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Relative Relation Matrix-Based Approaches for Updating Approximations in Multigranulation Rough Sets

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Abstract. Multigranulation rough set (MGRS) theory has attracted much attention. However, with the advent of big data era, the attribute values may often change dynamically, which leads to high computational complexity when handling large and complex data. How to effectively obtain useful knowledge from the dynamic information system becomes an important issue in MGRS. Motivated by this requirement, in this paper, we propose relative relation matrix approaches for computing approximations in MGRS and updating them dynamically. A simplified relative relation matrix is used to calculate approximations in MGRS, it is showed that the space and time complexities are no more than that of the original method. Furthermore, relative relation matrix-based approaches for updating approximations in MGRS while refining or coarsening attribute values are proposed. Several incremental algorithms for updating approximations in MGRS are designed. Finally, experiments are conducted to evaluate the efficiency and validity of the proposed methods.

1. Introduction

Rough sets theory, which was proposed by Pawlak in 1982 [19], has become an important mathematical tool for effectively processing uncertain and ambiguous information. It has been widely used in machine learning [12, 17, 18, 23, 30, 31], pattern recognition [2, 8, 11, 15, 26, 27, 39], decision making [43, 44], image processing [19, 20], data mining and etc. With respect to various requirements, many extensions have been proposed to overcome its limitation, such as covering based rough sets [10, 40], neighborhood rough sets [13, 33], variable precision rough sets [9, 46], fuzzy rough sets [7, 29, 32, 34–36], multigranulation rough sets [24] and etc.

Pawlak's rough sets (PRS) are constructed by a single equivalence relation, which is regarded as a single granular structure. Therefore, PRS is too restrictive to apply in other information systems. For example, it is difficult to extract decision rules from multi-source information system. To overcome this problem, Qian et al. [24] proposed the MGRS theory, in which the lower and upper approximations are no longer composed of a single relation, but are approximated by multiple binary relations. What's more,

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MGRS theory includes optimistic multigranulation rough sets (OMGRS) and pessimistic multigranulation rough sets (PMGRS). Based on the PMGRS model, Qian et al. [21] studied the attribute reduction from a decision table. Based on the MGRS theory, many models have been proposed to apply MGRS into a broader area, such as variable multigranulation rough sets [45], neighborhood multigranulation rough sets [14, 16], incomplete multigranulation rough sets [22], variable precision multigranulation decision-theoretic fuzzy rough sets [3] and local multigranulation decision-theoretic rough sets [25].

The incremental updating method can effectively acquire new knowledge on the basis of the previous knowledge, which has received a lot of attention. Hu et al. [6] investigated a matrix-based incremental method to update knowledge in neighborhood multigranulation rough sets while adding or deleting granular structures. Using the dominant rough set method, Sang et al. [28] studied the incremental attribute reduction approach when adding a single object to the data, and proposed an incremental algorithm to updating the dominant conditional entropy. Based on the matrix-based incremental knowledge updating method, Xu et al. [37] proposed incremental algorithms of positive domain, negative domain and boundary domain to update the neighborhood multigranulation rough set knowledge. Before applying MGRS in real life situation, we must calculate the approximations. However, in an information explosion era, the structures of data sets become more and more complex, the attributes often increase or decrease, attribute values may change, and the objects may often change. At this time, computing the approximations of MGRS by dynamic method is an effective way to solve these problems. Yang et al. [38] proposed a dynamic update method when the granular structures increase. Yu et al. [41, 42] proposed vector-based and matrix-based approaches to compute the approximations in MGRS, respectively. Hu et al. [4] paid attention to the dynamic updating approximations in MGRS while refining or coarsening attribute values. Hu et al. [5] proposed the matrix-based approaches to dynamically update approximations in MGRS when a single granular structure changes.

In real world applications, attribute values are variable during the knowledge update process. For example, some attribute values in the information system will be out of date, so they need to be updated in time. Therefore, some new attribute values are added in the domain to improve the timeliness of the data. When the attribute values are coarsened or refined, the knowledge granular may change. In order to improve the efficiency of knowledge acquisition, it is important to update the approximations in MGRS while refining or coarsening attribute values. In this paper, we propose relative relation matrix approaches for computing approximations in the context of MGRS while refining or coarsening attribute values. By reducing the calculation of elements that are not related to the target concept, we obtain a relative relation matrix, which reduces the dimension of the equivalence relation matrix. Its calculation time and storage space are no more than the original matrix. In other words, the relative relation matrix is obtained by simplifying the equivalence relation matrix proposed in [5]. Finally, experiments are conducted to evaluate the efficiency and validity of the proposed methods. From the experimental results, the relative relation matrix-based approaches have a better performance than matrix-based approaches.

The rest of this paper is organized as follows. In Section 2, some basic concepts about MGRS are reviewed. In Section 3, the relative relation matrix-based approaches are proposed, and we design a static algorithm to calculate lower and upper approximations in MGRS. In Section 4, dynamic approaches for updating approximations in MGRS while refining or coarsening attribute values are proposed. In Section 5, several dynamic algorithms are designed for updating approximations in MGRS. In Section 6, experiments are conducted to show the efficiency and validity of the proposed methods. Finally, some conclusions are given in Section 7.

2. Preliminaries

In this section, some basic concepts about MGRS are briefly reviewed.

Definition 2.1. *([19])* Let $IS = \left(U, AT, V_{AT}, f\right)$ be an information system, where $U = \{x_1, x_2, \cdots, x_n\}$ is a nonempty finite set of objects called the universe. $AT = \{A_1, A_2, \cdots, A_m\}$ is a nonempty finite family of attribute sets, the element $A_k \in AT$ is called an attribute set, for any $k \in \{1, 2, \cdots, m\}$. $V_{AT} = \bigcup\limits_{A \in AT} V_A$ is a domain of attributes values, where V_A is the domain of attribute set A. $f : U \times AT \to V_{AT}$ is a decision function such that $\forall A \in AT, x \in U, f(x, A) \in V_A$.

Definition 2.2. *([24])* Let $IS = (U, AT, V_{AT}, f)$ be an information system. $\forall X \subseteq U$, the optimistic multigranulation lower and upper approximations of X are denoted by $\underline{\sum_{k=1}^{m} A_k}^{O}(X)$ and $\overline{\sum_{k=1}^{m} A_k}^{O}(X)$, respectively.

$$\underline{\sum_{k=1}^{m} A_k}^{O}(X) = \left\{x \in U | [x]_{A_1} \subseteq X \vee [x]_{A_2} \subseteq X \vee \cdots \vee [x]_{A_m} \subseteq X\right\},$$

$$\overline{\sum_{k=1}^{m} A_k}^{O}(X) = \sim \underline{\sum_{k=1}^{m} A_k}^{O}(\sim X),$$

where $[x]_{A_k}$ is the equivalence class containing x in terms of the attribute set A_k and $\sim X$ is the complement of the given X.

Theorem 2.3. *([24]) Let $IS = (U, AT, V_{AT}, f)$ be an information system. $\forall X \subseteq U$, then*

$$\overline{\sum_{k=1}^{m} A_k}^{O}(X) = \left\{x \in U | [x]_{A_1} \cap X \neq \emptyset \wedge [x]_{A_2} \cap X \neq \emptyset \wedge \cdots \wedge [x]_{A_m} \cap X \neq \emptyset\right\}.$$

Definition 2.4. *([24])* Let $IS = (U, AT, V_{AT}, f)$ be an information system. $\forall X \subseteq U$, the pessimistic multigranulation lower and upper approximations of X are denoted by $\underline{\sum_{k=1}^{m} A_k}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k}^{P}(X)$, respectively.

$$\underline{\sum_{k=1}^{m} A_k}^{P}(X) = \left\{x \in U | [x]_{A_1} \subseteq X \wedge [x]_{A_2} \subseteq X \wedge \cdots \wedge [x]_{A_m} \subseteq X\right\},$$

$$\overline{\sum_{k=1}^{m} A_k}^{P}(X) = \sim \underline{\sum_{k=1}^{m} A_k}^{P}(\sim X).$$

Theorem 2.5. *([24]) Let $IS = (U, AT, V_{AT}, f)$ be an information system. $\forall X \subseteq U$, then*

$$\overline{\sum_{k=1}^{m} A_k}^{P}(X) = \left\{x \in U | [x]_{A_1} \cap X \neq \emptyset \vee [x]_{A_2} \cap X \neq \emptyset \cdots \vee [x]_{A_m} \cap X \neq \emptyset\right\}.$$

Definition 2.6. *([1])* Let $IS = (U, AT, V_{AT}, f)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. $f(x_i, A_k)$ is the value of x_i with respect to the attribute set A_k. Then $U_{A_k} = \left\{x_{i^*} | f(x_{i^*}, A_k) = f(x_i, A_k)\right\}$. Suppose $f(x_{i^*}, A_k) = V$, where $V \notin V_{A_k}$ and $x_{i^*} \in U_{A_k}$. Then the attribute value set $f(x_{i^*}, A_k)$ of object x_{i^*} is refined to V.

Definition 2.7. *([1])* Let $IS = (U, AT, V_{AT}, f)$ be an information system, where $A_k \in AT$ for any $k \in \{1, 2, \cdots, m\}$. $f(x_i, A_k)$ is the value of x_i with respect to the attribute set A_k, $f(x_j, A_k)$ is the value of x_j with respect to the attribute set A_k, and $f(x_i, A_k) \neq f(x_j, A_k)$. Then $U_{A_k} = \left\{x_{i^*} | f(x_{i^*}, A_k) = f(x_i, A_k)\right\}$. Suppose $f(x_{i^*}, A_k) = f(x_j, A_k)$, $\forall x_{i^*} \in U_{A_k}$. Then the attribute value set $f(x_i, A_k)$ of object x_i is coarsened to $f(x_j, A_k)$.

3. On computation of approximations in MGRS based on the relative relation matrix

In this section, a static approach is proposed to calculate the approximations in MGRS, which is based on the relative relation matrix.

Definition 3.1. *([5])* Let $U=\{x_1,x_2,\cdots,x_n\}$ be an universe, the boolean column matrix of $X\subseteq U$ is denoted as $G^U(X)=[\chi_X^U(x_1),\chi_X^U(x_2),\cdots,\chi_X^U(x_n)]^T$, and $G_{-1}^U\left(G^U(X)\right)=X$, where "$T$" denotes the transpose operation and $\chi_X^U(x)$ is known as the characteristic function:

$$\chi_X^U(x)=\begin{cases}1 & x\in X\\ 0 & x\notin X\end{cases}.$$

Definition 3.2. Let $IS=\left(U,AT,V_{AT},f\right)$ be an information system. $[x]_{A_k}$ denotes the equivalence class containing x with respect to the granular structure A_k on U. The relative approximation space with respect to X is defined as $W_k=\bigcup\limits_{x\in X}[x]_{A_k}$. Then $RIS=\left(\Omega,AT,V_{AT},f\right)$ is called a relative information system such that $\Omega=\{W_1,W_2,\cdots,W_m\}$.

Definition 3.3. Let $RIS=\left(\Omega,AT,V_{AT},f\right)$ be a relative information system. $\forall X\subseteq U$ and given the granular structure A_k, the boolean column matrix of X in W_k and U are denoted by $G^{W_k}(X)$ and $G^U(X)$, respectively. $L:U\to W_k$ is a bijective mapping and L^{-1} is an inverse mapping of L satisfies that $L(G^U(X))=G^{W_k}(X)$ and $L^{-1}(G^{W_k}(X))=G^U(X)$.

Definition 3.3 shows that the two information systems $IS=\left(U,AT,V_{AT},f\right)$ and $RIS=\left(\Omega,AT,V_{AT},f\right)$ can be converted to each other via the mapping.

Definition 3.4. Let $RIS=\left(\Omega,AT,V_{AT},f\right)$ be a relative information system. $\forall X\subseteq U$ and given the granular structure A_k, denote $W_k=\left\{w_1,w_2,\cdots,w_{r_k}\right\}$ and $r_k=|W_k|$, then we call $M_{A_k}=(m_{ij})_{r_k\times r_k}$ the relative relation matrix of A_k with respect to X, where

$$m_{ij}=\begin{cases}1 & w_i\in[w_j]_{A_k}\\ 0 & w_i\notin[w_j]_{A_k}\end{cases}\qquad i,j\in\{1,2,\cdots,r_k\}.$$

Table 1. A decision information system.

U	a_1	a_2	a_3	d
x_1	2	2	1	1
x_2	1	3	3	2
x_3	2	2	1	1
x_4	1	1	3	2
x_5	2	1	2	1
x_6	3	3	3	1

Example 3.5. Let $IS=\left(U,AT,V_{AT},f\right)$ be an information system with $U=\{x_1,x_2,x_3,x_4,x_5,x_6\}$, $A_1=\{a_1,a_3\}$ and $A_2=\{a_2,a_3\}$ (see Table 1). Let $X=\{x_1,x_4,x_5\}$, according to Definition 3.2, $W_1=\{x_1,x_2,x_3,x_4,x_5\}=\{w_1,w_2,w_3,w_4,w_5\}$, $W_2=\{x_1,x_3,x_4,x_5\}=\{w_1,w_2,w_3,w_4\}$. By Definition 3.3, $G^U(X)=[1,0,0,1,1,0]^T$, $G^{W_1}(X)=[1,0,0,1,1]^T$, $G^{W_2}(X)=[1,0,1,1]^T$, $L^{-1}(G^{W_1}(X))=L^{-1}(G^{W_2}(X))=G^U(X)$. From Definition 3.4, we have that for any $k\in\{1,2\}$, the relative relation matrix can be calculated as follows:

$$M_{A_1}=\begin{bmatrix}1&0&1&0&0\\0&1&0&1&0\\1&0&1&0&0\\0&1&0&1&0\\0&0&0&0&1\end{bmatrix},M_{A_2}=\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

According to [5], the relation matrix can be calculated as follows:

$$M^*_{A_1} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, M^*_{A_2} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It shows that the dimension of the relative relation matrix is not higher than that of the equivalence relation matrix. Therefore, $M^*_{A_1}$ and $M^*_{A_2}$ occupy more storage space than M_{A_1} and M_{A_2}. Reducing the calculation of elements unrelated to X can simplify the relation matrix proposed in [5].

Definition 3.6. Let $RIS = (\Omega, AT, V_{AT}, f)$ be a relative information system. The diagonal matrix D_{A_k} of granular structure A_k induced by the relative relation matrix M_{A_k} can be defined as follows.

$$D_{A_k} = diag(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_{r_k}}),$$

where $\lambda_i = \sum_{j=1}^{r_k} m_{ij}$ for any $i \in \{1, 2, \cdots, r_k\}$.

From the viewpoint of probabilistic rough sets, the definition of inclusion degree can be introduced. Let $P_i = \frac{|[w_i]_{A_k} \cap X|}{|\lambda_i|}$, P_i is the degree of equivalence class $[w_i]_{A_k}$ included in X, where $0 \leq P_i \leq 1 (1 \leq i \leq r_k)$. If $P_i = 1$, then w_i belongs to the element in the lower approximation; if $P_i > 0$, then w_i belongs to the element in the upper approximation.

Definition 3.7. Let $RIS = (\Omega, AT, V_{AT}, f)$ be a relative information system. $\forall X \subseteq U$, $G^{W_k}(X)$ represents the boolean column matrix of X in W_k, $M_{A_k} = (m_{ij})_{r_k \times r_k}$ denotes the relative relation matrix of A_k, D_{A_k} denotes the diagonal matrix of granular structure A_k induced by the relative relation matrix. The column matrix $H_{A_k}(X)$ of the granular structure A_k can be calculated as follows.

$$H_{A_k}(X) = D_{A_k} \cdot (M_{A_k} \cdot G^{W_k}(X)),$$

where " $\cdot$ " denotes the product of matrix.

Through the matrix multiplication, $H_{A_k}(X))$ can induce the upper and the lower approximations.

Example 3.8. *(Continuation of Example 3.5)* By Definition 3.7, the column matrix $H_{A_1}(X)$ and $H_{A_2}(X)$ can be calculated as follows.

$$H_{A_1}(X) = D_{A_1} \cdot (M_{A_1} \cdot G^{W_1}(X)) = [1/2, 1/2, 1/2, 1/2, 1]^T,$$
$$H_{A_2}(X) = D_{A_2} \cdot (M_{A_2} \cdot G^{W_2}(X)) = [1/2, 1/2, 1, 1]^T.$$

In order to further describe the approximations in MGRS, the column matrix $H_{A_k}(X) = [h^1_{A_k}, h^2_{A_k}, \cdots, h^{r_k}_{A_k}]^T$ is used to define two types of cut matrices.

Definition 3.9. *([5])* Let $RIS = (\Omega, AT, V_{AT}, f)$ be a relative information system. $\forall X \subseteq U$ and $0 \leq \alpha \leq \beta \leq 1$, $H_{A_k}(X) = [h^1_{A_k}, h^2_{A_k}, \cdots, h^{r_k}_{A_k}]^T$ denotes the column matrix of A_k. The two cut matrices of A_k can be denoted as follows.

1. $H^{[\alpha,\beta]}_{A_k}(X) = (h^{i\downarrow}_{A_k})_{r_k \times 1}$, where

$$h^{i\downarrow}_{A_k} = \begin{cases} 1 & \alpha \leq h^i_{A_k} \leq \beta \\ 0 & otherwise \end{cases} \quad i \in \{1, 2, \cdots, r_k\}.$$

2. $H_{A_k}^{(\alpha,\beta]}(X) = (h_{A_k}^{i\uparrow})_{r_k \times 1}$,where

$$h_{A_k}^{i\uparrow} = \begin{cases} 1 & \alpha < h_{A_k}^{i} \leq \beta \\ 0 & otherwise \end{cases} \quad i \in \{1, 2, \cdots, r_k\}.$$

Definition 3.10. Let $P = [p_1, p_2, ..., p_n]^T$ is an n-dimensional column matrix. $C_\gamma(P) = [c_1, c_2, ..., c_n]^T$ is a boolean column matrix called the γ-cut of P, where

$$c_i = \begin{cases} 1 & p_i \geq \gamma \\ 0 & p_i < \gamma \end{cases} \quad i \in \{1, 2, \cdots, n\}.$$

Theorem 3.11. *Let $RIS = (\Omega, AT, V_{AT}, f)$ be a relative information system. $\forall X \subseteq U$, the following results hold.*

$$G^U(\underline{\sum_{k=1}^{m} A_k}^{O}(X)) = C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{[1,1]}(X))), G^U(\overline{\sum_{k=1}^{m} A_k}^{O}(X)) = C_m(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{(0,1]}(X))).$$

Proof. Suppose that

$$G^{W_k}(\underline{\sum_{k=1}^{m} A_k}^{O}(X)) = [\chi^{W_k}_{\underline{\sum_{k=1}^{m} A_k}^{O}(X)}(w_1), \chi^{W_k}_{\underline{\sum_{k=1}^{m} A_k}^{O}(X)}(w_2), \cdots, \chi^{W_k}_{\underline{\sum_{k=1}^{m} A_k}^{O}(X)}(w_{r_k})].$$

$\forall s \in \{1, 2, \cdots, r_k\}$, we have

$$\begin{aligned} \chi^{W_k}_{\underline{\sum_{k=1}^{m} A_k}^{O}(X)}(w_s) = 1 &\Leftrightarrow w_s \in \underline{\sum_{k=1}^{m} A_k}^{O}(X) \\ &\Leftrightarrow \exists k \in \{1, 2, \cdots, m\}, [w_s]_{A_k} \subseteq X \\ &\Leftrightarrow \forall w_t \in [w_s]_{A_k}, (w_t, w_s) \in R_k, w_t \in X \\ &\Leftrightarrow m_{st} = 1, \chi_X^{W_k}(w_t) = 1 \\ &\Leftrightarrow h_{A_k}^{s\downarrow} = \frac{\sum_{t=1}^{r_k} m_{st} \times \chi_X^{W_k}(w_t)}{\sum_{t=1}^{r_k} m_{st}} = 1 \\ &\Leftrightarrow c_1^s(\sum_{k=1}^{m} h_{A_k}^{s\downarrow}) = 1. \end{aligned}$$

Therefore, $G^U(\underline{\sum_{k=1}^{m} A_k}^{O}(X)) = C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{[1,1]}(X)))$. Similarly, we can also have that $G^U(\overline{\sum_{k=1}^{m} A_k}^{O}(X)) = C_m(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{(0,1]}(X)))$. □

Theorem 3.12. *Let $RIS = (\Omega, AT, V_{AT}, f)$ be a relative information system. $\forall X \subseteq U$, the following results hold.*

$$G^U(\underline{\sum_{k=1}^{m} A_k}^{P}(X)) = C_m(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{[1,1]}(X))), G^U(\overline{\sum_{k=1}^{m} A_k}^{P}(X)) = C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{(0,1]}(X))).$$

Proof. The proof is similar to that of Theorem 3.11. □

Example 3.13. *(Continuation of Example 3.8)* The boolean column matrix of lower and upper approximations in OMGRS can be computed as follows.

$$G^U(\underline{\sum_{k=1}^{2} A_k}^{O}(X)) = C_1(\sum_{k=1}^{2} L^{-1}(H_{A_k}^{[1,1]}(X))) = C_1([0,0,0,1,2,0]^T) = [0,0,0,1,1,0]^T .$$

Similarly, we have

$$G^U(\overline{\sum_{k=1}^{2} A_k}^{O}(X)) = C_2(\sum_{k=1}^{2} L^{-1}(H_{A_k}^{(0,1]}(X))) = [1,0,1,1,1,0]^T .$$

The boolean column matrix of lower and upper approximations in PMGRS can be computed as follows.

$$G^U(\underline{\sum_{k=1}^{2} A_k}^{P}(X)) = C_2(\sum_{k=1}^{2} L^{-1}(H_{A_k}^{[1,1]}(X))) = [0,0,0,0,1,0]^T ,$$

$$G^U(\overline{\sum_{k=1}^{2} A_k}^{P}(X)) = C_1(\sum_{k=1}^{2} L^{-1}(H_{A_k}^{(0,1]}(X))) = [1,1,1,1,1,0]^T .$$

Thus, according to Definition 3.1, we can obtain that

$$\underline{\sum_{k=1}^{2} A_k}^{O}(X) = \{x_4, x_5\} , \overline{\sum_{k=1}^{2} A_k}^{O}(X) = \{x_1, x_3, x_4, x_5\} , \underline{\sum_{k=1}^{2} A_k}^{P}(X) = \{x_5\} , \overline{\sum_{k=1}^{2} A_k}^{P}(X) = \{x_1, x_2, x_3, x_4, x_5\} .$$

Based on the above results, we design a static algorithm to compute the lower and upper approximations in MGRS.

Algorithm 1 Relative relation matrix-based static algorithm for approximations in MGRS (RRMS).

Input: $IS = (U, AT, V_{AT}, f)$ and $X \subseteq U$.
Output: Approximations in MGRS.

1: **for** each $k \in |AT|$ **do**
2: Compute $W_k = \bigcup_{x\in X} [x]_{A_k}$;
3: Let $r_k = |W_k|$;
4: **for** each $i, j \in r_k$ **do**
5: **if** $\chi^{W_K}_{[w_j]_{A_k}}(w_i) = 1$ and $\chi^{W_K}_{X}(w_j) = 1$ **then** $m_{ij} = 1$
6: **end if**
7: **end for**
8: Compute the diagonal matrix D_{A_k};
9: Let $G^{W_k}(X) = L(G^U(X))$, $H_{A_k}(X) = D_{A_k} \cdot (M_{A_k} \cdot G^{W_k}(X))$.
10: **end for**
11: $G^U(\underline{\sum_{k=1}^{m} A_k}^{O}(X)) = C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{[1,1]}(X)))$; $G^U(\overline{\sum_{k=1}^{m} A_k}^{O}(X)) = C_m(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{(0,1]}(X)))$.
12: $G^U(\underline{\sum_{k=1}^{m} A_k}^{P}(X)) = C_m(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{[1,1]}(X)))$; $G^U(\overline{\sum_{k=1}^{m} A_k}^{P}(X)) = C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k}^{(0,1]}(X)))$.

Suppose that $|R(X)| = max\{|W_k|\}$. The time complexity of Step 2 is $O(|AT||X|| \sim X|))$. Steps 4-7 are to calculate M_{A_k} with time complexity $O(|AT||R(X)|^2))$. Step 9 is to calculate $H_{A_k}(X)$ with time complexity

$O(|AT||R(X)|^2)$). Steps 11-12 are to compute the approximations of MGRS with time complexity $O(|AT||U|)$. The total time complexity of Algorithm 1 is $O(|AT||R(X)|^2)$. Similarly, the total space complexity of Algorithm 1 is $S(|R(X)|^2)$. The total time and space complexities of the static algorithm in [5] are $O(|AT||U|^2)$ and $S(|U|^2)$, respectively. Based on the relative relation matrix approach, it shows that the time and space complexities of Algorithm 1 are no more than that of the static algorithm in [5].

4. Relative relation matrix-based dynamic approaches for updating approximations in MGRS while refining or coarsening attribute values

4.1. Relative relation matrix-based approaches for updating approximations while refining attribute values

In this subsection, we present the relative relation matrix-based theorems for dynamic updating approximations in MGRS while attribute values refining. For convenience of description, for any $X \subseteq U$, a list of symbols are shown in Table 2 at times t and $t+1$. For all $V_{A_k^t} \in V_{AT^t}$ $(k \le m)$, $\exists V_{A_k^{t+1}} \in V_{AT^{t+1}}$ such that $V_{A_k^t} \subseteq V_{A_k^{t+1}}$ for any $k \in \{1, 2, \cdots, m\}$. When the attribute values of x_i with respect to granular structure A_k are refined, there exists equivalence class $[x_i]^t_{A_k}$ is divided into $[x_i]^{t+1}_{A_k}$ and $[x_j]^{t+1}_{A_k}$ such that $[x_i]^{t+1}_{A_k} \bigcap [x_j]^{t+1}_{A_k} = \emptyset$ and $[x_i]^t_{A_k} = [x_i]^{t+1}_{A_k} \cup [x_j]^{t+1}_{A_k}$. What's more, according to [4], we have the following results.

Table 2. A list of symbols.

Variable Implication	Time t	Time $t+1$
Information system	(U, AT, V_{AT^t}, f^t)	$(U, AT, V_{AT^{t+1}}, f^{t+1})$
Attribute values	$V_{A_k^t}$	$V_{A_k^{t+1}}$
Equivalence class of x	$[x]^t_{A_k}$	$[x]^{t+1}_{A_k}$
Lower approximations of OMGRS	$\underline{\sum_{k=1}^m A_k^t}^O(X)$	$\underline{\sum_{k=1}^m A_k^{t+1}}^O(X)$
Upper approximations of OMGRS	$\overline{\sum_{k=1}^m A_k^t}^O(X)$	$\overline{\sum_{k=1}^m A_k^{t+1}}^O(X)$
Lower approximations of PMGRS	$\underline{\sum_{k=1}^m A_k^t}^P(X)$	$\underline{\sum_{k=1}^m A_k^{t+1}}^P(X)$
Upper approximations of PMGRS	$\overline{\sum_{k=1}^m A_k^t}^P(X)$	$\overline{\sum_{k=1}^m A_k^{t+1}}^P(X)$

Lemma 4.1. *([4]) Let $IS^t = (U, AT, V_{AT^t}, f^t)$ and $IS^{t+1} = (U, AT, V_{AT^{t+1}}, f^{t+1})$ be the information systems at times t and $t+1$, respectively. For any $X \subseteq U$, the following results hold.*

$$\underline{\sum_{k=1}^m A_k^t}^O(X) \subseteq \underline{\sum_{k=1}^m A_k^{t+1}}^O(X), \overline{\sum_{k=1}^m A_k^t}^O(X) \supseteq \overline{\sum_{k=1}^m A_k^{t+1}}^O(X);$$
$$\underline{\sum_{k=1}^m A_k^t}^P(X) \subseteq \underline{\sum_{k=1}^m A_k^{t+1}}^P(X), \overline{\sum_{k=1}^m A_k^t}^P(X) \supseteq \overline{\sum_{k=1}^m A_k^{t+1}}^P(X).$$

We can find that Lemma 4.1 represents the relations of lower and upper approximations in MGRS between times t and $t+1$. The lower approximations of MGRS at time t are included in the lower approximations at time $t+1$, and the upper approximations of MGRS at time t include the upper approximation at time $t+1$, which means that the lower approximations of MGRS become larger, and the upper approximations of MGRS become smaller. However, if we use Lemma 4.1 for updating approximations in MGRS directly, the search region is too large. When the attribute values are refined, the searching region will be $BN^P(X) = \overline{\sum_{k=1}^m A_k^t}^P(X) - \underline{\sum_{k=1}^m A_k^t}^P(X)$, which is out of the smallest approximation and is in the biggest approximation. Therefore, we only need to find the lower approximations increasing elements and the upper approximations decreasing elements in $BN^P(X)$. By the following theorem, we can further reduce the search region which makes the algorithm more efficient.

Theorem 4.2. *Let $IS^t = (U, AT, V_{AT^t}, f^t)$ and $IS^{t+1} = (U, AT, V_{AT^{t+1}}, f^{t+1})$ be the information systems at times t and $t+1$, respectively. For any $X \in U$, if $BN^P(X) = \overline{\sum_{k=1}^m A_k^t}^P(X) - \underline{\sum_{k=1}^m A_k^t}^P(X)$, the following results hold.*

$$G^U(\overline{\sum_{k=1}^m A_k^{t+1}}^O(X)) = G^U(\overline{\sum_{k=1}^m A_k^t}^O(X)) \wedge (1 - C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X))))),$$
$$G^U(\overline{\sum_{k=1}^m A_k^{t+1}}^P(X)) = G^U(\overline{\sum_{k=1}^m A_k^t}^P(X)) \wedge (1 - C_m(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X))))),$$

where $1 - G^U(X) = [1 - \chi_X^U(x_1), 1 - \chi_X^U(x_2), \cdots, 1 - \chi_X^U(x_n)]^T$.

Proof. Suppose that $H^{[0,0]}_{A_k^{t+1}}(BN^P(X)) = [h^{1\downarrow}_{A_k}, h^{2\downarrow}_{A_k}, ..., h^{n\downarrow}_{A_k}]$. For any $x_s \in \overline{\sum_{k=1}^m A_k^{t+1}}^O(X)$, by Lemma 4.1, $\overline{\sum_{k=1}^m A_k^{t+1}}^O(X) \subseteq \overline{\sum_{k=1}^m A_k^t}^O(X)$, then $x_s \in \overline{\sum_{k=1}^m A_k^t}^O(X)$. What's more, if $x_s \in G^U_{-1}(C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X)))))$, then $c_1^s(\sum_{k=1}^m h^{s\downarrow}_{A_k}) = \frac{\sum_{t=1}^n m_{st} \times \chi_X^U(x_t)}{\sum_{t=1}^n m_{st}} = 0$, and $[x_s]^{t+1}_{A_k} \cap X = \emptyset$. Therefore, $G^U_{-1}(C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X))))) \subseteq U - \overline{\sum_{k=1}^m A_k^{t+1}}^O(X)$. Thus, we can conclude that $\overline{\sum_{k=1}^m A_k^{t+1}}^O(X) \subseteq U - G^U_{-1}(C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X)))))$. Hence, $\overline{\sum_{k=1}^m A_k^{t+1}}^O(X) \subseteq \overline{\sum_{k=1}^m A_k^t}^O(X) \cap (U - G^U_{-1}(C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X))))))$. If $x_s \in G^U_{-1}(C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X)))))$, for any $s \in \{1, 2, ..., n\}$, $\exists k \in \{1, 2, ..., m\}$, then $c_1^s(\sum_{k=1}^m h^{s\downarrow}_{A_k}) = \frac{\sum_{t=1}^n m_{st} \times \chi_X^U(x_t)}{\sum_{t=1}^n m_{st}} = 0$, $[x_s]^{t+1}_{A_k} \cap X = \emptyset$. In other words, if $x_s \in U - G^U_{-1}(C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X)))))$, then $[x_s]^{t+1}_{A_k} \cap X \neq \emptyset$, for any $k \in \{1, 2, ..., m\}$. Thus, we have that $x_s \in \overline{\sum_{k=1}^m A_k^{t+1}}^O(X)$. Therefore, $U - G^U_{-1}(C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X))))) \subseteq \overline{\sum_{k=1}^m A_k^{t+1}}^O(X)$. This implies that $\overline{\sum_{k=1}^m A_k^t}^O(X) \cap (U - G^U_{-1}(C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X)))))) \subseteq \overline{\sum_{k=1}^m A_k^{t+1}}^O(X)$. Form above, $\overline{\sum_{k=1}^m A_k^{t+1}}^O(X) = \overline{\sum_{k=1}^m A_k^t}^O(X) \cap (U - G^U_{-1}(C_1(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X))))))$. Thus, the result holds. Similarly, we have

$$G^U(\overline{\sum_{k=1}^m A_k^{t+1}}^P(X)) = G^U(\overline{\sum_{k=1}^m A_k^t}^P(X)) \wedge (1 - C_m(\sum_{k=1}^m L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X))))). \square$$

Theorem 4.3. *Let $IS^t = (U, AT, V_{AT^t}, f^t)$ and $IS^{t+1} = (U, AT, V_{AT^{t+1}}, f^{t+1})$ be the information systems at times t and $t+1$, respectively. For any $X \in U$, if $BN^P(X) = \overline{\sum_{k=1}^m A_k^t}^P(X) - \underline{\sum_{k=1}^m A_k^t}^P(X)$, the following results hold.*

$$G^U(\underline{\sum_{k=1}^m A_k^{t+1}}^O(X)) = G^U(\underline{\sum_{k=1}^m A_k^t}^O(X)) \vee C_1(\sum_{k=1}^m L^{-1}(H^{[1,1]}_{A_k^{t+1}}(BN^P(X)))),$$
$$G^U(\underline{\sum_{k=1}^m A_k^{t+1}}^P(X)) = G^U(\underline{\sum_{k=1}^m A_k^t}^P(X)) \vee C_m(\sum_{k=1}^m L^{-1}(H^{[1,1]}_{A_k^{t+1}}(BN^P(X)))).$$

Proof. Suppose that $H^{[1,1]}_{A_k^{t+1}}(BN^P(X)) = [h^{1\downarrow}_{A_k}, h^{2\downarrow}_{A_k}, ..., h^{n\downarrow}_{A_k}]$. For any $s \in \{1, 2, ..., n\}$, it is easy to see that

$$x_s \in \underline{\sum_{k=1}^m A_k^{t+1}}^O(X) \Leftrightarrow [x_s]^{t+1}_{A_k} \subseteq X, [x_s]^t_{A_k} \subseteq X, \text{or } [x_s]^{t+1}_{A_k} \subseteq X, [x_s]^t_{A_k} \nsubseteq X.$$

By Lemma 4.1, $x_s \in \underline{\sum_{k=1}^{m} A_k^t}^O(X) \Leftrightarrow [x_s]_{A_k}^{t+1} \subseteq X$ and $[x_s]_{A_k}^{t} \subseteq X$. From the definition of $BN^P(X)$, we can obtain that $[x_s]_{A_k}^{t} \nsubseteq X \Rightarrow x_s \in BN^P(X)$. If $x_s \in G_{-1}^U(C_1(\sum\limits_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[1,1]}(BN^P(X)))))$, then $c_1^s(\sum\limits_{k=1}^{m} h_{A_k}^{s\downarrow}) = \frac{\sum_{t=1}^{n} m_{st} \times \chi_X^U(x_t)}{\sum_{t=1}^{n} m_{st}} = 1$, and $[x_s]_{A_k}^{t+1} \subseteq X$. Thus, we have

$$x_s \in G_{-1}^U(C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[1,1]}(BN^P(X))))) \Leftrightarrow [x_s]_{A_k}^{t+1} \subseteq X \text{ and } [x_s]_{A_k}^{t} \nsubseteq X.$$

Therefore, we can conclude that

$$\begin{aligned} x_s \in \underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X) &\Leftrightarrow x_s \in \underline{\sum_{k=1}^{m} A_k^{t}}^O(X) \text{ or } x_s \in G_{-1}^U(C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[1,1]}(BN^P(X))))) \\ &\Leftrightarrow x_s \in \underline{\sum_{k=1}^{m} A_k^{t}}^O(X) \cup G_{-1}^U(C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[1,1]}(BN^P(X))))). \end{aligned}$$

Form above, one can see that

$$\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X) = \underline{\sum_{k=1}^{m} A_k^{t}}^O(X) \cup G_{-1}^U(C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[1,1]}(BN^P(X))))).$$

Hence, we have

$$G^U(\underline{\sum_{k=1}^{m} A_k^{t+1}}^O(X)) = G^U(\underline{\sum_{k=1}^{m} A_k^{t}}^O(X)) \vee C_1(\sum_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[1,1]}(BN^P(X)))),$$

Similarly, we can prove that

$$G^U(\underline{\sum_{k=1}^{m} A_k^{t+1}}^P(X)) = G^U(\underline{\sum_{k=1}^{m} A_k^{t}}^P(X)) \vee C_m(\sum_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[1,1]}(BN^P(X)))).\square$$

Table 3. A refined decision information system.

U	a_1	a_2	a_3	d
x_1	2	2	1	1
x_2	1	3	3	2
x_3	4	4	1	1
x_4	1	1	3	2
x_5	2	1	2	1
x_6	3	3	3	1

Example 4.4. *(Continuation of Example 3.5)* The refined information system is show in Table 3. We denote $W_k^P = \bigcup\limits_{x \in BN^P(X)} [x]_{A_k}$ as the relative approximation space with respect to $BN^P(X)$. Then we can obtain that

$$\begin{aligned} &[x_1]_{A_1}^{t+1} = [x_1]_{A_2}^{t+1} = \{x_1\}, [x_3]_{A_1}^{t+1} = [x_3]_{A_2}^{t+1} = \{x_3\}, BN^P(X) = \{x_1, x_2, x_3, x_4\}, \\ &W_1^P = \{x_1, x_2, x_3, x_4\}, W_2^P = \{x_1, x_2, x_3, x_4, x_6\}. \end{aligned}$$

By Definition 3.7, the column matrix $H_{A_1^{t+1}}(BN^P(X))$ and $H_{A_2^{t+1}}(BN^P(X))$ can be calculated as follows.

$$H_{A_1^{t+1}}(BN^P(X)) = D_{A_1} \cdot (M_{A_1} \cdot G^{W_1^P}(X)) = [1, 1/2, 0, 1/2]^T,$$
$$H_{A_2^{t+1}}(BN^P(X)) = D_{A_2} \cdot (M_{A_2} \cdot G^{W_2^P}(X)) = [1, 0, 0, 1, 0]^T.$$

According to Theorems 4.2 and 4.3, we have

$$\begin{aligned} G^U(\overline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X)) &= G^U(\overline{\sum_{k=1}^{2} A_k^{t}}^{O}(X)) \wedge (1 - C_1(\sum_{k=1}^{2} L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X))))). \\ &= [1, 0, 1, 1, 1, 0]^T \wedge (1 - C_1([0, 1, 2, 0, 0, 2]^T)) \\ &= [1, 0, 1, 1, 1, 0]^T \wedge [1, 0, 0, 1, 1, 0]^T \\ &= [1, 0, 0, 1, 1, 0]^T. \end{aligned}$$

Similarly, we have

$$G^U(\overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X)) = G^U(\overline{\sum_{k=1}^{2} A_k^{t}}^{P}(X)) \wedge (1 - C_2(\sum_{k=1}^{2} L^{-1}(H^{[0,0]}_{A_k^{t+1}}(BN^P(X))))) = [1, 1, 0, 1, 1, 0]^T.$$

$$G^U(\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X)) = G^U(\underline{\sum_{k=1}^{2} A_k^{t}}^{O}(X)) \vee C_1(\sum_{k=1}^{2} L^{-1}(H^{[1,1]}_{A_k^{t+1}}(BN^P(X)))) = [1, 0, 0, 1, 1, 0]^T.$$

$$G^U(\underline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X)) = G^U(\underline{\sum_{k=1}^{2} A_k^{t}}^{P}(X)) \vee C_2(\sum_{k=1}^{2} L^{-1}(H^{[1,1]}_{A_k^{t+1}}(BN^P(X)))) = [1, 0, 0, 0, 0, 0]^T.$$

Thus, by Definition 3.1, we can obtain that

$$\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) = \{x_1, x_4, x_5\}, \overline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) = \{x_1, x_4, x_5\},$$
$$\underline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) = \{x_1, x_5\}, \overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) = \{x_1, x_2, x_4, x_5\}.$$

4.2. Relative relation matrix-based approaches for updating approximations while coarsening attribute values

In this section, we present the relative relation matrix-based theorems for dynamic updating approximations in MGRS while coarsening attribute values. For convenience of description, for any $X \subseteq U$, a list of symbols are also shown in Table 2 at times t and $t+1$. For all $V_{A_k^t} \in V_{AT^t}$ $(k \leq m)$, $\exists V_{A_k^{t+1}} \in V_{AT^{t+1}}$ such that $V_{A_k^{t+1}} \subseteq V_{A_k^t}$ for any $k \in \{1, 2, \cdots, m\}$. When the attribute values of x_i with respect to granular structure A_k are coarsened, $[x_i]^{t+1}_{A_k}$ is the equivalence class after coarsening. Namely, $[x_i]^t_{A_k} \cup [x_j]^t_{A_k} = [x_i]^{t+1}_{A_k}$. What's more, according to [4], we have the following results.

Lemma 4.5. *([4]) Let $IS^t = \left(U, AT, V_{AT^t}, f^t\right)$ and $IS^{t+1} = \left(U, AT, V_{AT^{t+1}}, f^{t+1}\right)$ be the information systems at times t and $t+1$, respectively. For any $X \subseteq U$, the following results hold.*

$$\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X) \subseteq \underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X), \overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X) \subseteq \overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X);$$
$$\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X) \subseteq \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X), \overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) \subseteq \overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X).$$

Similar to that of Lemmas 4.1, Lemmas 4.5 shows that the lower approximations of MGRS at time t include the lower approximations at time $t+1$, and the upper approximations of MGRS at time t are included in the upper approximation at time $t+1$. It reveals that the lower approximations of MGRS become smaller, and the upper approximations of MGRS become larger. We find the optimistic boundary region $BN^O(X) = \underline{\sum_{k=1}^{m} A_k^t}^O(X) \cup (U - \overline{\sum_{k=1}^{m} A_k^t}^O(X)$, the searching region will be the union of the biggest lower approximation and the complement set of the smallest upper approximation. Therefore, we only need to find the elements that decrease and increase in the lower and upper approximations, respectively. The following theorems provide a more effective method to update the approximates of MGRS while the attribute values are coarsened.

Theorem 4.6. *Let $IS^t = \left(U, AT, V_{AT^t}, f^t\right)$ and $IS^{t+1} = \left(U, AT, V_{AT^{t+1}}, f^{t+1}\right)$ be the information systems at times t and $t+1$, respectively. For any $X \in U$, if $BN^O(X) = \underline{\sum_{k=1}^{m} A_k^t}^O(X) \cup (U - \overline{\sum_{k=1}^{m} A_k^t}^O(X)$, the following results hold.*

$$G^U(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)) = G^U(\overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)) \vee C_m(\sum_{k=1}^{m} L^{-1}(H^{(0,1]}_{A_k^{t+1}}(BN^O(X)))),$$
$$G^U(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)) = G^U(\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)) \vee C_1(\sum_{k=1}^{m} L^{-1}(H^{(0,1]}_{A_k^{t+1}}(BN^O(X)))).$$

Proof. The proof is similar to that of Theorem 4.3. □

Theorem 4.7. *Let $IS^t = \left(U, AT, V_{AT^t}, f^t\right)$ and $IS^{t+1} = \left(U, AT, V_{AT^{t+1}}, f^{t+1}\right)$ be the information systems at times t and $t+1$, respectively. For any $X \in U$, if $BN^O(X) = \underline{\sum_{k=1}^{m} A_k^t}^O(X) \cup (U - \overline{\sum_{k=1}^{m} A_k^t}^O(X)$, the following results hold.*

$$G^U(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)) = G^U(\underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)) \wedge (1 - C_m(\sum_{k=1}^{m} L^{-1}(H^{[0,1)}_{A_k^{t+1}}(BN^O(X))))),$$
$$G^U(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)) = G^U(\underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)) \wedge (1 - C_1(\sum_{k=1}^{m} L^{-1}(H^{[0,1)}_{A_k^{t+1}}(BN^O(X))))).$$

Proof. The proof is similar to that of Theorem 4.2. □

Example 4.8. *(Continuation of Example 3.5)* The coarsened information system is showed in Table 3. We denote $W_k^O = \bigcup_{x \in BN^O(X)} [x]_{A_k}$ as the relative approximation space with respect to $BN^O(X)$. Then we have

$$[x_5]_{A_1}^{t+1} = [x_6]_{A_1}^{t+1} = \{x_5, x_6\}, [x_2]_{A_2}^{t+1} = [x_5]_{A_2}^{t+1} = [x_6]_{A_2}^{t+1} = \{x_2, x_5, x_6\},$$
$$BN^O(X) = \{x_2, x_4, x_5, x_6\}, W_1^O = W_2^O = \{x_2, x_4, x_5, x_6\}.$$

Table 4. A coarsened decision information system.

U	a_1	a_2	a_3	d
x_1	2	2	1	1
x_2	1	3	3	2
x_3	2	2	1	1
x_4	1	1	3	2
x_5	3	3	3	1
x_6	3	3	3	1

According to Definition 3.7, the column matrix $H_{A_1^{t+1}}(BN^O(X))$ and $H_{A_2^{t+1}}(BN^O(X))$ can be calculated as follows.

$$H_{A_1^{t+1}}(BN^O(X)) = D_{A_1} \cdot (M_{A_1} \cdot G^{W_1^O}(X)) = [1/2, 1/2, 1/2, 1/2]^T ,$$
$$H_{A_2^{t+1}}(BN^O(X)) = D_{A_2} \cdot (M_{A_2} \cdot G^{W_2^O}(X)) = [1/3, 1/3, 1, 1/3]^T .$$

From Theorems 4.6 and 4.7, we have

$$G^U(\overline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X)) = G^U(\overline{\sum_{k=1}^{2} A_k^{t}}^{O}(X)) \vee C_2(\sum_{k=1}^{2} L^{-1}(H^{(0,1]}_{A_k^{t+1}}(BN^O(X)))) = [1, 1, 1, 1, 1, 1]^T ,$$
$$G^U(\overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X)) = G^U(\overline{\sum_{k=1}^{2} A_k^{t}}^{P}(X)) \vee C_1(\sum_{k=1}^{2} L^{-1}(H^{(0,1]}_{A_k^{t+1}}(BN^O(X)))) = [1, 1, 1, 1, 1, 1]^T ,$$
$$G^U(\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X)) = G^U(\underline{\sum_{k=1}^{2} A_k^{t}}^{O}(X)) \wedge (1 - C_2(\sum_{k=1}^{2} L^{-1}(H^{[0,1)}_{A_k^{t+1}}(BN^O(X))))) = [0, 0, 0, 1, 0, 0]^T ,$$
$$G^U(\underline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X)) = G^U(\underline{\sum_{k=1}^{2} A_k^{t}}^{P}(X)) \wedge (1 - C_1(\sum_{k=1}^{2} L^{-1}(H^{[0,1)}_{A_k^{t+1}}(BN^O(X))))) = [0, 0, 0, 0, 0, 0]^T .$$

Thus, according to Definition 3.1, we can obtain that

$$\underline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) = \{x_4\}, \overline{\sum_{k=1}^{2} A_k^{t+1}}^{O}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6\},$$
$$\underline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) = \emptyset, \overline{\sum_{k=1}^{2} A_k^{t+1}}^{P}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6\}.$$

5. Relative matrix-based dynamic algorithms for updating approximations while refining or coarsening attribute values

When the attribute values are refined, the granular structure will change, refining attribute values can be seen as adding attributes. Based on Theorems 4.2 and 4.3, we propose Algorithm 2 for updating approximations in MGRS and compare it with the matrix-based dynamic algorithm in [5].

Algorithm 2 Relative relation matrix-based dynamic algorithm for approximations in MGRS while refining attribute values (RRMDR).

Input: $IS^t = (U, AT, V_{AT^t}, f^t)$, $IS^{t+1} = \big(U, AT, V_{AT^{t+1}}, f^{t+1}\big)$ and $X \subseteq U$.

Output: $\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$,$\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)$,$\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$ and $\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)$.

1: Let $BN^P(X) = \overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X) - \underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)$.
2: **for** each $k \in |AT|$ **do**
3:     Compute W_k^P;
4:     Let $r_k = |W_k^P|$;
5:     **for** each $i, j \in r_k$ **do**
6:         **if** $\chi_{[w_j]_{A_k}}^{W_k^P}(w_i) = 1$ and $\chi_X^{W_k^P}(w_j) = 1$ **then** $m_{ij} = 1$
7:         **end if**
8:     **end for**
9:     Compute the diagonal matrix D_{A_k};
10:     Let $G^{W_k^P}(X) = L(G^U(X))$,$H_{A_k^{t+1}}(BN^P(X)) = D_{A_k} \cdot (M_{A_k} \cdot G^{W_k^P}(X))$.
11: **end for**
12: $G^U(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)) = G^U(\overline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)) \wedge (1 - C_1(\sum\limits_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[0,0]}(BN^P(X)))))$.
13: $G^U(\overline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)) = G^U(\overline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)) \wedge (1 - C_m(\sum\limits_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[0,0]}(BN^P(X)))))$.
14: $G^U(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{O}(X)) = G^U(\underline{\sum_{k=1}^{m} A_k^{t}}^{O}(X)) \vee C_1(\sum\limits_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[1,1]}(BN^P(X))))$.
15: $G^U(\underline{\sum_{k=1}^{m} A_k^{t+1}}^{P}(X)) = G^U(\underline{\sum_{k=1}^{m} A_k^{t}}^{P}(X)) \vee C_m(\sum\limits_{k=1}^{m} L^{-1}(H_{A_k^{t+1}}^{[1,1]}(BN^P(X))))$.

Suppose that $|R(BN^P(X))| = max\{|W_k^P|\}$. Step 3 needs the time complexity $O(|AT||X| \sim X|)$. Steps 5-8 are to compute M_{A_k} with time complexity $O(|AT||R(BN^P(X))|^2)$. The time complexity of Step 10 is $O(|AT||R(BN^P(X))|^2)$. Steps 12-15 can be done with time complexity $O(|AT||U|)$. The total time complexity of Algorithm 2 is $O(|AT||R(BN^P(X))|^2)$.

Similarly, coarsening attribute values can be seen as deleting attributes. Based on Theorems 4.6 and 4.7, we propose Algorithm 3 for updating approximations in MGRS while coarsening attribute values. The total time complexity of Algorithm 3 is $O(|AT||R(BN^O(X))|^2)$, where $|R(BN^O(X))| = max\{|W_k^O|\}$.

The space complexities of Algorithms 2 and 3 are $S(|R(BN^P(X))|^2)$ and $S(|R(BN^O(X))|^2)$, respectively. The space complexities of the static and dynamic algorithms based on the relation matrix are both $S(|U|^2)$ in [5]. Therefore, the space complexity of the algorithm based on the relative relation matrix is no more than that of the algorithm based on the relation matrix.

Algorithm 3 Relative relation matrix-based dynamic algorithm for approximations in MGRS while coarsening attribute values (RRMDC).

Input: $IS^t = (U, AT, V_{AT^t}, f^t)$, $IS^{t+1} = \big(U, AT, V_{AT^{t+1}}, f^{t+1}\big)$ and $X \subseteq U$.

Output: $\underline{\sum_{k=1}^m A_k^{t+1}}^O(X)$, $\overline{\sum_{k=1}^m A_k^{t+1}}^O(X)$, $\underline{\sum_{k=1}^m A_k^{t+1}}^P(X)$ and $\overline{\sum_{k=1}^m A_k^{t+1}}^P(X)$.

Let $BN^O(X) = \underline{\sum_{k=1}^m A_k^t}^O(X) \cup (U - \overline{\sum_{k=1}^m A_k^t}^O(X)$.
for each $k \in |AT|$ **do**
    Compute W_k^O;
    Let $r_k = |W_k^O|$.
    for each $i, j \in r_k$ **do**
        if $\chi^{W_k^O}_{[w_j]_{A_k}}(w_i) = 1$ and $\chi^{W_k^O}_X(w_j) = 1$ **then** $m_{ij} = 1$
        end if
    end for
    Compute the diagonal matrix D_{A_k};
    Let $G^{W_k^O}(X) = L(G^U(X))$, $H_{A_k^{t+1}}(BN^O(X)) = D_{A_k} \cdot (M_{A_k} \cdot G^{W_k^O}(X))$.
end for
$G^U(\overline{\sum_{k=1}^m A_k^{t+1}}^O(X)) = G^U(\overline{\sum_{k=1}^m A_k^t}^O(X)) \vee C_m(\sum_{k=1}^m L^{-1}(H^{(0,1]}_{A_k^{t+1}}(BN^O(X))))$.
$G^U(\overline{\sum_{k=1}^m A_k^{t+1}}^P(X)) = G^U(\overline{\sum_{k=1}^m A_k^t}^P(X)) \vee C_1(\sum_{k=1}^m L^{-1}(H^{(0,1]}_{A_k^{t+1}}(BN^O(X))))$.
$G^U(\underline{\sum_{k=1}^m A_k^{t+1}}^O(X)) = G^U(\underline{\sum_{k=1}^m A_k^t}^O(X)) \wedge (1 - C_m(\sum_{k=1}^m L^{-1}(H^{[0,1)}_{A_k^{t+1}}(BN^O(X)))))$.
$G^U(\underline{\sum_{k=1}^m A_k^{t+1}}^P(X)) = G^U(\underline{\sum_{k=1}^m A_k^t}^P(X)) \wedge (1 - C_1(\sum_{k=1}^m L^{-1}(H^{[0,1)}_{A_k^{t+1}}(BN^O(X)))))$.

6. Experimental results

In this section, we conduct several experiments to show the validity of the proposed algorithms. We compare the performance of the relative relation matrix-based static algorithm (RRMS), the relative relation matrix-based dynamic algorithms (RRMDR and RRMDC), the matrix-based static algorithm (MBS) and the matrix-based dynamic algorithms (MBDA and MBDD) [5]. In order to verify the efficiency of RRMS, RRMDR and RRMDC, eight data sets are chosen from UCI. They are Las Vegas Trip Advisor Reviews Data (LVTAR), BS, Whole Sale Customers Data (WSC), Blood Transfusion (BT), Facebook Metrics (FM), Student Mat (SM), Solar Flare (SF) and German Credit Data (GCD). The details of data sets are described in Table 5. All the experiments are carried out on a personal computer with 64-bit windows 10, AMD A10-7300 Radeon R6, 10 Compute Cores 4C+6G, and 8GB memory. The program lauguage is Matlab R2015b.

Table 5. Data sets used in the experiments.

No.	Data sets	Samples	Attributes
1	LVTAR	504	20
2	BS	625	4
3	WSC	440	8
4	BT	748	5
5	FM	500	19
6	SM	396	33
7	SF	1398	13
8	GCD	1000	21

6.1. The comparison of static and dynamic updating algorithms with different size of data sets

When the size of data set increases gradually, computational times are compared among RRMS, RRMDR, MBS and MBDA in MGRS while refining attribute values, and computational times are compared among RRMS, RRMDC, MBS and MBDD in MGRS while coarsening attribute values. Firstly, when refining the attribute values, one attribute value is randomly added into an attribute set A_k. When coarsening attribute values, one attribute value is randomly deleted from an attribute set A_k. Secondly, every data set U is randomly divided into ten subsets, which is denoted by $\{U_1, U_2, \cdots, U_{10}\}$. Then, selecting U_1 as the first temporary data set, the combination of the first part and the second part is regarded as the second temporary data set. Namely, $U_1 \cup U_2$ is the second temporary data set, and so on. The composition of the target concept X is randomly selected from each temporary data set, and its size is approximately 0.95 times the temporary data set. We calculate the lower and upper approximations in MGRS by RRMS, RRMDR, RRMDC, MBS, MBDA and MBDD ten times and compare these averages.

Experimental results of RRMS, RRMDR (RRMDC), MBS and MBDA (MBDD) while refining and coarsening attribute values in MGRS are shown in Figs.1 and 2, respectively. The x-coordinate represents the size of data sets, while the y-coordinate represents the computational time. In each figure, by comparing the computational times of the static and dynamic updating algorithms, in general, we can see that RRMS and RRMDR (or RRMDC) have a better performance on MBS and MBDA (or MBDD), respectively. In each sub-figure of Fig.1, RRMDR is faster than RRMS on updating approximations in most situation, especially when the data set increases to a large size, the computational time difference between RRMS and RRMDR based on the relative relation matrix approach becomes more obvious. What's more, we have that the computational time of RRMDR is less than or equal to any other algorithms in most situation. In Fig.2, the computational times of RRMS and RRMDC stay close, in general, these results show the superior computational efficiency of the RRMS and RRMDC over the MBS and MBDD, respectively.

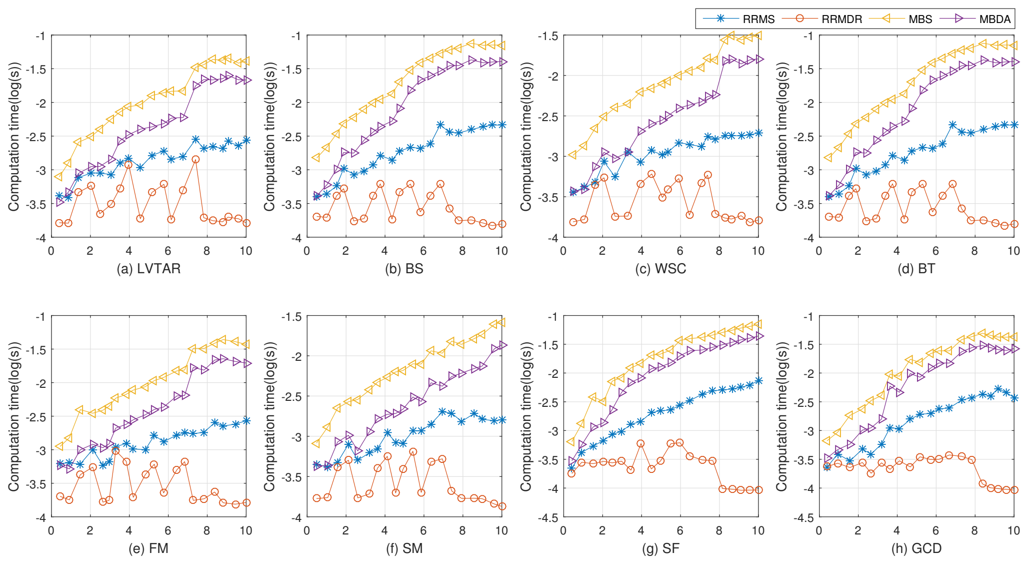

Fig 1. Computational times of RRMS, RRMDR, MBS and MBDA when the size of U increases gradually

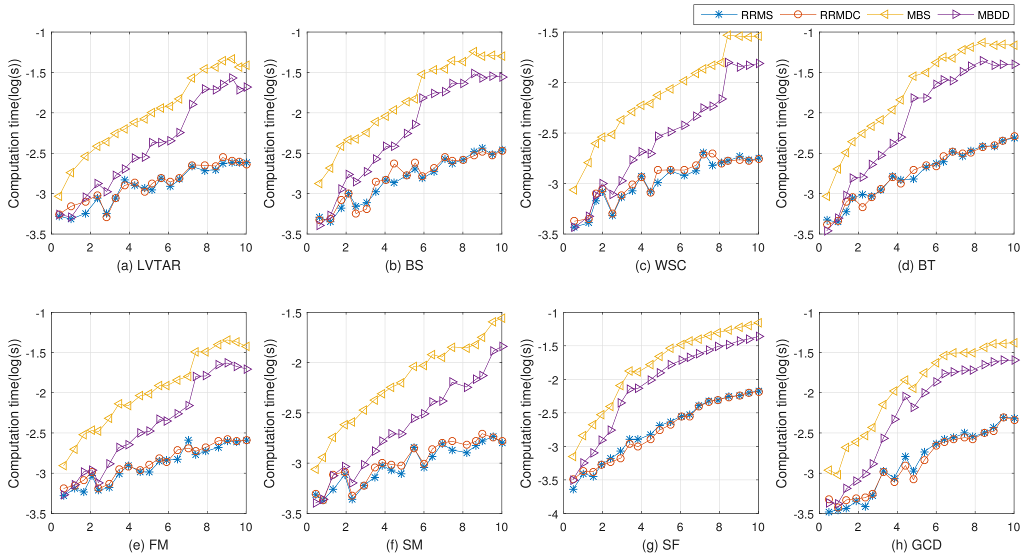

Fig 2. Computational times of RRMS, RRMDC, MBS and MBDD when the size of U increases gradually

6.2. The comparison of static and dynamic updating algorithms with different size of target concept X

When the size of target concepts X increases gradually, the computational times are compared among RRMS, RRMDR (RRMDC), MBS and MBDA (MBDD) in MGRS. The process of constructing temporary target concepts X is similar to that of subsection 6.1. When the size of target concept X gradually increases, we choose U_1 as the first temporary target concept, $U_1 \cup U_2$ is the second temporary target concept, and so on. Experimental results are shown in Figs.3 and 4, which show more detailed change trend lines of RRMS, RRMDR (RRMDC), MBS and MBDA (MBDD) with the increasing size of target concepts while refining or coarsening attribute values, the x-coordinate represents the size of target concepts, while the y-coordinate represents the computational time. We can find that the computational times of RRMS and RRMDR (RRMDC)) are more efficient in comparison with MBS and MBDA (MBDD), respectively. In Fig.3, the computational time of RRMDR is the lowest among the four algorithms. In Fig.4, the computational times of RRMS and RRMDC stay close, but the performance of the relative relation matrix-based algorithms is better than that of the matrix-based algorithms.

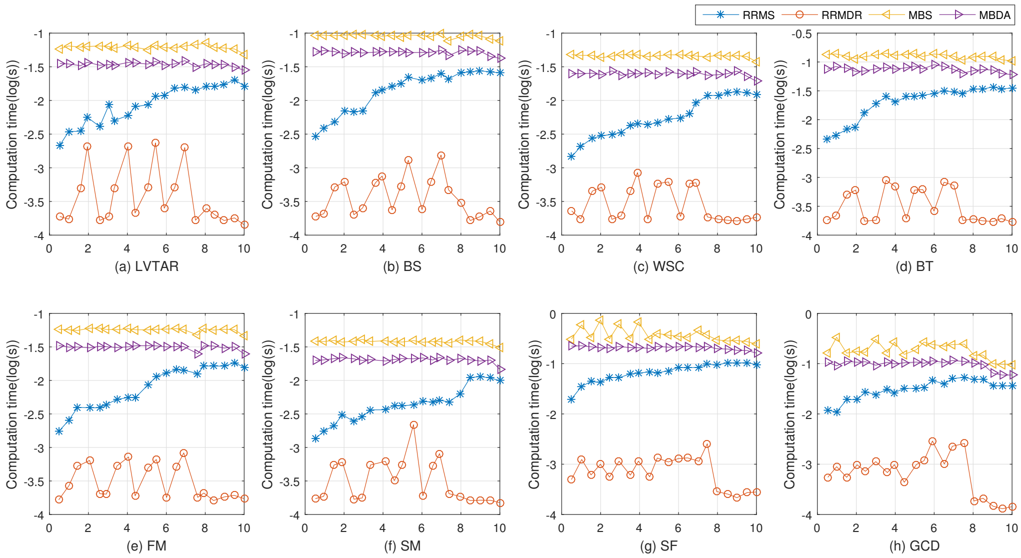

Fig 3. Computational times of RRMS, RRMDR, MBS and MBDA when the size of X increases gradually

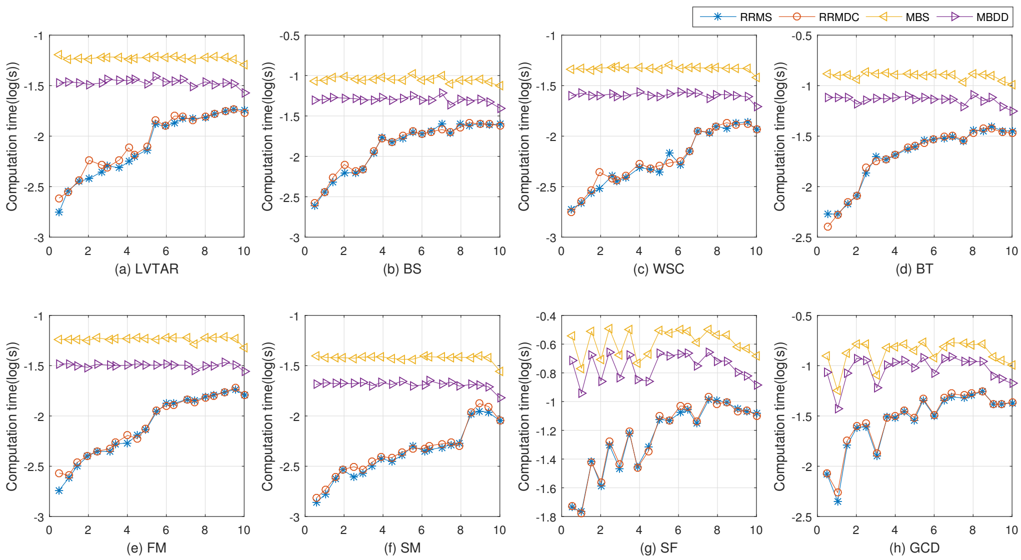

Fig 4. Computational times of RRMS, RRMDC, MBS and MBDD when the size of X increases gradually

7. Conclusion

In this paper, we discussed the problem of updating approximations in MGRS while refining or coarsening attribute values. Relative relation matrix-based approaches for updating approximations were proposed. The results indicated that the storage and time complexities of the relative relation matrix are less than that of the relation matrix. Furthermore, we designed two types of dynamic algorithms RRMDR and RRMDC. The experimental results demonstrated that the computational times of RRMDR (RRMDC) are no more than that of RRMS, MBS, and MBDA (MBDD) in most situations. In our future studies, we plan to investigate updating approximations with the variation of objects based on the proposed approaches in this paper.

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