta_n}{1-\gamma_n} d(u_n,\ f_n u_n) + d\left(\frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n,\ z_n\right) \\ &\le \frac{b}{1-a} d(u_n,\ f_n u_n) + d\left(\frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n,\ z_n\right). \end{aligned}$$

It follows from (26) and (27) that

$$\lim_{n\to\infty} d(u_n,\ z_n) = 0. \tag{28}$$

Also, from (8) we have

$$\begin{aligned} d(x_{n+1},\ u_n) &\le (1-\gamma_n) d\left(\frac{\alpha_n}{1-\gamma_n} u_n \oplus \frac{\beta_n}{1-\gamma_n} f_n u_n,\ u_n\right) + \gamma_n d(z_n,\ u_n) \\ &= \beta_n d(u_n,\ f_n u_n) + \gamma_n d(z_n,\ u_n) \to 0 \ \text{ (as } n \to \infty). \end{aligned} \tag{29}$$

Let $q \in \mathcal{F}$. Since J_Φ is firmly nonexpansive, then

$$\begin{aligned} d^2(u_n,\ q) &= d^2(J_\Phi x_n,\ J_\Phi q) \\ &\le \langle \overrightarrow{J_\Phi x_n\ J_\Phi q}, \overrightarrow{x_n q} \rangle \\ &= \langle \overrightarrow{u_n q}, \overrightarrow{x_n q} \rangle \\ &= \frac{1}{2}(d^2(u_n,\ q) + d^2(x_n,\ q) - d^2(u_n,\ x_n)) \end{aligned}$$

and hence

$$d^2(u_n,\ q) \le d^2(x_n,\ q) - d^2(u_n,\ x_n).$$

This inequality and (22) give us

$$d^2(x_{n+1},\ q) \le d^2(u_n,\ q) \le d^2(x_n,\ q) - d^2(u_n,\ x_n)$$

and hence

$$d^2(u_n,\ x_n) \le d^2(x_{n+1},\ q) - d^2(x_n,\ q).$$

Since the limit of $d(x_n,\ q)$ exists, then

$$\lim_{n\to\infty} d(u_n,\ x_n) = 0. \tag{30}$$

Utilizing (29) and (30) we get

$$d(u_{n+1},\ u_n) \le d(u_{n+1},\ x_{n+1}) + d(x_{n+1},\ u_n) \to 0 \quad \text{(as } n \to \infty).$$

Clearly, this shows that

$$\lim_{n\to\infty} d(u_{n+i},\ u_n) = 0, \quad \forall i \in \{1, 2, \dots,\ N\}. \tag{31}$$

Applying (30) and (31) we obtain that

$$d(x_{n+1},\ x_n) \le d(x_{n+1},\ u_{n+1}) + d(n_{n+1},\ u_n) + d(u_n,\ x_n) \to 0 \quad (\text{as } n \to \infty).$$

This also implies that

$$\lim_{n\to\infty} d(x_{n+i},\ x_n) = 0, \quad \forall i \in \{1, 2, \dots, N\}. \tag{32}$$

Step 3. Now, we prove for each $i \in \{1, 2, \dots, N\}$,

$$\lim_{n\to\infty} \operatorname{dist}(u_n,\ T_i u_n) = \lim_{n\to\infty} d(u_n,\ f_i u_n) = 0$$

and

$$\lim_{n\to\infty} \operatorname{dist}(x_n,\ T_i x_n) = \lim_{n\to\infty} d(x_n,\ f_i x_n) = 0.$$

It follows from (28) that

$$\lim_{n\to\infty} \operatorname{dist}(u_n,\ T_n u_n) \le \lim_{n\to\infty} d(u_n,\ z_n) = 0.$$

Observe that

$$\begin{aligned} \operatorname{dist}(u_n,\ T_{n+i}u_n) &\le d(u_n,\ u_{n+i}) + \operatorname{dist}(u_{n+i},\ T_{n+i}u_{n+i}) + H(T_{n+i}u_{n+i},\ T_{n+i}u_n) \\ &\le 2d(u_n,\ u_{n+i}) + \operatorname{dist}(u_{n+i},\ T_{n+i}u_{n+i}) \to 0 \end{aligned}$$

which implies that the sequence

$$\bigcup_{i=1}^{N} \{\operatorname{dist}(u_n,\ T_{n+i}u_n)\}_{n\ge 0} \to 0 \text{ as } n \to \infty.$$

Furthermore, observe that for $i = 1, 2, \dots, N$ we have

$$\begin{aligned} \{\operatorname{dist}(u_n,\ T_i u_n)\}_{n\ge 0} &= \{\operatorname{dist}(u_n,\ T_{n+(i-n)}u_n)\}_{n\ge 0} \\ &= \{\operatorname{dist}(u_n,\ T_{n+i_n}u_n)\}_{n\ge 0} \subset \bigcup_{i=1}^{N} \{\operatorname{dist}(u_n,\ T_{n+i}u_n)\}_{n\ge 0}, \end{aligned}$$

where $i - n =: i_n \pmod N$ and $i_n \in \{1, 2, \dots, N\}$. Therefore $\lim_{n\to\infty} \operatorname{dist}(u_n,\ T_i u_n) = 0$, for $i = 1, 2, \dots, N$. Since f_{n+i} is nonspreading, we have

$$\begin{aligned} d^2(f_{n+i}u_{n+i},\ f_{n+i}u_n) &\le d^2(u_{n+i},\ u_n) + 2\langle \overrightarrow{u_{n+i}f_{n+i}u_{n+i}}, \overrightarrow{u_n f_{n+i}u_n}\rangle \\ &\le d^2(u_{n+i},\ u_n) + 2d(u_{n+i},\ f_{n+i}u_{n+i})d(u_n,\ f_{n+i}u_n) \to 0 \end{aligned} \tag{33}$$

as $n \to \infty$. This together with (26) and (31) implies that

$$d(u_n,\ f_{n+i}u_n) \le d(u_n,\ u_{n+i}) + d(u_{n+i},\ f_{n+i}u_{n+i}) + d(f_{n+i}u_{n+i},\ f_{n+i}u_n) \to 0 \ (\text{as } n \to \infty)$$

for any $i \in \{1, 2, \dots, N\}$, which gives us

$$\lim_{n\to\infty} d(u_n,\ f_i u_n) = 0, \quad \forall i \in \{1, 2, \dots, N\}. \tag{34}$$

For each $i = 1, 2, \dots, N$, we obtain that

$$\begin{aligned}\operatorname{dist}(x_n,\ T_i x_n) &\le d(x_n,\ u_n) + \operatorname{dist}(u_n,\ T_i u_n) + H(T_i u_n,\ T_i x_n)\\ &\le 2d(u_n,\ x_n) + \operatorname{dist}(u_n,\ T_i u_n) \to 0.\end{aligned}$$

By a similar argument as in (33) and considering (34) we have

$$\lim_{n\to\infty} d(f_i u_n,\ f_i x_n) = 0. \tag{35}$$

Hence

$$d(x_n,\ f_i x_n) \le d(x_n,\ u_n) + d(u_n,\ f_i u_n) + d(f_i u_n,\ f_i x_n)$$

Using (30), (34) and (35) we conclude that

$$\lim_{n\to\infty} d(x_n,\ f_i x_n) = 0.$$

Step 4. In this step, we show that if $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ that Δ-converges to q, then $q \in \mathcal{F}$. Applying (30) we see that $\{u_{n_i}\}$ also Δ-converges to q. First, we show $q \in \mathcal{E}(C,\ \Phi)$. Since $u_n = J_\Phi x_n$ we have

$$\Phi(u_n,\ y) - \langle \overrightarrow{u_n x_n}, \overrightarrow{u_n y}\rangle \ge 0,\ \forall y \in C.$$

From $(A2)$, we have

$$\Phi(y,\ u_n) \le \langle \overrightarrow{x_n u_n}, \overrightarrow{u_n y}\rangle$$

and hence

$$\Phi(y,\ u_{n_i}) \le \langle \overrightarrow{x_{n_i} u_{n_i}}\ ,\ \overrightarrow{u_{n_i} y}\rangle \le d(x_{n_i}\ ,\ u_{n_i}) d(u_{n_i}\ ,\ y).$$

It follow from (30), $(A4)$ and Lemma 2.9 that

$$\Phi(y,\ q) \le 0,\ \forall y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t = ty \oplus (1-t)q$. Since C is convex we have $y_t \in C$ and hence $\Phi(y_t,\ q) \le 0$. So, from $(A1)$ and $(A4)$ we have

$$0 = \Phi(y_t,\ y_t) \le t\Phi(y_t,\ y) + (1-t)\Phi(y_t,\ q) \le t\Phi(y_t,\ y),$$

which gives $\Phi(y_t,\ y) \ge 0$. From $(A3)$ we have $\Phi(q,\ y) \ge 0, \forall y \in C$ and hence $q \in \mathcal{E}(C,\ \Phi)$. Also, demiclosed principles in Lemmas 3.3 and 3.2 imply that $q \in \bigcap_{i=1}^N F(T_i) \cap F(f_i)$. Therefore, $q \in \mathcal{F}$.

Step 5. Finally, we show that $\{x_n\}$ and $\{u_n\}$ are Δ-convergent to an element of $\mathcal{F}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that Δ-converges to q. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that Δ-converges to q_1 and q_2. To complete the proof, we show $q_1 = q_2$. We know from preceding step that $q_1, q_2 \in \mathcal{F}$ and from step 1 that $\lim_{n\to\infty} d(x_n,\ q_1)$ and $\lim_{n\to\infty} d(x_n,\ q_2)$ exist. If $q_1 \ne q_2$, then from (6) we conclude that

$$\begin{aligned}\lim_{n\to\infty} d(x_n,\ q_1) &= \limsup_{i\to\infty} d(x_{n_i},\ q_1) < \limsup_{i\to\infty} d(x_{n_i},\ q_2)\\ &= \lim_{n\to\infty} d(x_n,\ q_2) = \limsup_{j\to\infty} d(x_{n_j},\ q_2)\\ &< \limsup_{j\to\infty} d(x_{n_j},\ q_1) = \lim_{n\to\infty} d(x_n,\ q_1),\end{aligned}$$

which is a contradiction. Hence, $q_1 = q_2$. Thus $\{x_n\}$ Δ-converges to q. It follows from (30) that $\{u_n\}$ also Δ-converges to q.

Put $y_n = P_{\mathcal{F}} x_n$. We show that $q = \lim_{n\to\infty} y_n$. Since $q \in \mathcal{F}$, it follows from Theorem 2.7 that

$$\langle \overrightarrow{y_n x_n}, \overrightarrow{q y_n} \rangle \geq 0.$$

By Lemma 2.8, $\{y_n\}$ converges strongly to some $y \in \mathcal{F}$. Also,

$$\begin{aligned} 0 &\leq \langle \overrightarrow{y_n x_n}, \overrightarrow{q y_n} \rangle \\ &= \langle \overrightarrow{y_n q}, \overrightarrow{q y_n} \rangle + \langle \overrightarrow{q x_n}, \overrightarrow{q y} \rangle + \langle \overrightarrow{q x_n}, \overrightarrow{y y_n} \rangle \\ &\leq \langle \overrightarrow{y_n q}, \overrightarrow{q y_n} \rangle + \langle \overrightarrow{q x_n}, \overrightarrow{q y} \rangle + d(q, x_n) d(y, y_n). \end{aligned}$$

Taking $\limsup_{n\to\infty}$, using Lemma 2.3 and the fact that x_n Δ-converges to q and $y_n \to y$, we obtain

$$0 \leq \langle \overrightarrow{q y}, \overrightarrow{y q} \rangle = -d^2(q,\ y)\ ,$$

which gives us $q = y$ and the proof is complete. □

Theorem 3.5. *Let C be a nonempty closed convex subset of an Hadamard space X and $N \geq 1$ be an integer. Let, for $i = 1, 2, \ldots, N$, $f_i : C \to C$ be a finite family of nonspreading mappings and $T_i : C \to \mathcal{K}(C)$ be a finite family of nonexpansive multivalued mappings. Assume that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(f_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, \ldots, N)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the sequence generated initially by an arbitrary element $x_1 \in C$ and then by*

$$x_{n+1} = \alpha_n xn\ \oplus \beta_n f_n x_n \oplus \gamma_n z_n,\ \forall n \geq 1$$

where $z_n \in T_n x_n, T_n = T_{n(\mathrm{mod}}(N)), f_n = f_{n(\mathrm{mod}}(N))$ and $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ satisfy the condition $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a,\ b] \subset (0, 1)$. Then, the sequence $\{x_n\}$ is Δ-convergent to an element of $q \in \mathcal{F}$, where $q = \lim\limits_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. Putting $\Phi(x,\ y) = 0$ for all $x, y \in C$ in Theorem 3.4, we have $u_n = x_n$. Then the sequence $\{x_n\}$ is Δ-convergent to an element of $q \in \mathcal{F}$, where $q = \lim_{n\to\infty} P_{\mathcal{F}} x_n$. □

If in Theorem 3.5 $X = H$ is a Hilbert space, $N = 1$ and T_1 be singe valued mapping we have the following theorem as a corollary.

Theorem 3.6. *[17, Theorem 4.1] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and let T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} x_1 \in \mathrm{C} \\ x_{\mathrm{n}+1} = (1 - \alpha_{\mathrm{n}}) x_{\mathrm{n}} + \alpha_{\mathrm{n}} (\beta_{\mathrm{n}} \mathrm{S} x_{\mathrm{n}} + (1 - \beta_{\mathrm{n}}) T x_{\mathrm{n}}) \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. If $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$, then $\{x_n\}$ converges weakly to $v \in F(S) \cap F(T)$.

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