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Approximate Controllability of Impulsive System Involving State-Dependent Delay and Variable Delay in Control via Fundamental Solution

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Abstract. This article is concerned with the approximate controllability for a new class of impulsive semilinear control systems involving state-dependent delay and variable delay in control in Hilbert spaces. We formulate new sufficient conditions which guarantee the existence of solution to the considered system. We use the theory of fundamental solution, Krasnoselskii's and Schauder's fixed point theorems to establish our major results. Finally, two examples are constructed which demonstrate the effectiveness of obtained results.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{Z}$ be Hilbert spaces, and $\mathscr{L}(\mathscr{Z};\mathscr{H})$ the space of all bounded linear operators from $\mathscr{Z}$ into $\mathscr{H}$. Consider the following semilinear functional differential equation involving state-dependent delay and variable delay in control given by

$$\begin{cases} \dfrac{d\xi(t)}{dt} = A\xi(t) + L(\xi_t) + B_1(t)z(t) + B_2(t)z(h(t)) + F(t,\xi_{\rho(t,\xi_t)}),\ t \in (0,\mathcal{M}],\ t \neq t_i, \\ \xi_0 = \psi \in \mathscr{D}, \\ \Delta\xi(t_k) = I_k(\xi_{t_k}),\ \ k = 1,\cdots,m, \end{cases} \tag{1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_m < \mathcal{M}$ are fixed, the state variable $\xi(\cdot) \in \mathscr{H}$, $z(\cdot) \in L^2([0,\mathcal{M}];\mathscr{Z})$ is the control variable, and $B_1, B_2 \in \mathscr{L}(\mathscr{Z};\mathscr{H})$. Define $h(t) = t - h_1(t)$ ($h_1(t)$ is positive) is strictly increasing and continuously differentiable on $[0,\mathcal{M}]$. Let $\mathscr{D}$ be an abstract phase space, which is defined later. Then $\xi_t\colon (-\infty,0] \to \mathscr{H}$, given by $\xi_t(\kappa) = \xi(t+\kappa)$ for $\kappa \leq 0$, belongs to $\mathscr{D}$. We assume that $L \in \mathscr{L}(\mathscr{D};\mathscr{H})$ and A is a closed linear operator (not necessarily bounded) from $\mathscr{H}$ into itself that generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$. The function $F : [0,\mathcal{M}] \times \mathscr{D} \to \mathscr{H}$ is specified later, and $\rho : [0,\mathcal{M}] \times \mathscr{D} \to [0,\infty)$ is a continuous function. For convenience, choose $r(t)$ such that $r(h(t)) = h(r(t)) = t$.

In various fields of engineering input or output delays emerge naturally in different modeling and dynamical control systems. However, it is important to achieve the satisfactory control systems for the

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modeling of framework involving variable delays because variable delays represent more positive and powerful characteristics and behavior as compared to fixed time delays. At different point of applications, the existence of delay in variable in a flexible spacecraft is very common because of actuators physical design and energy consumptions. Therefore, to make the prediction regarding valuable system dynamics, it is required that system must involve these variable delays. Significant worsening of performance and instability of the system generally leads to the presence and requirement of variable delays in dynamical systems. In a delayed system both the present and past states directly or indirectly effects the future state of the system. We generally deal with dynamical control systems involving variable delay in control whenever there is any delay in input function. It is remarkable that several mathematical models which showcase dynamical systems involving delay in control are of special significance in control theory. Thus, it is important to discuss controllability for delayed semilinear systems involving variable delay in control. In particular, we can easily found different equations involving state-dependent delay in different practical control models. Indeed, it is natural to involve state-dependent delay in system dynamics as apparent large number of models representing real world problem may need the past states of the system for effective output. Delayed differential equations emerge in various biological and physical applications, because of this authors generally attract towards the consideration of variable or state-dependent delay. Moreover, it comes out that in various problems system dependency on time delay is only an additional assumption for making the study easier.

Controllability played a pivotal role in every part of the history of modern control theory. Systematically the study of controllability was initiated at the starting of 60's last century. After that, several controllability results were accomplished extensively in finite and infinite dimensional spaces using various approaches [6, 14, 22, 31]. Some basic concepts of control theory are introduced by Barnett [5] and Curtain *et al.* [6]. Mokkedem *et al.* [18, 19] discussed the approximate controllability of dynamical control systems by using the technique of fundamental solution. Controllability of various systems involving delay in control has broadly discussed by few authors. Klamka [12, 13] discussed controllability of linear system involving delays in control. Sikora and Klamka [29, 30] developed some interesting results by assuming constrained controls (that is the control functions are restricted to take their values in a prescribed admissible set) for linear and semilinear fractional systems with multiple delays in control in finite dimensional spaces. Balachandran [3] concerned with relative controllability of dynamical control system involving delay in control. Shen and Sun [28] studied relative controllability of nonlinear system involving variable delays in control. Kumar and Sukavanam [15] discussed controllability of semilinear systems involving fixed delay in control. Shen [27] proved relative approximate controllability of semilinear functional systems involving infinite delay and variable delay in control. Arora and Sukavanam [2] established approximate controllability for a semilinear differential equation of second order involving variable delay in control.

On the other hand, the theory of impulsive differential equations attracted many researchers because of its presence in several fields such as, in pharmacokinetics, population dynamics [4, 32], mathematical in epidemiology [8], fed-batch culture in fermentative production [9], among others. However, if we compare the development of control theory for ordinary differential equation and impulsive differential equation, then the second one is not yet adequately studied in relation with the first one. Theory of impulsive differential equations involving state-dependent delay has discussed by several authors. Muthukumar and Rajivganthi [21] determined the approximate controllability of stochastic neutral semilinear system involving state-dependent delay and impulse. Sakthivel and Anandhi [25] established sufficient conditions for the approximate controllability to a semilinear differential equation involving state-dependent delay for the impulsive process. Selvarasu *et al.* [26] obtained a set of sufficient conditions for the approximate controllability of impulsive fractional semilinear system involving state-dependent delay and poisson jump. Zhang *et al.* [33] investigated the approximate controllability of fractional stochastic semilinear system involving state-dependent delay and impulse.

Motivated by the above cited work and discussion, the foremost purpose of this article is to construct new sufficient conditions for the approximate controllability to the system (1). For this we formulate an appropriate control function associated to the system (1). By using this control function, fundamental solution, Krasnoselskii's and Schauder's fixed point theorems, we show that the system (1) has a mild solution. Finally, approximate controllability is reported for the system (1) under the assumption that the

linear system ($F \equiv 0$ in (1)) is approximately controllable. Nevertheless, we would point out here, that to the best of our knowledge the approximate controllability of semilinear systems involving state-dependent delay and variable delay in control for the impulsive case is not considered in the literature yet. In this article, we try to fill this gap which is the novelty of our work.

The set up of the remaining paper is as follows: We introduce the abstract phase space $\mathscr{D}$, basic notations, definitions and results in Sect. 2. We establish the existence of a mild solution to the system (1) by using the resolvent operator and fixed point technique in Sect. 3. We also show approximate controllability result for impulsive semilinear differential equation having state-dependent delay and variable delay in control. In Sect. 4, the obtained results are demonstrated with the help of two examples.

2. Preliminaries

Initially, we introduce the phase space $\mathscr{D}$. Next, we evolve the concept of fundamental solution and provide some results associated with it. The section is closed by raising an expression for a mild solution to the system (1) followed by resolvent operators. Let $\mathscr{H}$ endowed with the norm $\|\cdot\|$. For more details related to semigroup of operators, we suggest our readers to see [7, 24].

Define

$$PC = \{\xi : [0, \mathcal{M}] \to \mathscr{H} : \xi \text{ is continuous at } t \neq t_k,\ \xi(t_k^-) = \xi(t_k) \text{ and } \xi(t_k^+) \text{ exists for } k = 1, \cdots, m\}.$$

Clearly, $(PC; \|\cdot\|_{PC})$ is a Banach space with the norm $\|\xi\|_{PC} = \sup_{s\in[0,\mathcal{M}]} \|\xi(s)\|$. Initially, Hale and Kato [10] gave an axiomatic definition of phase space $\mathscr{D}$. Let $\mathscr{D}$ be the collection of all mapping from $(-\infty, 0]$ into $\mathscr{H}$ with the seminorm $\|\cdot\|_{\mathscr{D}}$ (see [11]), and the following axioms holds in $\mathscr{D}$:

(A) If $\xi : (-\infty, \omega + d] \to \mathscr{H}$ ($\omega \geq 0$ and $d > 0$) is such that $\xi_\omega \in \mathscr{D}$ and $\xi|_{[\omega,\omega+d]} \in PC([\omega, \omega + d]; \mathscr{H})$, then for any $t \in [\omega, \omega + d]$, we have

  (i) $\xi_t \in \mathscr{D}$

  (ii) $\|\xi(t)\| \leq \widetilde{\mathcal{B}}\|\xi_t\|_{\mathscr{D}}$ where $\widetilde{\mathcal{B}} \geq 0$ is a constant which is independent of $\xi(\cdot)$

  (iii) $\|\xi_t\|_{\mathscr{D}} \leq K(t - \omega) \sup\{\|\xi(s)\| : \omega \leq s \leq t\} + R(t - \omega)\|\xi_\omega\|_{\mathscr{D}}$, where K, R maps $[0, \infty)$ into itself. Also $K(\cdot)$ is continuous and $R(\cdot)$ is locally bounded, and both $K(\cdot)$ and $R(\cdot)$ are independent of $\xi(\cdot)$.

(B) The phase space $\mathscr{D}$ is complete.

To setup an expression for fundamental solution, we require following assumptions:

(a) Define the function ψ_η^0 by

$$\psi_\eta^0(\kappa) = \begin{cases} \eta, & \kappa = 0, \\ 0, & \kappa < 0, \end{cases} \tag{2}$$

which belongs to $\mathscr{D}$ for any $\eta \in \mathscr{H}$, and $\|\psi_\eta^0\|_{\mathscr{D}} \leq \|\eta\|$.

(b) The maps K and R appeared in axiom (A) are bounded on $[0, \infty)$. Let $K_{\mathcal{M}}$ and $R_{\mathcal{M}}$ be constants such that

$$K_{\mathcal{M}} = \max_{s\in[0,\mathcal{M}]} K(s) \text{ and } R_{\mathcal{M}} = \sup_{s\in[0,\mathcal{M}]} R(s).$$

Consider the system

$$\begin{cases} \dfrac{d\xi(t)}{dt} = A\xi(t) + L(\xi_t),\ t > 0 \\ \xi_0 = \psi \in \mathscr{D}, \end{cases} \tag{3}$$

then

$$\xi(t,\psi) = \begin{cases} S(t)\psi(0) + \int_0^t S(t-s)L(\xi_s(\cdot,\psi))ds, & t \geq 0, \\ \psi(t), & t \leq 0, \end{cases}$$

is the mild solution of (3).
The fundamental solution $Q(t) \in \mathscr{L}(\mathscr{H})$ of equation (3) is an operator valued function defined by

$$Q(t) = \begin{cases} S(t) + \int_0^t S(t-s)L(Q_s)ds, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where $Q_t(\kappa) := Q(t+\kappa)$, $\kappa \leq 0$ (see [18]). Clearly, $Q(t)$ is the unique solution of (3). For any $\eta \in \mathscr{H}$, we have

$$Q(t)\eta = \begin{cases} \xi(t,\psi_\eta^0), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where ψ_η^0 is defined by (2) and belongs to $\mathscr{D}$ due to assumption (a).

Definition 2.1. *For given $z(\cdot) \in L^2([0,\mathcal{M}];\mathscr{Z})$, $\xi(\cdot,\psi,z) : (-\infty,\mathcal{M}] \to \mathscr{H}$ is reffered to as a mild solution to the system* (1) *with initial data $\psi \in \mathscr{D}$, if it is in PC and the following intergal equation holds:*

$$\xi(t) = \begin{cases} Q(t)\psi(0) + \displaystyle\int_0^t Q(t-s)[L(\tilde{\psi}_s) + B_1(s)z(s) + B_2(s)z(h(s)) + F(s,\xi_{\rho(s,\xi_s)})]ds \\ + \displaystyle\sum_{0<t_k<t} Q(t-t_k)I_k(\xi_{t_k}),\ t \in [0,\mathcal{M}], \\ \psi(t),\ -\infty < t \leq 0, \end{cases}$$

where

$$\tilde{\psi}(s) = \begin{cases} \psi(s), & s \leq 0, \\ 0, & s > 0. \end{cases}$$

Remark 2.2. *Throughout the remaining paper we assume that $h(\mathcal{M}) > 0$, as the system* (1) *has no delay in control whenever $h(\mathcal{M})$ is negative. Also $z(s) = 0$, $s \in [h(0), 0]$.*

By keeping in mind that there is a delay in control, we redefine the mild solution as follows:

Definition 2.3. *For given $z(\cdot) \in L^2([0,\mathcal{M}];\mathscr{Z})$, $\xi(\cdot,\psi,z) : (-\infty,\mathcal{M}] \to \mathscr{H}$ is referred to as a mild solution to the system* (1) *with initial data $\psi \in \mathscr{D}$, if it is in PC and the following integral equation is satisfied:*

$$\xi(t) = \begin{cases} Q(t)\psi(0) + \displaystyle\int_0^{h(t)} [Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)]z(s)ds + \int_{h(t)}^t Q(t-s)B_1(s)z(s)ds \\ + \displaystyle\int_0^t Q(t-s)[L(\tilde{\psi}_s) + F(s,\xi_{\rho(s,\xi_s)})]ds + \sum_{0<t_k<t} Q(t-t_k)I_k(\xi_{t_k}),\ t \in [0,\mathcal{M}], \\ \psi(t),\ -\infty < t \leq 0. \end{cases}$$

Definition 2.4. *We say that the system* (1) *is approximately controllable on $[0,\mathcal{M}]$, if for any initial data $\psi \in \mathscr{D}$, $\mathscr{R}(\mathcal{M},\psi)$ is dense in the Hilbert space $\mathscr{H}$. That is,*

$$\overline{\mathscr{R}(\mathcal{M},\psi)} = \mathscr{H},$$

where $\mathscr{R}(\mathcal{M},\psi) = \{\xi(\mathcal{M},\psi,z) :\ z(\cdot) \in L^2([0,\mathcal{M}];\mathscr{Z})\}$.

Let Q^*, B_1^* and B_2^* be the adjoint of Q, B_1 and B_2, respectively. Then introduce the following operators: The controllability linear map $L_{\mathcal{M}} : L^2([0, \mathcal{M}], \mathscr{Z}) \to \mathscr{H}$ of the system (1) is defined by

$$L_{\mathcal{M}}z = \int_0^{h(\mathcal{M})} (Q(\mathcal{M} - \tau)B_1(\tau) + Q(\mathcal{M} - r(\tau))B_2(r(\tau))r'(\tau))z(\tau)d\tau + \int_{h(\mathcal{M})}^{\mathcal{M}} Q(\mathcal{M} - \tau)B_1(\tau)z(\tau)d\tau,$$

and

$$(L_{\mathcal{M}}^*\eta)(\tau) = \begin{cases} (B_1^*(\tau)Q^*(\mathcal{M} - \tau) + B_2^*(r(\tau))Q^*(\mathcal{M} - r(\tau))r'(\tau))\eta, \ \tau \in [0, h(\mathcal{M})], \\ B_1^*(\tau)Q^*(\mathcal{M} - \tau)\eta, \ \tau \in (h(\mathcal{M}), \mathcal{M}]. \end{cases}$$

The controllability Gramian map on the interval $[s, \mathcal{M}]$ to the system (1) is defined by

$$\Gamma_s^{\mathcal{M}} = L_{\mathcal{M}}L_{\mathcal{M}}^* = \begin{cases} \int_s^{h(\mathcal{M})} (r'(\tau)Q(\mathcal{M} - r(\tau))B_2(r(\tau))B_2^*(r(\tau))Q^*(\mathcal{M} - r(\tau))r'(\tau) \\ +Q(\mathcal{M} - \tau)B_1(\tau)B_1^*(\tau)Q^*(\mathcal{M} - \tau))d\tau \\ + \int_{h(\mathcal{M})}^{\mathcal{M}} Q(\mathcal{M} - \tau)B_1(\tau)B_1^*(\tau)Q^*(\mathcal{M} - \tau)d\tau, \ s \in [0, h(\mathcal{M})]; \\ \int_s^{\mathcal{M}} Q(\mathcal{M} - \tau)B_1(\tau)B_1^*(\tau)Q^*(\mathcal{M} - \tau)d\tau, \ s \in (h(\mathcal{M}), \mathcal{M}]. \end{cases}$$

For $\alpha > 0$ and $s \in [0, \mathcal{M}]$, the resolvent map $R(\alpha, \Gamma_s^{\mathcal{M}})$ is defined by $R(\alpha, \Gamma_s^{\mathcal{M}}) = (\alpha I + \Gamma_s^{\mathcal{M}})^{-1}$.

3. Approximate Controllability

This section is devoted to the approximate controllability of the impulsive system (1) containing state-dependent delay and delay in control. In what follows, we assume that $0 \le \rho(t, \psi) \le t$ for all $\psi \in \mathscr{D}$. To show the solvability and the approximate controllability for the system (1), the following hypotheses are required:

(*G*1) There exist constants $\theta \in \mathbb{R}$ and $P_\theta \ge 1$ such that for all $s \ge 0$, $\|S(s)\| \le P_\theta e^{\theta s}$. Particularly, for every $0 \le s \le \mathcal{M}$,

$$\|S(s)\| \le P, \text{ for some } P \ge 1.$$

(*G*2) There exists $l > 0$ such that $\|L\| = l$.

(*G*3) For each $t \in [0, \mathcal{M}]$, the function $F(t, \cdot) : \mathscr{D} \to \mathscr{H}$ is continuous and for each $\psi \in \mathscr{D}$, the function $F(\cdot, \psi) : [0, \mathcal{M}] \to \mathscr{H}$ is strongly measurable, and satisfies the following conditions:

(a) There exist positive constants L_F and $\tilde{L}_F$ such that

$$\|F(t, \xi) - F(t, \eta)\| \le L_F\|\xi - \eta\|_{\mathscr{D}}, \ \ \|F(t, \xi)\| \le \tilde{L}_F(1 + \|\xi\|_{\mathscr{D}}).$$

(b) There exists $L > 0$ such that for $t_1, t_2 \in [0, \mathcal{M}]$ we have

$$\|F(t, \xi_{t_1}) - F(t, \xi_{t_2})\| \le L|t_1 - t_2|.$$

(c) The function $\rho : [0, \mathcal{M}] \times \mathscr{D} \to [0, \infty)$ is such that the function $t \to \rho(t, \xi)$ is continuous for every $\xi \in \mathscr{D}$, and there exists a constant $L_\rho > 0$ such that

$$|\rho(t, \xi) - \rho(t, \eta)| \le L_\rho\|\xi - \eta\|, \ \xi, \eta \in \mathscr{D} \text{ and for all } t \in [0, \mathcal{M}].$$

(G4) The functions $I_k : \mathscr{D} \to \mathscr{H}$ are continuous, and there exist constants q_k and $\tilde{q}_k$ for $k = 1, 2, \cdots, m$, such that

$$\|I_k(\xi) - I_k(\eta)\| \leq q_k \|\xi - \eta\|_{\mathscr{D}}, \ \ \|I_k(\xi)\| \leq \tilde{q}_k(1 + \|\xi\|_{\mathscr{D}}).$$

(G5) The operator $\alpha R(\alpha, \Gamma_0^{\mathcal{M}})$ tends to 0 as $\alpha \to 0^+$ in the strong operator topology.

Remark 3.1. *[18, Theorem 3.2]. For $Q(t)$, $t \in \mathbb{R}$, we have the following*

(i) The family of bounded linear operators $Q(t)$ is strongly continuous which is defined on $\mathscr{H}$ and there exist contants $\gamma \in \mathbb{R}$ and $c_1 > 0$ such that

$$\|Q(s)\| \leq c_1 e^{\gamma s}, \ s \geq 0.$$

Particularly, for every $0 \leq s \leq \mathcal{M}$, we have

$$\|Q(s)\| \leq \widetilde{P}, \text{ for some } \widetilde{P} \geq 1.$$

(ii) If $\{S(s)\}_{s>0}$ is compact, then $\{Q(s)\}_{s>0}$ is compact.

(iii) For each $0 < s \leq \mathcal{M}$, $Q(s)$ is uniformly continuous.

Lemma 3.2. *The following are equivalent:*

(a) $\Gamma_0^{\mathcal{M}} > 0$.

(b) Assumption (G5) holds.

(c) The System

$$\begin{cases} \frac{d\xi(t)}{dt} = A\xi(t) + L(\tilde{\psi}_t) + B_1(t)z(t) + B_2(t)z(h(t)), \ t \in [0, \mathcal{M}] \\ \xi(0) = \psi(0), \end{cases} \tag{4}$$

is approximately controllable.

Proof. The proof is straightforward, for more details see [27, Lemma 2.1]. □

For further development, we construct an expression for the control function. That is, for any $\xi_{\mathcal{M}} \in \mathscr{H}$ and $\alpha > 0$, the control function is defined by

$$z^{\alpha}(t) = \begin{cases} [r'(t)B_2^*(r(t))Q^*(\mathcal{M} - r(t)) + B_1^*(t)Q^*(\mathcal{M} - t)]R(\alpha, \Gamma_0^{\mathcal{M}})[\xi_{\mathcal{M}} - Q(\mathcal{M})\psi(0)] \\ -[r'(t)B_2^*(r(t))Q^*(\mathcal{M} - r(t)) + B_1^*(t)Q^*(\mathcal{M} - t)]\Big\{ \int_0^t R(\alpha, \Gamma_s^{\mathcal{M}})Q(\mathcal{M} - s)[L(\tilde{\psi}_s) + F(s, \xi_{\rho(s,\xi_s)})]ds \\ +R(\alpha, \Gamma_0^{\mathcal{M}}) \sum_{k=1}^{m} Q(\mathcal{M} - t_k)I_k(\xi_{t_k})\Big\}, \ t \in [0, h(\mathcal{M})], \\ B_1^*(t)Q^*(\mathcal{M} - t)R(\alpha, \Gamma_0^{\mathcal{M}})(\xi_{\mathcal{M}} - Q(\mathcal{M})\psi(0)) - B_1^*(t)Q^*(\mathcal{M} - t)\Big\{ \int_0^t R(\alpha, \Gamma_s^{\mathcal{M}})Q(\mathcal{M} - s)[L(\tilde{\psi}_s) \\ +F(s, \xi_{\rho(s,\xi_s)})]ds + R(\alpha, \Gamma_0^{\mathcal{M}}) \sum_{k=1}^{m} Q(\mathcal{M} - t_k)I_k(\xi_{t_k})\Big\}, \ \ t \in [h(\mathcal{M}), \mathcal{M}]. \end{cases} \tag{5}$$

Also introduce the operator

$$(\Phi^{\alpha}\xi)(t) = \begin{cases} Q(t)\psi(0) + \int_0^{h(t)} [Q(t - s)B_1(s) + Q(t - r(s))B_2(r(s))(r'(s))]z^{\alpha}(s)ds \\ + \int_{h(t)}^t Q(t - s)B_1(s)z^{\alpha}(s)ds + \int_0^t Q(t - s)[L(\tilde{\psi}_s) + F(s, \xi_{\rho(s,\xi_s)})]ds \\ + \sum_{0<t_k<t} Q(t - t_k)I_k(\xi_{t_k}), \ \ t \in [0, \mathcal{M}], \\ \psi(t), \ -\infty < t \leq 0. \end{cases} \tag{6}$$

For the sack of convenience, let $N = \max\{\|B_1(\tau)\|, \|B_2(\tau)\| : \tau \in [0, \mathcal{M}]\}$ and $r = \max\{\|r'(\tau)\| : \tau \in [0, \mathcal{M}]\}$.

Theorem 3.3. *Suppose that $\psi \in \mathscr{D}$. If (G1) − (G4) hold, then for $\alpha > 0$, there is a mild solution to the system (1) on $(-\infty, \mathcal{M}]$ provided that*

$$\frac{1}{\alpha}\mathcal{M}[\widetilde{P}N + \widetilde{P}Nr]^2\Big\{\mathcal{M}\widetilde{P}\widetilde{L}_F K_{\mathcal{M}} + \mathcal{M}\widetilde{P}LL_\rho K_{\mathcal{M}} + \widetilde{P}\sum_{k=1}^{m}\tilde{q}_k K_{\mathcal{M}}\Big\} + \mathcal{M}\widetilde{P}\widetilde{L}_F K_{\mathcal{M}} + \sum_{k=1}^{m}\widetilde{P}\tilde{q}_k K_{\mathcal{M}} \leq 1. \tag{7}$$

Proof. Define the map $\eta(\cdot) : (-\infty, \mathcal{M}] \to \mathscr{H}$ by

$$\eta(t) = \begin{cases} Q(t)\psi(0), & t \geq 0, \\ \psi(t), & -\infty < t \leq 0. \end{cases}$$

Clearly for any $0 \leq t \leq \mathcal{M}$, $\eta_t \in \mathscr{D}$ with $\eta_0 = \psi$ and $t \to \eta_t$ is a continuous map on $[0, \mathcal{M}]$, which leads to the continuity of the map $t \to \eta_t$ in $\|\cdot\|_{\mathscr{D}}$.

Consider the set $C^0_{\mathcal{M}} = \{u \in PC : u_0 = 0\}$ with the norm

$$\|u\|_{\mathcal{M}} = \sup\{\|u(s)\| : 0 \leq s \leq \mathcal{M}\}.$$

For $\delta > 0$, set $B[0;\delta] = \Big\{u \in C^0_{\mathcal{M}} : \|u\|_{\mathcal{M}} \leq \delta\Big\}$. Clearly, $B[0;\delta]$ is a non-empty, bounded, closed and convex subset of PC. For each $u \in B[0;\delta]$, define

$$\bar{u}(\tau) = \begin{cases} u(\tau), & 0 \leq \tau \leq \mathcal{M}, \\ 0, & -\infty < \tau < 0. \end{cases}$$

If $\xi(\cdot)$ satisfies (1), then split it as $\xi(t) = u(t) + \eta(t)$, $t \in [0, \mathcal{M}]$, which yields that $\xi_t = \bar{u}_t + \eta_t$ for $t \in [0, \mathcal{M}]$, and for each $z^\alpha \in L^2([0, \mathcal{M}]; \mathscr{Z})$ the function $u(\cdot)$ has the form

$$\begin{aligned} u(t) = &\int_0^{h(t)} [Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)]z^\alpha(s)ds + \int_{h(t)}^{t} Q(t-s)B_1(s)z^\alpha(s)ds \\ &+ \int_0^t Q(t-s)[L(\tilde{\psi}_s) + F(s, \bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)})]ds + \sum_{0<t_k<t} Q(t-t_k)I_k(\bar{u}_{t_k} + \eta_{t_k}), \quad t \in [0, \mathcal{M}]. \end{aligned}$$

Define Φ on $B[0;\delta]$ by

$$\begin{aligned} (\Phi u)(t) = &\int_0^{h(t)} [Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)]z^\alpha(s)ds + \int_{h(t)}^{t} Q(t-s)B_1(s)z^\alpha(s)ds \\ &+ \int_0^t Q(t-s)[L(\tilde{\psi}_s) + F(s, \bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)})]ds + \sum_{0<t_k<t} Q(t-t_k)I_k(\bar{u}_{t_k} + \eta_{t_k}). \end{aligned}$$

Then clearly Φ is well-defined on $B[0;\delta]$ for each $\delta > 0$. Also the operator Φ^α has a fixed point if and only if Φ has a fixed point. Let $\Phi = \Phi_1 + \Phi_2$ and, Φ_1 and Φ_2 are defined by

$$\begin{aligned} (\Phi_1 u)(t) = &\int_0^{h(t)} [Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)]z^\alpha(s)ds + \int_{h(t)}^{t} Q(t-s)B_1(s)z^\alpha(s)ds \\ (\Phi_2 u)(t) = &\int_0^t Q(t-s)[L(\tilde{\psi}_s) + F(s, \bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)})]ds + \sum_{0<t_k<t} Q(t-t_k)I_k(\bar{u}_{t_k} + \eta_{t_k}). \end{aligned}$$

Now, in order to understand the proof easily we break it into several parts.

Step (i): We claim that $\Phi B[0;\delta] \subseteq B[0;\delta]$. Suppose it does not hold, then for any $\delta > 0$, there is $u^\delta \in B[0;\delta]$ and $t^\delta \in [0, \mathcal{M}]$ such that $\delta < \|\Phi u^\delta(t^\delta)\|$. For $t \in [0, h(\mathcal{M})]$, it follows that

$$\begin{aligned}\|z^\alpha(t)\| =& \Big\| [r'(t)B_2^*(r(t))Q^*(\mathcal{M}-r(t)) + B_1^*(t)Q^*(\mathcal{M}-t)]R(\alpha,\Gamma_0^{\mathcal{M}})[\xi_{\mathcal{M}} - Q(\mathcal{M})\psi(0)] \\ & - [r'(t)B_2^*(r(t))Q^*(\mathcal{M}-r(t)) + B_1^*(t)Q^*(\mathcal{M}-t)] \times \Big\{ \int_0^t R(\alpha,\Gamma_s^{\mathcal{M}})Q(\mathcal{M}-s)[L(\tilde{\psi}_s) \\ & + F(s, \bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)})]ds + R(\alpha,\Gamma_0^{\mathcal{M}}) \sum_{k=1}^{m} Q(\mathcal{M}-t_k)I_k(\bar{u}_{t_k}+\eta_{t_k}) \Big\} \Big\|\end{aligned}$$

By axiom (A), we have

$$\|u_{\rho(s,\xi_s)}\|_{\mathscr{D}} \le K_{\mathcal{M}}\delta, \text{ for any } u \in B[0;\delta],$$

and hence

$$\begin{aligned}\|\xi_{\rho(s,\xi_s)}\|_{\mathscr{D}} &= \|\bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)}\|_{\mathscr{D}} \le \|\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}\|_{\mathscr{D}} + \|\eta_{\rho(s,\bar{u}_s+\eta_s)}\|_{\mathscr{D}} \\ &\le K_{\mathcal{M}}\delta + R_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + K_{\mathcal{M}}\widetilde{P}\|\psi(0)\| = \delta_1,\end{aligned}$$

and

$$\|\bar{u}_{t_k} + \eta_{t_k}\| \le K_{\mathcal{M}}\delta + R_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + K_{\mathcal{M}}\widetilde{P}\|\psi(0)\| = \delta_1.$$

Thus

$$\begin{aligned}\|z^\alpha(t)\| \le& \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}][\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}\tilde{L}_F \\ & \times (1 + \|\bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)})\|_{\mathscr{D}}) + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\sum_{k=1}^{m}\tilde{q}_k(1 + \|\bar{u}_{t_k} + \eta_{t_k}\|_{\mathscr{D}}) \\ \le& \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}][\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] \\ & + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}\tilde{L}_F(1+\delta_1) \\ & + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\sum_{k=1}^{m}\tilde{q}_k(1+\delta_1).\end{aligned}$$

Now, if $t \in [h(\mathcal{M}), \mathcal{M}]$, then

$$\begin{aligned}\|z^\alpha(t)\| =& \Big\| B_1^*(t)Q^*(\mathcal{M}-t)R(\alpha,\Gamma_0^{\mathcal{M}})(\xi_{\mathcal{M}} - Q(\mathcal{M})\psi(0)) - B_1^*(t)Q^*(\mathcal{M}-t)\Big\{ \int_0^t R(\alpha,\Gamma_s^{\mathcal{M}})Q(\mathcal{M}-s) \\ & \times [L(\tilde{\psi}_s) + F(s, \bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)}))]ds + R(\alpha,\Gamma_0^{\mathcal{M}}) \sum_{k=1}^{m} Q(\mathcal{M}-t_k)I_k(\bar{u}_{t_k}+\eta_{t_k}) \Big\} \Big\| \\ \le& \frac{1}{\alpha}N\widetilde{P}[\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] + \frac{1}{\alpha}N\widetilde{P}^2\mathcal{M}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \frac{1}{\alpha}N\widetilde{P}^2\mathcal{M}\tilde{L}_F(1+\delta_1) + \frac{1}{\alpha}N\widetilde{P}^2\sum_{k=1}^{m}\tilde{q}_k(1+\delta_1)\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta <\|\Phi u^{\delta}(t)\| &= \Big\| \int_0^{h(t)} [Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)]z^{\alpha}(s)ds \\
&+ \int_{h(t)}^{t} Q(t-s)B_1(s)z^{\alpha}(s)ds + \int_0^{t} Q(t-s)[L(\tilde{\psi}_s) \\
&+ F(s, \bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)})]ds + \sum_{0<t_k<t} Q(t-t_k)I_k(\bar{u}_{t_k} + \eta_{t_k})\Big\|_{\mathscr{H}} \\
&\leq \int_0^{h(t)} [\widetilde{P}N + \widetilde{P}Nr]\Big\{\frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}][\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] \\
&+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}\tilde{L}_F(1+\delta_1) \\
&+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\sum_{k=1}^{m}\tilde{q}_k(1+\delta_1)\Big\}ds \\
&+ \int_{h(t)}^{t} \widetilde{P}N\Big\{\frac{1}{\alpha}N\widetilde{P}[\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] + \frac{1}{\alpha}N\widetilde{P}^2\mathcal{M}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} \\
&+ \frac{1}{\alpha}N\widetilde{P}^2\mathcal{M}\tilde{L}_F(1+\delta_1) + \frac{1}{\alpha}N\widetilde{P}^2\sum_{k=1}^{m}\tilde{q}_k(1+\delta_1)\Big\}ds \\
&+ \int_0^{t} \widetilde{P}[lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \tilde{L}_F(1+\delta_1)]ds + \sum_{k=1}^{m}\widetilde{P}\tilde{q}_k(1+\delta_1) \\
&\leq \int_0^{t} [\widetilde{P}N + \widetilde{P}Nr]\Big\{\frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}][\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] \\
&+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}\tilde{L}_F(1+\delta_1) \\
&+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\sum_{k=1}^{m}\tilde{q}_k(1+\delta_1)\Big\}ds \\
&+ \mathcal{M}\widetilde{P}[lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \tilde{L}_F(1+\delta_1)] + \sum_{k=1}^{m}\widetilde{P}\tilde{q}_k(1+\delta_1).
\end{aligned}$$

Dividing both sides by δ and taking the limit as $\delta \to \infty$ we get

$$1 < \frac{1}{\alpha}\mathcal{M}[\widetilde{P}N + \widetilde{P}Nr]^2\Big\{\mathcal{M}\widetilde{P}\tilde{L}_F K_{\mathcal{M}} + \widetilde{P}\sum_{k=1}^{m}\tilde{q}_k K_{\mathcal{M}}\Big\} + \mathcal{M}\widetilde{P}\tilde{L}_F K_{\mathcal{M}} + \sum_{k=1}^{m}\widetilde{P}\tilde{q}_k K_{\mathcal{M}},$$

which is a contradiction to (7). Hence, we conclude that for each $\alpha > 0$, there is a $\delta > 0$ such that Φ maps $B[0;\delta]$ into itself.

Step (ii): Φ_1 is a contraction map. By substituting the value of z^α in Φ_1, we have

$$\begin{aligned}
(\Phi_1 u)(t) = &\int_0^{h(t)} [Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)] \\
&\times [r'(s)B_2^*(r(s))Q^*(\mathcal{M}-r(s)) + B_1^*(s)Q^*(\mathcal{M}-s)]R(\alpha,\Gamma_0^{\mathcal{M}}) \\
&\times [\xi_{\mathcal{M}} - Q(\mathcal{M})\psi(0)]ds \\
&- \int_0^{h(t)} [Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)] \\
&\times [r'(s)B_2^*(r(s))Q^*(\mathcal{M}-r(s)) + B_1^*(s)Q^*(\mathcal{M}-s)] \\
&\times \Big\{ \int_0^t R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)[L(\tilde{\psi}_r) + F(r,\bar{u}_{\rho(r,\bar{u}_r+\eta_r)} + \eta_{\rho(r,\bar{u}_r+\eta_r)}))]dr \\
&+ R(\alpha,\Gamma_0^{\mathcal{M}}) \sum_{k=1}^m Q(\mathcal{M}-t_k)I_k(\bar{u}_{t_k}+\eta_{t_k})\Big\}ds \\
&+ \int_{h(t)}^t Q(t-s)B_1(s)B_1^*(s)Q^*(\mathcal{M}-s)R(\alpha,\Gamma_0^{\mathcal{M}})(\xi_{\mathcal{M}} - Q(\mathcal{M})\psi(0))ds \\
&- \int_{h(t)}^t Q(t-s)B_1(s)B_1^*(s)Q^*(\mathcal{M}-s)\Big\{ \int_0^t R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)[L(\tilde{\psi}_r) \\
&+ F(r,\bar{u}_{\rho(r,\bar{u}_r+\eta_r)} + \eta_{\rho(r,\bar{u}_r+\eta_r)}))]dr + R(\alpha,\Gamma_0^{\mathcal{M}}) \sum_{k=1}^m Q(\mathcal{M}-t_k)I_k(\bar{u}_{t_k}+\eta_{t_k})\Big\}ds.
\end{aligned}$$

For $u, v \in \mathscr{H}$, we have

$$\begin{aligned}
&\|(\Phi_1 u)(t) - (\Phi_1 v)(t)\| \\
\le &\int_0^{h(t)} \|Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)\| \\
&\times \|r'(s)B_2^*(r(s))Q^*(\mathcal{M}-r(s)) + B_1^*(s)Q^*(\mathcal{M}-s)\|\Big\{ \int_0^s \|R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)\| \\
&\times \|F(r,\bar{u}_{\rho(r,\bar{u}_r+\eta_r)} + \eta_{\rho(r,\bar{u}_r+\eta_r)}) - F(r,\bar{v}_{\rho(r,\bar{v}_r+\eta_r)} + \eta_{\rho(r,\bar{v}_r+\eta_r)})\|dr \\
&+ \|R(\alpha,\Gamma_0^{\mathcal{M}})\| \sum_{k=1}^m \|Q(\mathcal{M}-t_k)(I_k(\bar{u}_{t_k}+\eta_{t_k}) - I_k(\bar{v}_{t_k}+\eta_{t_k})\|\Big\}ds \\
&+ \int_{h(t)}^t \|Q(t-s)B_1(s)B_1^*(s)Q^*(\mathcal{M}-s)\|\Big\{ \int_0^s \|R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)\| \\
&\times \|F(r,\bar{u}_{\rho(r,\bar{u}_r+\eta_r)} + \eta_{\rho(r,\bar{u}_r+\eta_r)}) - F(r,\bar{v}_{\rho(r,\bar{v}_r+\eta_r)} + \eta_{\rho(r,\bar{v}_r+\eta_r)})\|dr \\
&+ \|R(\alpha,\Gamma_0^{\mathcal{M}})\| \sum_{k=1}^m \|Q(\mathcal{M}-t_k)(I_k(\bar{u}_{t_k}+\eta_{t_k}) - I_k(\bar{v}_{t_k}+\eta_{t_k})\|\Big\}ds \\
\le &\int_0^{h(t)} \|Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)\| \\
&\times \|r'(s)B_2^*(r(s))Q^*(\mathcal{M}-r(s)) + B_1^*(s)Q^*(\mathcal{M}-s)\|\Big\{ \int_0^s \|R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)\| \\
&\times [\|F(r,\bar{u}_{\rho(r,\bar{u}_r+\eta_r)} + \eta_{\rho(r,\bar{u}_r+\eta_r)}) - F(r,\bar{v}_{\rho(r,\bar{u}_r+\eta_r)} + \eta_{\rho(r,\bar{u}_r+\eta_r)})\| \\
&+ \|F(r,\bar{v}_{\rho(r,\bar{u}_r+\eta_r)} + \eta_{\rho(r,\bar{u}_r+\eta_r)}) - F(r,\bar{v}_{\rho(r,\bar{v}_r+\eta_r)} + \eta_{\rho(r,\bar{v}_r+\eta_r)})\|]dr
\end{aligned}$$

$$
\begin{aligned}
&+\|R(\alpha,\Gamma_0^{\mathcal{M}})\|\sum_{k=1}^{m}\|Q(\mathcal{M}-t_k)(I_k(\bar{u}_{t_k}+\eta_{t_k})-I_k(\bar{v}_{t_k}+\eta_{t_k})\|\Big\}ds\\
&+\int_{h(t)}^{t}\|Q(t-s)B_1(s)B_1^*(s)Q^*(\mathcal{M}-s)\|\Big\{\int_0^s\|R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)\|\\
&\times[\|F(r,\bar{u}_{\rho(r,\bar{u}_r+\eta_r)}+\eta_{\rho(r,\bar{u}_r+\eta_r)})-F(r,\bar{v}_{\rho(r,\bar{u}_r+\eta_r)}+\eta_{\rho(r,\bar{u}_r+\eta_r)})\|\\
&+\|F(r,\bar{v}_{\rho(r,\bar{u}_r+\eta_r)}+\eta_{\rho(r,\bar{u}_r+\eta_r)})-F(r,\bar{v}_{\rho(r,\bar{v}_r+\eta_r)}+\eta_{\rho(r,\bar{v}_r+\eta_r)})\|]dr\\
&+\|R(\alpha,\Gamma_0^{\mathcal{M}})\|\sum_{k=1}^{m}\|Q(\mathcal{M}-t_k)(I_k(\bar{u}_{t_k}+\eta_{t_k})-I_k(\bar{v}_{t_k}+\eta_{t_k}))\|\Big\}ds\\
\le&\int_0^{h(t)}\frac{1}{\alpha}[\widetilde{P}N+\widetilde{P}Nr]^2\Big\{\int_0^s\widetilde{P}[L_FK_{\mathcal{M}}\|u-v\|_{\mathcal{M}}+LL_\rho K_{\mathcal{M}}\|u-v\|_{\mathcal{M}}]dr+\widetilde{P}\sum_{k=1}^{m}q_kK_{\mathcal{M}}\|u-v\|_{\mathcal{M}}\Big\}ds\\
&+\int_{h(t)}^{t}\frac{1}{\alpha}\widetilde{P}^2N^2\Big\{\int_0^s\widetilde{P}[L_FK_{\mathcal{M}}\|u-v\|_{\mathcal{M}}+LL_\rho K_{\mathcal{M}}\|u-v\|_{\mathcal{M}}]dr+\widetilde{P}\sum_{k=1}^{m}q_kK_{\mathcal{M}}\|u-v\|_{\mathcal{M}}\Big\}ds\\
\le&\int_0^{t}\frac{1}{\alpha}[\widetilde{P}N+\widetilde{P}Nr]^2\Big\{\int_0^s\widetilde{P}[L_FK_{\mathcal{M}}\|u-v\|_{\mathcal{M}}+LL_\rho K_{\mathcal{M}}\|u-v\|_{\mathcal{M}}]dr+\widetilde{P}\sum_{k=1}^{m}q_kK_{\mathcal{M}}\|u-v\|_{\mathcal{M}}\Big\}ds\\
\le&\frac{1}{\alpha}\mathcal{M}[\widetilde{P}N+\widetilde{P}Nr]^2\Big\{\mathcal{M}\widetilde{P}[L_FK_{\mathcal{M}}\|u-v\|_{\mathcal{M}}+LL_\rho K_{\mathcal{M}}\|u-v\|_{\mathcal{M}}]+\widetilde{P}\sum_{k=1}^{m}q_kK_{\mathcal{M}}\|u-v\|_{\mathcal{M}}\Big\}\\
=&\Big\{\frac{1}{\alpha}\mathcal{M}[\widetilde{P}N+\widetilde{P}Nr]^2\Big\{\mathcal{M}\widetilde{P}L_FK_{\mathcal{M}}+\mathcal{M}\widetilde{P}LL_\rho K_{\mathcal{M}}+\widetilde{P}\sum_{k=1}^{m}q_kK_{\mathcal{M}}\Big\}\Big\}\|u-v\|_{\mathcal{M}},
\end{aligned}
$$

where $\frac{1}{\alpha}\mathcal{M}[\widetilde{P}N+\widetilde{P}Nr]^2\Big\{\mathcal{M}\widetilde{P}L_FK_{\mathcal{M}}+\mathcal{M}\widetilde{P}LL_\rho K_{\mathcal{M}}+\widetilde{P}\sum_{k=1}^{m}q_kK_{\mathcal{M}}\Big\}<1$ (by assumption (7)), which yield that Φ_1 is the contraction map.

Step (iii): Φ_2 is continuous on $B[0;\delta]$.

Let $\{u^n\}_{n\in\mathbb{N}}\subset B[0;\delta]$ be a sequence such that $u^n\to u$ as $n\to\infty$. Then for any $s\in[0,\mathcal{M}]$, $u^n_{\rho(s,u_s^n)}\to u_{\rho(s,u_s)}$ as $n\to\infty$. Hence

$$
\begin{aligned}
&\|F(s,\bar{u}^n_{\rho(s,\bar{u}_s^n+\eta_s)}+\eta_{\rho(s,\bar{u}_s^n+\eta_s)})-F(s,\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}+\eta_{\rho(s,\bar{u}_s+\eta_s)})\|\\
\le&\|F(s,\bar{u}^n_{\rho(s,\bar{u}_s^n+\eta_s)}+\eta_{\rho(s,\bar{u}_s^n+\eta_s)})-F(s,\bar{u}_{\rho(s,\bar{u}_s^n+\eta_s)}+\eta_{\rho(s,\bar{u}_s^n+\eta_s)})\|\\
&+\|F(s,\bar{u}_{\rho(s,\bar{u}_s^n+\eta_s)}+\eta_{\rho(s,\bar{u}_s^n+\eta_s)})-F(s,\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}+\eta_{\rho(s,\bar{u}_s+\eta_s)})\|\\
&\to 0,\ \text{as } n\to\infty.
\end{aligned}
$$

Further notice that for $s\in[0,\mathcal{M}]$

$$
\|F(s,\bar{u}^n_{\rho(s,\bar{u}_s^n+\eta_s)}+\eta_{\rho(s,\bar{u}_s^n+\eta_s)})-F(s,\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}+\eta_{\rho(s,\bar{u}_s+\eta_s)})\|\le 2f_{\delta_1}(s).
$$

Thus by the Lebesgue dominated convergence theorem

$$
\begin{aligned}
\|(\Phi_2u^n)(t)-(\Phi_2u)(t)\|\le&\int_0^t\|Q(t-s)[F(s,\bar{u}^n_{\rho(s,\bar{u}_s^n+\eta_s)}+\eta_{\rho(s,\bar{u}_s^n+\eta_s)})-F(s,\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}+\eta_{\rho(s,\bar{u}_s+\eta_s)})]ds\|\\
&+\sum_{0<t_k<t}\|Q(t-s)[I_k(\bar{u}^n_{t_k}+\eta_{t_k})-I_k(\bar{u}_{t_k}+\eta_{t_k})]\|\\
\le&\widetilde{P}\int_0^t\|F(s,\bar{u}^n_{\rho(s,\bar{u}_s^n+\eta_s)}+\eta_{\rho(s,\bar{u}_s^n+\eta_s)})-F(s,\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}+\eta_{\rho(s,\bar{u}_s+\eta_s)})\|ds
\end{aligned}
$$

$$+\widetilde{P}\sum_{k=1}^{m} q_k\|\bar{u}^n_{t_k}-\bar{u}_{t_k}\|\to 0,\ \text{as } n\to\infty.$$

Therefore, Φ_2 is continuous on $B[0;\delta]$ for $t\in B[0;\delta]$.

Step (iv): Φ_2 is equicontinuous. First we check the equicontinuity of the family

$$Z(t)=\{(\Phi_2 u)(t): u\in B[0;\delta]\}$$

on $(0,\mathcal{M}]$. Let $\epsilon>0$ be given and $0<t_1<t_2<\mathcal{M}$, then

$$\begin{aligned}
&\|(\Phi_2u)(t_2)-(\Phi_2u)(t_1)\|\\
&\le\int_0^{t_1-\epsilon}\|Q(t_2-s)-Q(t_1-s)\|\|L(\tilde{\psi}_s)+F(s,\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}+\eta_{\rho(s,\bar{u}_s+\eta_s)})\|ds\\
&+\sum_{0<t_k<t_1-\epsilon}\|Q(t_2-t_k)-Q(t_1-t_k)\|\|I_k(\bar{u}_{t_k}+\eta_{t_k})\|\\
&+\int_{t_1-\epsilon}^{t_1}\|Q(t_2-s)-Q(t_1-s)\|\|L(\tilde{\psi}_s)+F(s,\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}+\eta_{\rho(s,\bar{u}_s+\eta_s)})\|ds\\
&+\sum_{t_1-\epsilon<t_k<t_1}\|Q(t_2-t_k)-Q(t_1-t_k)\|\|I_k(\bar{u}_{t_k}+\eta_{t_k})\|\\
&+\int_{t_1}^{t_2}\|Q(t_2-s)\|\|L(\tilde{\psi}_s)+F(s,\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}+\eta_{\rho(s,\bar{u}_s+\eta_s)})\|ds+\sum_{t_1<t_k<t_2}\|Q(t_2-t_k)\|\|I_k(\bar{u}_{t_k}+\eta_{t_k})\|.
\end{aligned}$$

Thus we see that for each $u\in B[0;\delta]$, $\|(\Phi_2u)(t_2)-(\Phi_2u)(t_1)\|\to 0$ as $t_2-t_1\to 0$ for sufficiently small $\epsilon>0$ as for each $t\in(0,\mathcal{M}]$, $Q(t)$ is uniformly continuous. At $t=0$ the equicontinuity of $Z(\cdot)$ is trivial. Hence $Z(\cdot)$ is the equicontinuous family.

Step (v): We now explain that Φ_2 maps the ball $B[0;\delta]$ into a precompact subset of $\mathscr{H}$. Clearly $Z(0)$ is precompact in $\mathscr{H}$. Now, we break Φ_2 that is $\Phi_2=\Psi_1+\Psi_2$ as

$$\begin{aligned}
(\Psi_1u)(t)&=\int_0^t Q(t-s)[L(\tilde{\psi}_s)+F(s,\bar{u}_{\rho(s,\bar{u}_s+\eta_s)}+\eta_{\rho(s,\bar{u}_s+\eta_s)})]ds,\ t\in(0,\mathcal{M}],\\
(\Psi_2u)(t)&=\sum_{0<t_k<t}Q(t-t_k)I_k(\bar{u}_{t_k}+\eta_{t_k}),\ t\in(0,\mathcal{M}].
\end{aligned}$$

Using Lemma 3.1 in [20], one can easily show that

$$\Psi_1(B[0;\delta](t))=\{(\Psi_1u)(t): u\in B[0;\delta]\}$$

is precompact in $\mathscr{H}$ for all $t\in[0,\mathcal{M}]$. We can also easily prove that $\Psi_1(B[0;\delta])$ is uniformly bounded. Hence using the Arzela–Ascoli theorem, we deduce that Ψ_1 is the compact operator as Φ_2 is equicontinuous. Next, it remains to prove that $\Psi_2(B[0;\delta])$ is precompact for each $t\in(0,\mathcal{M}]$. It is trivial whenever $t\in(0,t_1]$. Now it is required to determine that $U=\left\{\sum_{0<t_k<t}Q(t-t_k)I_k(\bar{u}_{t_k}+\eta_{t_k}): t\in(t_i,t_{i+1}], u\in B[0;\delta]\right\}$ is precompact in $\mathscr{H}$ for $t\in(t_i,t_{i+1}]$, $i=1,2,\cdots,m$ and $u\in B[0;\delta]$. By using the compactness of $\{Q(t)\}_{t>0}$ and assumption on I_k, it yields that the set U is precompact in $\mathscr{H}$. Obviously, the elements of U are equicontinuous. Therefore, compactness of Ψ_2 is implied by the Arzela–Ascoli theorem. Thus, $\Phi_2=\Psi_1+\Psi_2$ is a compact operator. Hence, $Z(t)$ is precompact in $\mathscr{H}$ for every $t\in[0,\mathcal{M}]$. By the Arzela–Ascoli theorem, Φ_2 is completely continuous.

Hence, we conclude that the operator Φ has a fixed point as all hypotheses of Krasnoselskii's fixed point theorem are satisfied, consequently Φ^α has a fixed point, say ξ^α, which is a mild solution of the system (1). □

Theorem 3.4. *Suppose that the system* (4) *is approximately controllable and all conditions in Theorem 3.3 hold. Further, assume that the function F is uniformly bounded. Then the system* (1) *is approximately controllable on* $[0, \mathcal{M}]$.

Proof. By Theorem 3.3, it follows that $\xi^{\alpha}(\cdot)$ is a mild solution of the system (1) on $(-\infty, \mathcal{M}]$ corresponding to the control $z^{\alpha}(\cdot)$ and satisfying

$$\begin{aligned}
\xi^{\alpha}(\mathcal{M}) =& Q(\mathcal{M})\psi(0) + \int_0^{h(\mathcal{M})} [Q(\mathcal{M}-s)B_1(s) + Q(\mathcal{M}-r(s))B_2(r(s))r'(s)] \\
&\times [r'(s)B_2^*(r(s))Q^*(\mathcal{M}-r(s)) + B_1^*(s)Q^*(\mathcal{M}-s)]R(\alpha,\Gamma_0^{\mathcal{M}}) \\
&\times [\xi_{\mathcal{M}} - Q(\mathcal{M})\psi(0) - \sum_{k=1}^{m} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha})]ds \\
&- \int_0^{h(\mathcal{M})} [Q(\mathcal{M}-s)B_1(s) + Q(\mathcal{M}-r(s))B_2(r(s))r'(s)] \\
&\times [r'(s)B_2^*(r(s))Q^*(\mathcal{M}-r(s)) + B_1^*(s)Q^*(\mathcal{M}-s)] \\
&\times \int_0^{s} R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)[L(\tilde{\psi}_r) + F(r, \xi^{\alpha}{}_{\rho(r,\xi_r^{\alpha})})]drds \\
&+ \int_{h(\mathcal{M})}^{\mathcal{M}} Q(\mathcal{M}-s)B_1(s)B_1^*(s)Q^*(\mathcal{M}-s)R(\alpha,\Gamma_0^{\mathcal{M}})(\xi_{\mathcal{M}} - Q(\mathcal{M})\psi(0) \\
&- \sum_{k=1}^{m} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha}))ds \\
&- \int_{h(\mathcal{M})}^{\mathcal{M}} Q(\mathcal{M}-s)B_1(s)B_1^*(s)Q^*(\mathcal{M}-s) \int_0^{s} R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)[L(\tilde{\psi}_r) \\
&+ F(r, \xi^{\alpha}{}_{\rho(r,\xi_r^{\alpha})})]drds \\
&+ \int_0^{\mathcal{M}} Q(\mathcal{M}-s)[L(\tilde{\psi}_s) + F(s, \xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})]ds + \sum_{0<t_k<\mathcal{M}} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha}).
\end{aligned}$$

Then clearly

$$\begin{aligned}
\xi^{\alpha}(\mathcal{M}) =& Q(\mathcal{M})\psi(0) + \Gamma_0^{\mathcal{M}}R(\alpha,\Gamma_0^{\mathcal{M}})[\xi_{\mathcal{M}} - Q(\mathcal{M})\psi(0) - \sum_{k=1}^{m} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha})] \\
&- \int_0^{h(\mathcal{M})} \int_r^{h(\mathcal{M})} [Q(\mathcal{M}-s)B_1(s) + Q(\mathcal{M}-r(s))B_2(r(s))r'(s)] \\
&\times [r'(s)B_2^*(r(s))Q^*(\mathcal{M}-r(s)) + B_1^*(s)Q^*(\mathcal{M}-s)] \\
&\times R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)[L(\tilde{\psi}_r) + F(r, \xi^{\alpha}{}_{\rho(r,\xi_r^{\alpha})})]dsdr \\
&- \int_0^{h(\mathcal{M})} \int_{h(\mathcal{M})}^{\mathcal{M}} Q(\mathcal{M}-s)B_1(s)B_1^*(s)Q^*(\mathcal{M}-s)R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)[L(\tilde{\psi}_r) \\
&+ F(r, \xi^{\alpha}{}_{\rho(r,\xi_r^{\alpha})})]dsdr \\
&- \int_{h(\mathcal{M})}^{\mathcal{M}} \int_r^{\mathcal{M}} Q(\mathcal{M}-s)B_1(s)B_1^*(s)Q^*(\mathcal{M}-s)R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)[L(\tilde{\psi}_r) \\
&+ F(r, \xi^{\alpha}{}_{\rho(r,\xi_r^{\alpha})})]dsdr
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\mathcal{M}} Q(\mathcal{M}-s)[L(\tilde{\psi}_s)+F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})]ds+\sum_{0<t_k<\mathcal{M}} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha})\\
=&Q(\mathcal{M})\psi(0)+(I-\alpha R(\alpha;\Gamma_0^{\mathcal{M}})[\xi_{\mathcal{M}}-Q(\mathcal{M})\psi(0)-\sum_{k=1}^{m} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha})]\\
&-\int_0^{h(\mathcal{M})} \Gamma_r^{\mathcal{M}} R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)[L(\tilde{\psi}_r)+F(r,\xi^{\alpha}{}_{\rho(r,\xi_r^{\alpha})})]dr\\
&-\int_{h(\mathcal{M})}^{\mathcal{M}} \Gamma_r^{\mathcal{M}} R(\alpha,\Gamma_r^{\mathcal{M}})Q(\mathcal{M}-r)[L(\tilde{\psi}_r)+F(r,\xi^{\alpha}{}_{\rho(r,\xi_r^{\alpha})})]dr\\
&+\int_0^{\mathcal{M}} Q(\mathcal{M}-s)[L(\tilde{\psi}_s)+F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})]ds+\sum_{0<t_k<\mathcal{M}} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha})\\
=&Q(\mathcal{M})\psi(0)+(I-\alpha R(\alpha;\Gamma_0^{\mathcal{M}}))[\xi_{\mathcal{M}}-Q(\mathcal{M})\psi(0)-\sum_{k=1}^{m} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha})]\\
&-\int_0^{\mathcal{M}} (I-R(\alpha,\Gamma_r^{\mathcal{M}}))Q(\mathcal{M}-r)[L(\tilde{\psi}_r)+F(r,\xi^{\alpha}{}_{\rho(r,\xi_r^{\alpha})})]dr\\
&+\int_0^{\mathcal{M}} Q(\mathcal{M}-s)[L(\tilde{\psi}_s)+F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})]ds+\sum_{0<t_k<\mathcal{M}} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha}).
\end{aligned}
$$

$$
\begin{aligned}
\xi^{\alpha}(\mathcal{M}) =&\xi_{\mathcal{M}}-\alpha R(\alpha;\Gamma_0^{\mathcal{M}})[\xi_{\mathcal{M}}-Q(\mathcal{M})\psi(0)-\sum_{k=1}^{m} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha})]\\
&+\int_0^{\mathcal{M}} \alpha R(\alpha;\Gamma_s^{\mathcal{M}})Q(\mathcal{M}-s)[L(\tilde{\psi}_s)+F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})]ds.
\end{aligned}
$$

By the hypothesis, $F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})$ is uniformly bounded, for $\alpha>0$ a weakly convergent subsequence, represented by, $F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})$ exists such that for every $s\in[0,\mathcal{M}]$, $F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})\to f(s)$ say, weakly in $\mathscr{H}$. Now, since the family $\{Q(t)\}_{t>0}$ is compact, we assert that for every $s\in[0,\mathcal{M})$, $Q(\mathcal{M}-s)F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})\to Q(\mathcal{M}-s)f(s)$ in $\mathscr{H}$. Therefore, we deduce that

$$
\left\|\int_0^{\mathcal{M}} Q(\mathcal{M}-s)(F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})-f(s))ds\right\|\to 0,
$$

as $\alpha\to 0^+$. Finally, by the assumption that linear system (4) is approximately controllable, the operator $\alpha R(\alpha;\Gamma_0^{\mathcal{M}})\to 0$ strongly as $\alpha\to 0^+$, $\|\alpha R(\alpha;\Gamma_0^{\mathcal{M}})\|\leq 1$ and using the Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
\|\xi^{\alpha}(\mathcal{M})-\xi_{\mathcal{M}}\| \leq&\|\alpha R(\alpha;\Gamma_0^{\mathcal{M}})[\xi_{\mathcal{M}}-Q(\mathcal{M})\psi(0)-\sum_{k=1}^{m} Q(\mathcal{M}-t_k)I_k(\xi_{t_k}^{\alpha})]\|\\
&+\int_0^{\mathcal{M}} \|\alpha R(\alpha;\Gamma_s^{\mathcal{M}})Q(\mathcal{M}-s)[L(\tilde{\psi}_s)+f(s)]\|ds\\
&+\int_0^{\mathcal{M}} \|\alpha R(\alpha;\Gamma_s^{\mathcal{M}})Q(\mathcal{M}-s)[F(s,\xi^{\alpha}{}_{\rho(s,\xi_s^{\alpha})})-f(s)]\|ds\\
&\longrightarrow 0,\quad \text{as } \alpha\to 0^+,
\end{aligned}
$$

and that comes to the conclusion. □

We are now obtained the approximate controllability of the system (1) by Schauder's fixed point theorem under different hypotheses. In order to establish the approximate controllability results, we need the following hypothesis:

(*G*6) The function $F : [0, \mathcal{M}] \times \mathscr{D} \to \mathscr{H}$ satisfies the following condition:

(a) For each $r > 0$, there exists a continuous function $f_r : [0, \mathcal{M}] \to (0, \infty)$ such that for any $0 \leq t \leq \mathcal{M}$ and $\psi \in \mathscr{D}$

$$\sup_{\|\psi\|_{\mathscr{D}} \leq r} \|F(t, \psi)\| \leq f_r(t),$$

and there is a constant $\delta_2 > 0$ such that $\liminf_{\delta \to \infty} \frac{1}{\delta} \int_0^{\mathcal{M}} f_{\delta_1}(t)dt = \delta_2 < \infty$, where

$$\delta_1 = K_{\mathcal{M}}\delta + R_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + K_{\mathcal{M}}\widetilde{P}\|\psi(0)\|.$$

Theorem 3.5. *Suppose that hypotheses* (*G*1), (*G*2), (*G*4) *and* (*G*6) *are hold. Then the control system* (1) *has at least one mild solution on* $(-\infty, \mathcal{M}]$, *provided that*

$$\frac{1}{\alpha}\mathcal{M}[\widetilde{P}N + \widetilde{P}Nr]^2\widetilde{P}\delta_2 + \frac{1}{\alpha}\mathcal{M}[\widetilde{P}N + \widetilde{P}Nr]^2\widetilde{P}\sum_{k=1}^{m}\tilde{q}_k K_{\mathcal{M}} + \widetilde{P}\delta_2 + \widetilde{P}\sum_{k=1}^{m}\tilde{q}_k K_{\mathcal{M}} \leq 1.$$

Proof. Consider $B[0;\delta] = \left\{u \in C^0_{\mathcal{M}} : \|u\|_{\mathcal{M}} \leq \delta\right\}$ as given in Theorem 3.3. Define Φ on $B[0;\delta]$ by

$$\begin{aligned}(\Phi u)(t) = &\int_0^{h(t)} [Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)]z^{\alpha}(s)ds + \int_{h(t)}^{t} Q(t-s)B_1(s)z^{\alpha}(s)ds \\ &+ \int_0^t Q(t-s)[L(\tilde{\psi}_s) + F(s, \bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)})]ds + \sum_{0<t_k<t} Q(t-t_k)I_k(\bar{u}_{t_k} + \eta_{t_k}).\end{aligned}$$

Next we prove that Φ satisfies all conditions of Schauder's fixed point theorem. For $t \in [0, h(\mathcal{M})]$,

$$\begin{aligned}\|z^{\alpha}(t)\| \leq &\frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}][\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] \\ &+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\int_0^{\mathcal{M}} f_{\delta_1}(t)dt \\ &+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\sum_{k=1}^{m}\tilde{q}_k(1+\delta_1).\end{aligned}$$

For $t \in [h(\mathcal{M}), \mathcal{M}]$, then

$$\|z^{\alpha}(t)\| \leq \frac{1}{\alpha}N\widetilde{P}[\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] + \frac{1}{\alpha}N\widetilde{P}^2\mathcal{M}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \frac{1}{\alpha}N\widetilde{P}^2\int_0^{\mathcal{M}} f_{\delta_1}(t)dt + \frac{1}{\alpha}N\widetilde{P}^2\sum_{k=1}^{m}\tilde{q}_k(1+\delta_1).$$

We claim that $\Phi B[0;\delta] \subseteq B[0;\delta]$. Suppose it does not hold, then for any $\delta > 0$, there is $u^{\delta} \in B[0;\delta]$ and

$t^\delta \in [0, \mathcal{M}]$ such that $\delta < \|\Phi u^\delta(t^\delta)\|$. Then consider

$$\begin{aligned}
\delta <&\|\Phi u^\delta(t)\| = \Big\| \int_0^{h(t)} [Q(t-s)B_1(s) + Q(t-r(s))B_2(r(s))r'(s)]z^\alpha(s)ds \\
&+ \int_{h(t)}^t Q(t-s)B_1(s)z^\alpha(s)ds + \int_0^t Q(t-s)[L(\tilde{\psi}_s) \\
&+ F(s, \bar{u}_{\rho(s,\bar{u}_s+\eta_s)} + \eta_{\rho(s,\bar{u}_s+\eta_s)})]ds + \sum_{0<t_k<t} Q(t-t_k)I_k(\bar{u}_{t_k} + \eta_{t_k})\Big\|_{\mathscr{H}} \\
\leq& \int_0^{h(t)} [\widetilde{P}N + \widetilde{P}Nr]\Big\{\frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}][\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] \\
&+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\int_0^{\mathcal{M}} f_{\delta_1}(t)dt \\
&+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\sum_{k=1}^m \tilde{q}_k(1+\delta_1)\Big\}ds \\
&+ \int_{h(t)}^t \widetilde{P}N\Big\{\frac{1}{\alpha}N\widetilde{P}[\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] + \frac{1}{\alpha}N\widetilde{P}^2\mathcal{M}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} \\
&+ \frac{1}{\alpha}N\widetilde{P}^2\int_0^{\mathcal{M}} f_{\delta_1}(t)dt + \frac{1}{\alpha}N\widetilde{P}^2\sum_{k=1}^m \tilde{q}_k(1+\delta_1)\Big\}ds \\
&+ \int_0^t \widetilde{P}[lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + f_{\delta_1}(s)]ds + \sum_{k=1}^m \widetilde{P}\tilde{q}_k(1+\delta_1) \\
\leq& \int_0^t [\widetilde{P}N + \widetilde{P}Nr]\Big\{\frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}][\|\xi_{\mathcal{M}}\| + \widetilde{P}\|\psi(0)\|] \\
&+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\mathcal{M}\widetilde{P}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\int_0^{\mathcal{M}} f_{\delta_1}(t)dt \\
&+ \frac{1}{\alpha}[rN\widetilde{P} + N\widetilde{P}]\widetilde{P}\sum_{k=1}^m \tilde{q}_k(1+\delta_1)\Big\}ds \\
&+ \mathcal{M}\widetilde{P}lR_{\mathcal{M}}\|\psi\|_{\mathscr{D}} + \widetilde{P}\int_0^{\mathcal{M}} f_{\delta_1}(s)ds + \sum_{k=1}^m \widetilde{P}\tilde{q}_k(1+\delta_1).
\end{aligned}$$

Dividing both sides by δ and taking the limit as $\delta \to \infty$ we get

$$1 < \frac{1}{\alpha}\mathcal{M}[\widetilde{P}N + \widetilde{P}Nr]^2\widetilde{P}\delta_2 + \frac{1}{\alpha}\mathcal{M}[\widetilde{P}N + \widetilde{P}Nr]^2\widetilde{P}\sum_{k=1}^m \tilde{q}_k K_{\mathcal{M}} + \widetilde{P}\delta_2 + \widetilde{P}\sum_{k=1}^m \tilde{q}_k K_{\mathcal{M}},$$

which is a contradiction to our assumption. Hence, we conclude that there is $\delta > 0$ such that Φ maps $B[0;\delta]$ into itself. Moreover, using the methods in Theorem 3.3 above and the technique in Theorem 3.1 of [1], it is not difficult to verify that Φ is completely continuous. Hence, the operator Φ satisfies all conditions of Schauder's fixed point theorem. Therefore, there exists a fixed point ξ on $B[0;\delta]$. Hence, the control system (1) has at least one mild solution on $(-\infty, \mathcal{M}]$. □

Theorem 3.6. *Suppose that the system* (4) *is approximately controllable and all conditions in Theorem 3.5 hold. Further, assume that the function F is uniformly bounded. Then the system* (1) *is approximately controllable on* $[0, \mathcal{M}]$.

Proof. The proof is similar to Theorem 3.4. □

4. Applications

Example 4.1 Consider the following system

$$\begin{cases} \frac{\partial}{\partial t}u(t,\xi) = \frac{\partial^2}{\partial \xi^2}u(t,\xi) + \int_{-\infty}^{t-1}\int_0^{\pi} \omega(s-t,\xi,\eta)u(s,\eta)d\eta ds + B_1 z(t,\xi) + B_2 z(t/2,\xi) \\ \quad + \int_{-\infty}^{t} b(s-t)u(s-\rho_1(t)\rho_2(\|u(t)\|),\xi)ds, \\ \quad 0 \le t \le 2,\ 0 \le \xi \le \pi,\ t \ne t_i,\ i = 1,2,\cdots,m, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 2, \\ u(\kappa,\xi) = \psi(\kappa,\xi),\ \kappa \le 0,\ 0 \le \xi \le \pi, \\ \Delta u(t_j,\xi) = \int_{-\infty}^{t_j} \gamma_j(s-t_j)u(s,\xi)ds,\ j = 1,2,\cdots,m, \end{cases} \tag{8}$$

where $\omega(\cdot,\cdot,\cdot)$, $b(\cdot)$, $\rho_1(\cdot)$, $\rho_2(\cdot)$ and γ_j, $j = 1,2,\cdots,m$ are appropriate functions which will be defined below. The mathematical model of heat flow in materials is described by the system (8) and so-called retarded type ([16, 23]). At time t, the temperature of the point ξ is denoted by $u(t,\xi)$, where $0 < t_1 < t_2 < \cdots < t_m < \mathcal{M}$ are fixed and $\rho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$, are continuous functions.

Set $\mathscr{H} = L^2([0,\pi])$. Here, we take $h_1(t) = t/2$ which implies $h(t) = t/2$ and $r(t) = 2t$. Define $W(t)(\cdot) = W(t,\cdot)$ and $\psi(t)(\cdot) = \psi(t,\cdot)$. Let $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ be defined by

$$A\mu = \mu''$$

and

$$D(A) = \{\mu(\cdot) \in \mathscr{H} :\ \mu', \mu'' \text{ are absolutely continuous},\ \mu'' \in \mathscr{H},\ \mu(0) = \mu(\pi) = 0\}.$$

Then clearly A generates a compact C_0-semigroup $\{S(t)\}_{t\ge 0}$ which is also self adjoint. Clearly the eigenvalues of A are $-n^2$, $n \in \mathbb{N}$, and $s_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$ are the corresponding orthogonal eigenfunctions with norm 1. Now we have the following

(i) For $\mu \in D(A)$, we obtain

$$A\mu = -\sum_{n=1}^{\infty} n^2\langle \mu, s_n\rangle s_n.$$

(ii) If $\mu \in \mathscr{H}$, then

$$S(t)\mu = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \mu, s_n\rangle s_n, \tag{9}$$

and $\|S(t)\| \le e^{-t}$.

For more details one can see [18].

Consider the phase space $\mathscr{D} = PC_0 \times L^2(h,\mathscr{H})$ (where $h : (-\infty,0) \to \mathbb{R}$ is a positive function) (see [11]), with the norm

$$\|\psi\|_{\mathscr{D}} = \|\psi(0)\| + \left(\int_{-\infty}^{0} h(\zeta)\|\psi(\zeta)\|^2 d\zeta\right)^{1/2},$$

where h and $h\|\psi(\cdot)\|^2$ are real valued Lebesgue integrable functions on $(-\infty,0)$ and ψ is continuous at 0. It is well known that, $PC_0 \times L^2(h,\mathscr{H})$ satisfy the axioms (A) and (B) by choosing a proper function h. Clearly, assumptions (a) and (b) also hold (see [11]). Furthermore, for a proper choice of h, due to Hino *et al.* [11], we have $\widetilde{\mathcal{B}}$, $K(\cdot)$, $R(\cdot) \le 1$. Thus it follows that $\max\{R_{\mathcal{M}},\ K_{\mathcal{M}}\} \le 1$.

Suppose that for the system (8), the following conditions hold:

(i) For any $\kappa \leq 0$, $\omega(\kappa, 0, \cdot) = \omega(\kappa, \pi, \cdot) \equiv 0$ where $\omega(\kappa, \cdot, \cdot) \in C^1([0, \pi] \times [0, \pi])$ and

$$l = \int_0^{\pi} \Big(\int_{-\infty}^{-1} \frac{1}{h(\kappa)} \int_0^{\pi} |\omega(\kappa, \xi, \eta)|^2 d\eta d\kappa \Big) d\xi < \infty. \tag{10}$$

(ii) The function $b : \mathbb{R} \to \mathbb{R}$ is continuous and bounded such that

$$L_1 = \Big(\int_{-\infty}^{0} \frac{b^2(s)}{h(s)} ds \Big)^{1/2} < \infty.$$

(iii) The functions $\gamma_k : \mathbb{R} \to \mathbb{R}$ are continuous such that

$$L_k = \Big(\int_{-\infty}^{0} \frac{\gamma_k^2(s)}{h(s)} ds \Big)^{1/2} < \infty,$$

for every $k = 1, 2, \cdots, m$.

Define $L : \mathscr{D} \to \mathscr{H}$, $F(\cdot, \cdot) : [0, 2] \times \mathscr{D} \to \mathscr{H}$, $\rho(\cdot, \cdot) : [0, 2] \times \mathscr{D} \to [0, \infty)$ and $I_k : \mathscr{D} \to \mathscr{H}$ respectively, as

$$L(\psi)(u) = \int_{-\infty}^{-1} \int_0^{\pi} \omega(\kappa, u, \eta) \psi(\kappa)(\eta) d\eta d\kappa,$$

$$F(t, \psi)(u) = \int_{-\infty}^{0} b(s) \psi(s, u) ds,$$

$$\rho(t, \psi)(u) = t - \rho_1(t) \rho_2(\|\psi(0, u)\|),$$

$$I_i(\psi)(u) = \int_{-\infty}^{0} \gamma_i(s) \psi(s, u) ds, \ i = 1, 2, \cdots, m,$$

for $t \in [0, 2]$, $\psi \in \mathscr{D}$. Thus the system (8) can be written in the abstract form given by the system (1). Evidently, F verifies $(G3)$ which is guaranteed well by the assumption (ii). Furthermore, assumption (i) satisfies the hypothesis $(G2)$. Moreover, for any $\psi \in \mathscr{D}$,

$$\langle L(\psi), s_n \rangle = \frac{1}{n} \Big\langle \int_{-\infty}^{-1} \int_0^{\pi} \frac{\partial}{\partial \xi} \omega(\kappa, \xi, \eta) \psi(\kappa)(\eta) d\eta d\kappa, \tilde{s}_n(\xi) \Big\rangle,$$

where $\tilde{s}_n(\xi) = \sqrt{\frac{2}{\pi}} \cos(n\xi)$, $n = 1, 2, \cdots$. In fact, for any $\psi \in \mathscr{D}$, we obtain

$$\begin{aligned} \|L(\psi)\|^2 &\leq \int_0^{\pi} \Big| \int_{-\infty}^{-1} \int_0^{\pi} \omega(\kappa, \xi, \eta) \psi(\kappa)(\eta) d\eta d\kappa \Big|^2 d\xi \\ &\leq \int_0^{\pi} \Big(\int_{-\infty}^{-1} \int_0^{\pi} |\omega(\kappa, \xi, \eta) \psi(\kappa)(\eta)| d\eta d\kappa \Big)^2 d\xi. \end{aligned}$$

By Hölder's inequality, it follows that

$$\begin{aligned} \|L(\psi)\|^2 &\leq \int_0^{\pi} \Big[\int_{-\infty}^{-1} \Big(\int_0^{\pi} |\omega(\kappa, \xi, \eta)|^2 d\eta \Big)^{1/2} \Big(\int_0^{\pi} |\psi(\kappa)(\eta)|^2 d\eta \Big)^{1/2} d\kappa \Big]^2 d\xi \\ &\leq \int_0^{\pi} \Big(\int_{-\infty}^{-1} \frac{1}{h(\kappa)} \int_0^{\pi} |\omega(\kappa, \xi, \eta|^2 d\eta d\kappa \Big) \Big[\int_{-\infty}^{-1} h(\kappa) \Big(\int_0^{\pi} |\varphi(\kappa, \eta)|^2 d\eta \Big) d\kappa \Big] d\xi \\ &\leq \int_0^{\pi} \Big(\int_{-\infty}^{-1} \frac{1}{h(\kappa)} \int_0^{\pi} |\omega(\kappa, \xi, \eta)|^2 d\eta d\kappa \Big) d\xi \|\psi\|^2. \end{aligned}$$

Hence, ($G2$) holds with l given by (10).

Consider

$$\mathscr{Z} = \left\{ z = \sum_{n=2}^{\infty} z_n s_n : \sum_{n=2}^{\infty} z_n^2 < +\infty \right\},$$

and

$$\|z\| = \left(\sum_{n=2}^{\infty} z_n^2 \right)^{1/2}.$$

Then $\mathscr{Z}$ is a Hilbert space. If $B_1 = B_2 \equiv B$ and

$$Bz = 2z_2 s_1(\xi) + \sum_{n=2}^{\infty} z_n s_n(\xi), \text{ for } z = \sum_{n=2}^{\infty} z_n s_n \in \mathscr{Z},$$

then, $B \in \mathscr{L}(\mathscr{Z}, \mathscr{H})$ and the adjoint B^* is given by

$$B^* v = (2v_1 + v_2)s_2(\xi) + \sum_{n=3}^{\infty} v_n s_n(\xi), \tag{11}$$

with $v = \sum_{n=1}^{\infty} v_n s_n(\xi) \in \mathscr{H}$.

It remains to verify condition ($G5$). Observe that, for the associated linear system corresponding to (8), the explicit expression for the fundamental solution $Q(t)$ is quiet difficult to write. On the other hand, for any $t \in [0,1]$, the expression for $Q(t)$ can be written easily, which guarantee that the condition ($G5$) holds. Indeed, the solution of the system

$$\begin{cases} \frac{d}{dt} W(t) = -AW(t) + L(W_t) + f(t), \quad t \in [0,2], \\ W_0 = 0 \end{cases}$$

on the interval $[0,1]$ is

$$W(t) = \int_0^t S(t-s) f(s) ds, \quad t \in [0,1].$$

Thus for any $t \in [0,1]$, $Q(t) = S(t)$ and hence

$$Q^*(t) = Q(t) = S(t).$$

It is proved in [17] that

$$dx(t,\xi) = (x_{\xi\xi}(t,\xi) + B_1 z(t,\xi))dt$$

is approximately controllable on $[0,2]$. That is,

$$\int_0^2 Q(2-t) B_1 B_1^* Q^*(2-t) dt > 0,$$

and meanwhile

$$\int_0^{h(2)=1} r'(t) Q(2-2t) B_2 B_2^* Q^*(2-2t) r'(t) dt > 0,$$

from which and Lemma 3.2, one can deduce, the linear system corresponding to (8) is approximately controllable, therefore ($G5$) is satisfied. Moreover, $\|F\| \leq L_1$. Further we can impose suitable conditions on the above defined functions to verify the assumptions of Theorem 3.4. Hence the approximate controllability of the system (8) on $[0,2]$ follows from Theorem 3.4.

Example 4.2 In Example 4.1 set

$$F(t,\psi)(\xi) = e^{t+\xi}e^{-\sqrt{\|\psi\|_{\mathscr{D}}}}. \tag{12}$$

Clearly $F(t,\psi)(\xi)$ does not satisfy the Lipschitz condition $(G3)$, however it satisfies the hypothesis $(G6)$. Thus Theorem 3.4 can not be applied to the system (8) if F is given by (12). On the other hand, since all hypotheses in Theorem 3.6 are fulfilled, for the function F defined by (12), the system (8) is approximately controllable on $[0,2]$. Hence, we conclude that Theorem 3.4 provides only sufficient conditions for the approximate controllability of the system (1).

5. Conclusion

In this paper, we discussed approximate controllability of impulsive semilinear differential equation having both state-dependent delay and variable delay in control. It is significant to find the approximate controllability for such systems as they are important from the theoretical and real life application aspect. In this work, we use a new control function, and theory of fundamental solution to get our results. We also use the resolvent operator, Krasnoselskii's and Schauder's fixed point theorems. Finally, the developed theory is validated with the help of two examples. This problem can be extended for the stochastic case, which is our future work. It is also interesting to develop theory for constrained controllability of an abstract system involving delays in control and state-dependent delay.

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References

[1] S.M. Abdal, S. Kumar, Approximate controllability of semilinear system involving state-dependent delay via fundamental solution, Ric. Mat. 69 (2020), 261–282.
[2] U. Arora, N. Sukavanam, Approximate controllability of second order semilinear stochastic system with variable delay in control and with nonlocal conditions, Rend. Circ. Mat. Palermo, 65 (2016), 307–322
[3] K. Balachandran, Y. Zhou, J. Kokila, Relative controllability of fractional dynamical systems with delays in control, Commun. Nonlinear. Sci. Numer. Simulat. 17 (2012), 3508–3520.
[4] G. Ballinger, X. Liu, Boundness for impulsive delay differential equations and applications to population growth models, Nonlinear Anal. 53 (2003), 1041–1062.
[5] S. Barnett, Introduction to Mathematical Control Theory, Clarendon press, Oxford, 1975.
[6] R.F. Curtain, H. Zwart, An Introduction to Infinite Dimensional Linear Systems Theory, Springer, New York, 1995.
[7] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, New York, 2000.
[8] S. Gao, L. Chen, J.J. Nieto, A. Torres, Analysis of a delayed epidemic model with pulse vaccination and saturation incidence, Vaccine, 24 (2006), 6037–6045.
[9] C. Gao, K. Li, E. Feng, Z. Xiu, Nonlinear impulsive system of fed-batch culture in fermentative production and its properties, Chaos Solitons & Fractals, 28 (2006), 271–277.
[10] J.K. Hale, J. Kato, Phase space for retarded equations with infinite delay, Funk. Ekvac. 21 (1978), 11–41.
[11] Y. Hino, S. Murakami, T. Naito, Functional Differential Equations with Infinite Delay, Springer-Verlag, Berlin, 1991.
[12] J. Klamka, On the controllability of linear systems with variable delay in control, Int. J. Control, 25 (1977), 875–883.
[13] J. Klamka, Stochastic controllability of systems with variable delay in control, Bull. Polon. A: Tech. 56 (2008), 279–284.
[14] S. Kumar, N. Sukavanam, Approximate controllability of fractional order semilinear systems with bounded delay, J. Diff. Equ. 252 (2012), 6163–6174.
[15] S. Kumar, N. Sukavanam, Controllability of semilinear systems with fixed delay in control, Opusc. Math. 35 (2015), 71–83.
[16] A. Lunardi, On the linear heat equation with fading memory, SIAM J. Math. Anal. 21 (1990), 1213–1224.
[17] N.I. Mahmudov, Controllability of linear stochastic systems in Hilbert spaces, J. Math. Anal. Appl. 259 (2001), 64–82.
[18] F.Z. Mokkedem, X. Fu, Approximate controllability for a semilinear evolution system with infinite delay, J. Dyn. Control Syst. 22 (2016), 71–89.
[19] F.Z. Mokkedem, X. Fu, Approximate controllability for a semilinear stochastic evolution system with infinite delay in L_p space, App. Math. Optim. 75 (2017), 253–283.

[20] F.Z. Mokkedem, X. Fu, Optimal control problems for a semilinear evolution system with infinite delay, Appl. Math. Optim. 79 (2019), 41–67.
[21] P. Muthukumar, C. Rajivganthi, Approximate controllability of impulsive neutral stochastic functional differential system with state-dependent delay in Hilbert spaces, J. Control Theory Appl. 11 (2013), 351–358.
[22] K. Naito, Controllability of semilinear control systems dominated by the linear part, SIAM J. Control Optim. 25 (1987), 715–722.
[23] J.W. Nunziato, On heat conduction in materials with memory, Quart. Appl. Math. 29 (1971), 187–204.
[24] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[25] R. Sakthivel, E.R. Anandhi, Approximate controllability of impulsive differential equations with state-dependent delay, Int. J. Control, 83 (2010), 387–393.
[26] S. Selvarasu, P. Kalamani, M.M. Arjunan, Approximate controllability of impulsive fractional stochastic integro-differential systems with state-dependent delay and poisson jump, J. Fractional Calculus Appl. 9(2) (2018), 229–254.
[27] L. Shen, Relative approximate controllability of abstract functional systems with infinite delay and delayed control, Math. Meth. Appl. Sci. 39 (2016), 5223–5232.
[28] L. Shen, J. Sun, Relative controllability of stochastic nonlinear systems with delay in control, Nonlinear Analysis: Real World Appl. 13 (2012), 2880–2887.
[29] B. Sikora, J. Klamka, Cone-type constrained relative controllability of semilinear fractional systems with delays, Kybernetika, 53 (2017), 370–381.
[30] B. Sikora, J. Klamka, Constrained controllability of fractional linear systems with delays in control, Systems Control Lett. 106 (2017), 9–15.
[31] E.D. Sontag, Mathematical Control Theory: Deterministic Finite Dimensional Systems, Springer-Verlag, New York, 1998.
[32] S. Tang, L. Chen, Density-dependent birth rate, birth pulses and their population dynamic consequences, J. Math. Biology, 44 (2002), 185–199.
[33] X. Zhang, X. Zhu, C Yuan, Approximate controllability of impulsive fractional stochastic differential equations with state-dependent delay, Adv. Difference Equ. (2015), 2015:91.