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On Partial Metric Preserving Functions and their Characterization

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Abstract.
In 1981, J. Borsík and J. Doboš characterized those functions that allow to transform a metric into another one in such a way that the topology of the metric to be transformed is preserved. Later on, in 1994, S.G. Matthews introduced a new generalized metric notion known as partial metric. In this paper, motivated in part by the applications of partial metrics, we characterize partial metric-preserving functions, i.e., those functions that help to transform a partial metric into another one. In particular we prove that partial metric-preserving functions are exactly those that are strictly monotone and concave. Moreover, we prove that the partial metric-preserving functions preserving the topology of the transformed partial metric are exactly those that are continuous. Furthermore, we give a characterization of those partial-metric preserving functions which preserve completeness and contractivity. Concretely, we prove that completeness is preserved by those partial metric-preserving functions that are non-bounded, and contractivity is kept by those partial metric-functions that satisfy a distinguished functional equation involving contractive constants. The relationship between metric-preserving and partial metric-preserving functions is also discussed. Finally, appropriate examples are introduced in order to illustrate the exposed theory.

1. Introduction

Throughout this paper we assume that the reader is familiar with the basics of general topology and metric spaces. Our main references for that topics are [8] and [15], respectively.

Transforming a metric into a new one by means of a function plays a crucial role in many real applications. The reasons for doing this may be of a very diverse nature. A typical situation consists in clarifying if a particular property is fulfilled by a subset or a mapping when verifying such a property is an arduous task in the metric space to be transformed. Another motivation for transforming may be to reduce running time of computing of an algorithm used to solve a problem or to express certain cost measures when modelling

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logistics problems (see, for instance, [5, 6, 17, 19]). Of course, the transformation is made in such a way that the aforesaid function preserves the main properties of the metric space to be transformed. The first studies about this type of functions go back to the original works by T.K. Sreenivasan ([18]), W.A. Wilson ([20]) and J.L. Kelley ([8]). Let us recall that a function $f : [0, \infty) \to [0, \infty)$ is metric-preserving provided that, for each metric space (X, d), the function d_f is a metric on X, where $d_f(x, y) = f(d(x, y))$ for all $x, y \in X$.

The importance and applicability of metric-preserving functions motivated that J. Borsík and J. Dobŏs tried to characterize the aforementioned functions in [3]. In the first instance they provided a partial description of metric-preserving functions. In order to state it, let us recall, following [3, 5], that a function $f : [0, \infty) \to [0, \infty)$ is said to be amenable provided that $f(a) = 0 \Leftrightarrow a = 0$. Moreover, f is (strictly) monotone if $f(a) \leq f(b)$ ($f(a) < f(b)$) whenever $a \leq b$ ($a < b$). Furthermore, f is called concave provided that $f(\lambda a + (1 - \lambda)b) \geq \lambda f(a) + (1 - \lambda)f(b)$ for all $\lambda \in [0, 1]$ and for all $a, b \in [0, \infty)$ and subadditive provided that $f(a + b) \leq f(a) + f(b)$ for all $a, b \in [0, \infty)$.

The next results give us the aforesaid partial information of metric-preserving functions.

Proposition 1.1. *Let $f : [0, \infty) \to [0, \infty)$ be a metric-preserving function, then f is amenable.*

Proposition 1.2. *Let $f : [0, \infty) \to [0, \infty)$ be an amenable, monotone and subadditive function, then f is a metric-preserving function.*

Proposition 1.3. *Let $f : [0, \infty) \to [0, \infty)$ be an amenable and concave function, then f is a metric-preserving function.*

In the light of Propositions 1.2 and 1.3, it seems natural to ask whether the converse of any of both results is true in general. However, there are metric-preserving functions which are not either monotone or concave (see, [5]). Motivated by this fact, Borsík and Doboš proved the announced characterization, given by Theorem 1.4, of metric-preserving functions by means of the so-called triangle triplets, where a triplet $(a, b, c) \in [0, \infty)^3$ forms a triangle triplet if $a \leq b + c$, $b \leq a + c$ and $c \leq b + a$.

Theorem 1.4. *Let $f : [0, \infty) \to [0, \infty)$. The below assertions are equivalent:*

(1) *f is a metric-preserving function.*
(2) *f holds the following properties:*
  (2.1) *f is amenable,*
  (2.2) *If (a, b, c) is a triangle triplet, then $(f(a), f(b), f(c))$ is so.*

As a consequence of the preceding result, the following one is obtained.

Corollary 1.5. *Every metric-preserving function is subadditive.*

In [3, 5], Borsík and Doboš also gave sufficient and necessary conditions in order to guarantee that a metric-preserving function preserves the topology, i.e., the topology induced by the transformed metric coincides with the topology induced by the original metric to be transformed. This type of metric-preserving functions are known as strong metric-preserving functions. In particular, they proved the following result.

Theorem 1.6. *Let $f : [0, \infty) \to [0, \infty)$ be a metric-preserving function. The below assertions are equivalent:*

(1) *f is strong metric-preserving function.*
(2) *f is continuous.*
(3) *f is continuous at 0.*

In 1994, S.G. Matthews introduced a new metric notion, which is known as partial metric, with the aim of developing an appropriate mathematical framework for modelling different processes that arise in a natural way in Computer Science and Artificial Intelligence (see [10]). Nowadays, the applicability of partial metric spaces covers areas like denotational semantics for programming languages, parallel processing, complexity analysis and logic programming (see [1, 7, 10, 11, 16]. Recall that, following [10], a partial metric space is a pair (X, p), where X is a non-empty set and $p : X \times X \to [0, \infty)$ is a function such that, for all $x, y, z \in X$, the following axioms are fulfilled:

(P1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

(P2) $p(x,x) \leq p(x,y)$;

(P3) $p(x,y) = p(y,x)$;

(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x;\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x;\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$.

A simple, but illustrative, example of partial metric space is given by the pair $([0,\infty), p_{\max})$, where $p_{\max}(x,y) = \max\{x,y\}$ for all $x,y \in [0,\infty)$. Notice that the topology $\tau_{p_{\max}}$ has as a base the family of open balls $\{[0, x+\varepsilon) : x \in [0,\infty)$ and $\epsilon \in (0,\infty)\}$.

The aforementioned applications of partial metric spaces have been obtained through fixed point techniques which are based on the Matthews fixed point theorem for self-mappings defined in complete partial metric spaces (see [10] for a detailed treatment of the topic). Let us recall the concepts of completeness and contractivity in partial metric spaces. Following [10], a mapping f from a partial metric space (X,p) into itself is said to be a p-contraction if there exists $c \in]0,1[$ such that $p(f(x),f(y)) \leq cp(x,y)$ for all $x,y \in X$. The constant c is called the contractive constant of the contraction f. A few applications of p-contractions can be found in [1, 2, 16].

Taking into account how a partial metric induces a topology, a sequence $(x_n)_{n\in\mathbb{N}}$ in (X,p) converges to $x \in X$ with respect to τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$. Besides, a sequence $(x_n)_{n\in\mathbb{N}}$ in (X,p) is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exist in $[0,\infty)$. As one can expect, a partial metric space (X,p) is called complete when every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in X$ with respect to τ_p such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Notice that, every metric space (X,d) is always a partial metric space such that $d(x,x) = 0$ for all $x \in X$. Of course, completeness and contractivity for self-mappings defined in metric spaces can be retrieved as a particular case of their counterparts when the partial metric is just replaced by a metric.

Since Borsík and Doboš characterized metric-preserving functions, different authors have extended their study to the case of generalized metric spaces. Among others, such a study has been made in the approach of partial metric spaces. Concretely, in 2012, S. Massanet and O. Valero posed the transformation problem in the framework of partial metric spaces. Specifically, they introduced the notion of partial metric-preserving function as follows (see [9] and, also, [2]): a function $f : [0,\infty) \to [0,\infty)$ is a partial metric-preserving function (shortly *pmp*-function), if for each partial metric space (X,p), the function p_f is a partial metric on X, where $p_f(x,y) = f(p(x,y))$ for each $x,y \in X$. A characterization of *pmp*-functions was given, in the spirit of Theorem 1.4, as follows:

Theorem 1.7. *Let* $f : [0,\infty) \to [0,\infty)$*. The below assertions are equivalent:*

(1) *f is a pmp-function.*

(2) *f holds the following properties for all $a,b,c,d \in [0,\infty)$:*

  (2.1) *$f(a) + f(b) \leq f(c) + f(d)$ whenever $a + b \leq c + d$, $b \leq c$ and $b \leq d$.*

  (2.2) *If $\max\{b,c\} \leq a$ and $f(a) = f(b) = f(c)$, then $a = b = c$.*

From the preceding result one can derive the following corollary.

Corollary 1.8. *Every pmp-function is monotone and subadditive.*

In spite of *pmp*-functions share the subadditivity with metric-preserving functions, the preceding corollary shows that they are not the same type of functions because of we have exposed that there are metric-preserving functions that are not monotone. Moreover, another differences can be easily stated. In particular, Proposition 1.1 does not hold in the partial metric framework. Indeed, there are *pmp*-functions which are not amenable such as the example below shows.

Example 1.9. Consider the function $f : [0,\infty) \to [0,\infty)$ given by $f(a) = a + 1$. Then it is not hard to check that f satisfies conditions (2.1) and (2.2) in the statement of Theorem 1.7 and, thus, it is a *pmp*-function. However, it is not amenable.

The next example warranties that Propositions 1.2 and 1.3 neither are true in the partial metric approach.

Example 1.10. Consider the function $f : [0,\infty) \to [0,\infty)$ defined by

$$f(a) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}.$$

It is clear that f is amenable, monotone, subadditive and concave. Nevertheless, f is not a partial metric-preserving function. Indeed, set $a = 3$, $b = 2$ and $c = 1$. Then $b \leq a$ and $c \leq a$ and $f(a) = f(b) = f(c) = 1$ but $a \neq b$. By (2.2) in Theorem 1.7, f is not a partial metric-preserving function.

Inspired by the fact that Proposition 1.3 gives a method to generate metric-preserving functions and that such a result cannot be extended directly to the framework of partial metric spaces, we focus our efforts in exploring what is the relationship between concavity and *pmp*-functions. Concretely, we show that there exists a deep and surprising connection between concave functions and *pmp*-functions. In fact, we prove that *pmp*-functions are exactly those concave functions that are strictly monotone. Hence we provide a new characterization of *pmp*-functions that, with respect to Theorem 1.4, sheds light on the real look of the functions under consideration. Moreover, we also characterize those *pmp*-functions that preserve the topology induced by the partial metric to be transformed. In particular, we prove that such *pmp*-functions are exactly the continuous at 0. Furthermore, motivated, on the one hand, by the fact that in many cases one needs, in order to solve an applied problem, to show that a self-mapping is a contraction in a metric space obtained by means of a transformation (a metric-preserving function) because to check the contractivity of such a mapping in the original and natural metric with which the space is endowed is an arduos task (see, for instance, [13]) and, on the other hand, by the fact that fixed point theory plays an important role in the applications of partial metric spaces to Computer Science and Artificial Intelligence, we go deeper into the study of *pmp*-functions and we give a characterization of those *pmp*-functions which preserve completeness and contractivity. Specifically, we prove that completeness is preserved by *pmp*-functions that are non-bounded, and that contractivity is preserved by those *pmp*-functions that satisfy a distinguished functional equation involving contractive constants. In this way we provide a method to induce new metrics from an old one in such a way that the two main components of every "metric" fixed point result are preserved in order to be able to apply an appropriate fixed point result best fitted to the problem to be solved. Finally, the exposed theory is illustrated with appropriate examples.

2. The New Characterization of Partial Metric-Preserving Functions

In this section we provide a new characterization of *pmp*-functions. To this end, the following lemmata will be essential.

Lemma 2.1. *Let $f : [0,\infty) \to [0,\infty)$ be a pmp-function. Then f is strictly monotone.*

Proof. Assume for the purpose of contradiction that f is not strictly monotone. Then, there exist $a, b \in [0,\infty)$, with $a < b$, such that $f(a) \geq f(b)$. Consider the partial metric space $([0,\infty), p)$, where $p(x,y) = \max\{x,y\}$ for each $x, y \in [0,\infty)$. Since f is a *pmp*-function we deduce that $([0,\infty), p_f)$ is a partial metric. Moreover, $p_f(a,a) = f(\max\{a,a\}) = f(a) \geq f(b) = f(\max\{a,b\}) = p_f(a,b)$ and so $p_f(a,a) = p_f(a,b)$. Furthermore, $p_f(b,b) = f(b) = f(\max\{a,b\}) = p_f(a,b)$. Then $p_f(a,a) = p_f(b,b) = p_f(a,b)$ and, thus, $a = b$ which contradicts the fact $a < b$. □

Observe that Lemma 2.1 refines part of the information provided by Corollary 1.8 about *pmp*-functions.

Lemma 2.2. *Let $f : [0,\infty) \to [0,\infty)$ be a strictly monotone function and let $a, b, c \in [0,\infty)$. If $b \leq a$, $c \leq a$ and $f(a) = f(b) = f(c)$, then $a = b = c$.*

Proof. Clearly the thesis is derived from the fact that every strictly monotone function is injective. □

The next result provides a particular method to generate new strictly monotone, concave and *pmp*-functions from old ones.

Lemma 2.3. *Let $f, g : [0,\infty) \to [0,\infty)$ be two functions such that $g(a) = f(a) - f(0)$ for all $a \in [0,\infty)$. Then the following assertions are hold:*

(1) *If f is strictly monotone and concave, then g is strictly monotone and concave.*
(2) *If f is a pmp-function, then g is a pmp-function.*

Proof. (1) Obviously g is strictly monotone. Next we prove that g is concave too. To this end, let $a, b \in [0,\infty)$ and $\lambda \in]0,1[$. Then,

$$g(\lambda a + (1-\lambda)b) = f(\lambda a + (1-\lambda)b) - f(0) \geq \lambda f(a) + (1-\lambda)f(b) - f(0) =$$

$$\lambda f(a) - \lambda f(0) + (1-\lambda)f(b) - (1-\lambda)f(0) = \lambda g(a) + (1-\lambda)g(b).$$

It follows that g is concave.

(2) By assertion (1) in Lemma 2.3 we have that g is strictly monotone. So, by Lemma 2.2, g satisfies condition (2.2) of Theorem 1.7. It remains to prove that g satisfies condition (2.1) of Theorem 1.7 in order to see that g is a *pmp*-function. With this aim, let $a, b, c, d \in [0,\infty)$ such that $a + b \leq c + d$, $b \leq c$ and $b \leq d$. Then,

$$g(a) + g(b) = f(a) - f(0) + f(b) - f(0) \leq f(c) - f(0) + f(d) - f(0) = g(c) + g(d),$$

since f is a *pmp*-function. Therefore, by Theorem 1.7, g is a *pmp*-function. □

Following similar arguments we can obtain the next lemma whose proof we omit.

Lemma 2.4. *Let $\alpha \in [0,\infty)$ and let $f, h : [0,\infty) \to [0,\infty)$ be two functions such that $h(a) = f(a) + \alpha$ for all $a \in [0,\infty)$. If f is strictly monotone and concave, then h is strictly monotone and concave.*

The next result was proved in [5, Theorem 2 in Chapter 1].

Lemma 2.5. *Let $f : [0,\infty) \to [0,\infty)$ be an amenable function. Then the following assertions are equivalent:*

(1) *f is concave.*
(2) *If $a, b, c \in [0, d]$ with $d \in (0,\infty)$ and $a + d = b + c$, then $f(a) + f(d) \leq f(b) + f(c)$.*

The next result states a relationship between condition (2.1) in Theorem 1.7 and condition (2) in Lemma 2.5 when strictly monotone functions are under consideration.

Lemma 2.6. *Let $f : [0,\infty) \to [0,\infty)$ be a strictly monotone function. Then the following assertions are equivalent:*

(1) *If $a, b, c \in [0, d]$ with $d \in (0,\infty)$ and $a + d = b + c$, then $f(a) + f(d) \leq f(b) + f(c)$.*
(2) *If $a, b, c, d \in [0,\infty)$, then $f(a) + f(b) \leq f(c) + f(d)$ whenever $a + b \leq c + d$, $b \leq c$ and $b \leq d$.*

Proof. (1) $\Rightarrow$ (2) Let $a, b, c, d \in [0,\infty)$ such that $a + b \leq c + d$, $b \leq c$ and $b \leq d$. Take $t = c + d - b$. Since $b \leq c$ and $b \leq d$ we have that $t \in [0,\infty)$. However if $t = 0$, then an easy computation shows that $a = b = c = d = 0$ and, thus, condition (2) is hold. So we can assume that $t > 0$. Clearly, $b, c, d \in [0, t]$ and, in addition, $b + t = c + d$. So, by (1), we have that $f(b) + f(t) \leq f(c) + f(d)$. Moreover the facts that $a \leq c + d - b = t$ and f is strictly monotone give that $f(a) \leq f(t)$. Thus we conclude that

$$f(a) + f(b) \leq f(b) + f(t) \leq f(c) + f(d).$$

(2) $\Rightarrow$ (1) Let $d \in (0,\infty)$ and consider $a, b, c \in [0, d]$ such that $a + d = b + c$. Assume that $a > c$. Then $a + d > c + d \geq c + b = a + d$, which is a contradiction. Whence we deduce that $a \leq c$. Similarly one can prove that $a \leq b$. Therefore $a \leq c$ and $a \leq b$. Then $f(a) + f(d) \leq f(b) + f(c)$ as we claimed. □

In the light of the exposed results we are able to prove the promised new characterization of *pmp*-functions.

Theorem 2.7. *Let $f : [0,\infty) \to [0,\infty)$ be a function. Then the following assertions are equivalent:*

(1) *f is a pmp-function.*
(2) *f is strictly monotone and concave.*

Proof. Define the function $g : [0,\infty) \to [0,\infty)$ given by $g(a) = f(a) - f(0)$ for all $a \in [0,\infty)$.

(1) ⇒ (2) By assertion (2) of Lemma 2.3 we have that g is a *pmp*-function. Lemma 2.1 warrantees that g is strictly monotone. Since g is a *pmp*-function g satisfies condition (2.1) in Theorem 1.7 and, thus, Lemma 2.6 guarantees that, given $d \in (0,\infty)$, $g(a) + g(d) \leq g(b) + g(c)$ whenever $a, b, c \in [0, d]$ with $d \in (0,\infty)$ and $a + d = b + c$. This last condition, by Lemma 2.5, is equivalent to the concavity of g, since g is amenable. Now, by Lemma 2.4, f is concave and strictly monotone, since $f(a) = g(a) + f(0)$ for all $a \in [0,\infty)$.

(2) ⇒ (1) By assertion (1) in Lemma 2.3 we have that g is strictly monotone and concave. On the one hand, Lemma 2.5 gives that, given $d \in (0,\infty)$, $g(a) + g(d) \leq g(b) + g(c)$ whenever $a, b, c \in [0, d]$ with $d \in (0,\infty)$ and $a + d = b + c$. It follows that, given $d \in (0,\infty)$, $f(a) + f(d) \leq f(b) + f(c)$ whenever $a, b, c \in [0, d]$ with $d \in (0,\infty)$ and $a + d = b + c$. From Lemma 2.4, we deduce that f is strictly monotone. Then Lemma 2.6 yields that, for each $a, b, c, d \in [0,\infty)$, $f(a) + f(b) \leq f(c) + f(d)$ whenever $a + b \leq c + d$, $b \leq c$ and $b \leq d$. Whence we obtain that f fulfills condition (2.1) in Theorem 1.7. Moreover, by Lemma 2.2, f satisfies condition (2.2) in Theorem 1.7. Therefore, the aforesaid theorem warrantees that f is a *pmp*-function. □

It must be stressed that, when comparing with Propositions 1.2 and 1.3, Theorem 1.7 shows great differences between metric-preserving functions and *pmp*-functions.

Taking into account Theorems 1.7 and 2.7 we can unify both independent characterizations of *pmp*-functions.

Corollary 2.8. *Let $f : [0,\infty) \to [0,\infty)$ be a function. The the following assertions are equivalent:*

(1) *f is a pmp-function.*
(2) *f is strictly monotone and concave.*
(3) *f holds the following properties:*
   (3.1) *$f(a) + f(b) \leq f(c) + f(b)$ whenever $a + b \leq c + d$, $b \leq c$ and $b \leq d$.*
   (3.2) *If $b \leq a$, $c \leq a$ and $f(a) = f(b) = f(c)$, then $a = b = c$.*

In [5, Theorem 1 in Chapter 1], the next result for metric-preserving functions was proved.

Proposition 2.9. *Let $f : [0,\infty) \to [0,\infty)$ be an amenable function. If f is concave, then f is a metric-preserving function.*

From Proposition 2.9 and Theorem 2.7 we derive the next interesting relationship between *pmp*-functions and metric-preserving functions.

Corollary 2.10. *Let $f : [0,\infty) \to [0,\infty)$ be an amenable function. If f is a pmp-function, then f is a metric-preserving function.*

Notice that the preceding result is consistent with the fact that the same conclusion can be obtained from Corollary 1.8 and Proposition 1.2. Moreover, recall that Example 1.10 shows that the converse of Corollary 2.10 is not verified. Finally, it must be pointed out that amenable *pmp*-functions match up with those functions named metric transforms in the sense of L.M. Blumenthal (see [4]).

3. Strong Partial Metric-Preserving Functions

In this section we focus our attention to discern if *pmp*-functions are able to preserve the topology in the spirit of strong metric-preserving functions, i.e., the topology induced by the transformed partial metric space is equivalent to the topology induced by the partial metric space to be transformed. So the main target of this section is to get a version of Theorem 1.6 in the framework of partial metric spaces.

As pointed out in Section 1, each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x;\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x;\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$. Taking this fact into account, it can be proved easily that two partial metrics p_1 and p_2 on a set X are topologically equivalent (induce the same topology) if and only if for each $x \in X$ and each $\epsilon \in (0,\infty)$ there exists $\delta \in (0,\infty)$ such that

$$B_{p_1}(x;\delta) \subseteq B_{p_2}(x;\epsilon) \text{ and } B_{p_2}(x;\delta) \subseteq B_{p_1}(x;\epsilon).$$

It seems natural to wonder whether always a *pmp*-function preserves topologies, that is, the topology induced by the transformed partial metric space coincides with the topology induced by the partial metric space to be transformed through the *pmp*-function. Nevertheless, as in the classical metric case, this is not the case such as the next example shows.

Example 3.1. Consider the function $f : [0,\infty) \to [0,\infty)$ given by

$$f(a) = \begin{cases} 0 & \text{if } a = 0 \\ \frac{1+a}{2+a} & \text{if } a \in (0,\infty) \end{cases}.$$

It not hard to check that f is strictly monotone and concave. So, by Theorem 2.7, f is a *pmp*-function. Moreover, consider the partial metric space $([0,\infty), p_{\max})$ introduced in Section 1. Furthermore, the partial metric $p_{\max_f}$ is given by

$$p_{\max_f}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{1+\max\{x,y\}}{2+\max\{x,y\}} & \text{otherwise} \end{cases}.$$

Obviously $B_{p_{\max_f}}(0;\frac{1}{4}) = \{0\}$ and $B_{p_{\max}}(0;\delta) = [0,\delta)$ for each $\delta \in (0,\infty)$. Consequently, the topologies $\tau_{p_{\max}}$ and $\tau_{p_{\max_f}}$ are not the same. Whence we conclude that the partial metrics $p_{\max}$ and $p_{\max_f}$ are not topologically equivalent.

Since *pmp*-functions do not preserve, in general, topologies it makes sense that we introduce the following notion.

Definition 3.2. A partial metric-preserving function $f : [0,\infty) \to [0,\infty)$ is said to be strong if for each partial metric space (X,p) the partial metrics p and p_f are topologically equivalent.

Similarly to the classical metric case, the *pmp*-function introduced in Example 3.1 fails to be continuous at 0. So apparently the continuity will play a fundamental role so that a *pmp*-function to be strong. Inspired by this fact we discuss the continuity of a *pmp*-function in the result below.

Lemma 3.3. *Let $f : [0,\infty) \to [0,\infty)$ be a pmp-function. The following assertions are equivalent:*

(1) f is continuous.
(2) f is continuous at 0.

Proof. (1) ⇒ (2) Clearly if f is continuous, then f is continuous at 0.

(2) ⇒ (1) We will distinguish two possible cases:

Case 1. f is amenable. Then, by Corollary 2.10, f is a metric-preserving function continuous at 0. Then the continuity of f follows from Theorem 1.6.

Case 2. f is not amenable. Then, by assertion (2) in Lemma 2.3, the function $g : [0,\infty) \to [0,\infty)$, given by $g(a) = f(a) - f(0)$, for all $a \in [0,\infty)$ is a *pmp*-function. Clearly g is amenable and continuous at 0. Then, by Case 1, g is continuous and so, obviously, f is continuous. □

The next theorem characterizes strong *pmp*-functions extending Theorem 1.6 to the new context under consideration.

Theorem 3.4. *Let $f : [0,\infty) \to [0,\infty)$ be a pmp-function. Then the following assertions are equivalent:*

(1) *f is strong.*
(2) *f is continuous.*
(3) *f is continuous at* 0.

Proof. By Lemma 3.3, (2) $\Leftrightarrow$ (3). So it remains to prove (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2) Consider the partial metric space $([0,\infty), e)$, where $e(x, y) = |x - y|$ for each $x, y \in [0,\infty)$. Since f is strong we have that the partial metrics e_f and e are topologically equivalent. Next we show that f is continuous at 0. To this end, fix $\epsilon \in (0,\infty)$. Then, taking into account that e and e_f are topologically equivalent, there exists $\delta \in (0,\infty)$ such that $B_e(0;\delta) \subseteq B_{e_f}(0;\epsilon)$. Thus, given $y \in B_e(0;\delta)$ then $y \in B_{e_f}(0;\epsilon)$, i.e., $e(0, y) < e(0, 0) + \delta$ implies $f(e(0, y)) < f(e(0, 0)) + \epsilon$. Therefore, for each $y \in [0,\delta[$, we have that $f(y) < f(0) + \epsilon$. Whence f is continuous at 0. Hence f is continuous by Lemma 3.3.

(2) $\Rightarrow$ (1) Consider a partial metric space (X, p). Let $x \in X$ and $\epsilon \in (0,\infty)$.

First we show that there exists $\delta_1 \in (0,\infty)$ such that $B_p(x;\delta_1) \subseteq B_{p_f}(x;\epsilon)$. With this aim, set $a_0 = p(x, x) \in [0,\infty)$. The continuity of f at a_0 gives the existence of $\delta_1 \in (0,\infty)$ such that for each $b \in (a_0 - \delta_1, a_0 + \delta_1) \cap [0,\infty)$ we have that $|f(a) - f(a_0)| < \epsilon$. Take $y \in B_p(x;\delta_1)$. Then $p(x, y) < p(x, x) + \delta_1$ and so $p(x, y) \in [a_0, a_0 + \delta_1[$. It follows that $|f(p(x, y)) - f(p(x, x))| < \epsilon$. Since f is strictly monotone we have that $f(p(x, y)) < f(p(x, x)) + \epsilon$. Whence $B_p(x;\delta_1) \subseteq B_{p_f}(x;\epsilon)$.

Next we show that there exists $\delta_2 \in (0,\infty)$ such that $B_{p_f}(x;\delta_2) \subseteq B_p(x;\epsilon)$. The strictly monotony and continuity of f provides that $f([0,\infty))$ is the interval either $[f(0),\infty)$ or $[f(0), b]$ with $b \in (0,\infty)$. Moreover, we have that there exists the inverse f^{-1} of f which is continuous. Let $c_0 = f(p(x, x)) \in f([0,\infty))$. By continuity of f^{-1} at c_0 there exists $\delta_2 \in (0,\infty)$ such that for each $c \in]c_0 - \delta_2, c_0 + \delta_2[\cap f([0,\infty))$ we have that $|f^{-1}(c) - f^{-1}(c_0)| < \epsilon$. Take $y \in B_{p_f}(x;\delta_2)$. It follows that $f(p(x, y)) \in]c_0 - \delta_2, c_0 + \delta_2[\cap f([0,\infty))$ and, thus, that $|f^{-1}(f(p(x, y))) - f^{-1}(f(p(x, x)))| < \epsilon$. Whence we deduce that $|p(x, y) - p(x, x)| < \varepsilon$. Hence $B_{p_f}(x;\delta_2) \subseteq B_p(x;\epsilon)$.

Therefore, taking $\delta = \min\{\delta_1, \delta_2\}$, we have that $B_p(x;\delta) \subseteq B_{p_f}(x;\epsilon)$ and $B_{p_f}(x;\delta) \subseteq B_p(x;\epsilon)$. So, we conclude that f is strong. □

The next example provides instances of strong *pmp*-functions.

Example 3.5. Let $\alpha, \beta \in (0,\infty)$. The following functions $f_\alpha : [0,\infty) \to [0,\infty)$ are strong *pmp*-functions, where for all $a \in [0,\infty)$ they are defined as follows:

(1) $f_\alpha(a) = (a + \alpha)^\beta$ with $\beta \in (0, 1]$.
(2) $f_\alpha(a) = \alpha a + \beta$.
(3) $f_\alpha(a) = \frac{\alpha a}{1+a}$.
(4) $f_\alpha(a) = \frac{1+\alpha a}{2+\alpha a}$.
(5) $f_\alpha(a) = \log_\beta(\alpha + a)$ with $\alpha, \beta \in (1,\infty)$.
(6) $f_\alpha(a) = 1 - e^{-\alpha a}$.

Observe that in the preceding example the instances from (1) until (5) are strong *pmp*-functions that are not metric-preserving functions. Moreover, the instance (6) is a strong *pmp*-function that is, at the same time, strong metric-preserving. This last fact inspires the next result which discusses the relationship between strong *pmp*-functions and strong metric-preserving functions.

Corollary 3.6. *Let $f : [0,\infty) \to [0,\infty)$ be an amenable strong pmp-function. Then f is a strong metric-preserving function.*

Proof. By Corollary 2.10 we have that f is a metric-preserving function. Moreover, by Theorem 3.4, we obtain that f is continuous. Theorem 1.6 gives that f is a strong metric-preserving function. □

The converse of the preceding result is not satisfied.

Example 3.7. Consider the function $f : [0, \infty) \to [0, \infty)$ given by $f(x) = 3x - 2|x - 1| + |x - 2|$ for all $x \in [0, \infty)$. According to [5] (see also [3]), f is an amenable, monotone, subadditive and continuous function. So, By Theorem 1.6, we deduce that f is a strong metric-preserving function. However, it is not hard to check that f is not concave and, thus by Theorem 2.7, f is not a *pmp*-function.

4. Completeness, Contractions and Partial Metric-Preserving Functions

The objective of this section is twofold. On the one hand, we are interested in stating those conditions that a *pmp*-function f must satisfy in order to guarantee that, given a complete partial metric space (X, p), the new induced partial metric space (X, p_f) is, again, complete. So we want to study when a *pmp*-function preserves completeness. On the other hand, we focus our efforts on getting those conditions about *pmp*-functions that help us to induce a new partial metric from an old one in such a way that the contractivity condition for a self-mapping is kept.

4.1. Preserving Completeness

In order to discuss when *pmp*-functions preserve completeness, the next result will play a crucial role.

Lemma 4.1. *Let $f : [0, \infty) \to [0, \infty)$ be a strictly monotone continuous function. The following assertions are equivalent:*

(1) *f is surjective on $[f(0), \infty)$.*
(2) *f is non-bounded ($f([0, \infty)) = [f(0), \infty)$).*

Proof. (1) ⇒ (2) The strictly monotony and continuity of f provides that $f([0, \infty))$ is an interval. Clearly the sujectivity of f gives that such an interval is $[f(0), \infty)$. Indeed, let $M \in (f(0), \infty)$. Then there exists $a_M \in (0, \infty)$ such that $f(a_M) = M$. Notice that the strictly monotony of f guarantees that $a_M \neq 0$ whenever $M > f(0)$.

(2) ⇒ (1) Let $M \in (f(0), \infty)$. Then there exists $a_M \in (0, \infty)$ such that $f(a_M) > M$. Thus we have that $f(0) < M < f(a_M)$. Since f is continuous the Darboux's theorem provides the existence of $b_M \in (0, a_M)$ such that $f(b_M) = M$. Therefore f is surjective on $[f(0), \infty)$. □

The next result fixes the condition that must be taken under consideration in order to guarantee that a *pmp*-function preserves completeness.

Theorem 4.2. *Let $f : [0, \infty) \to [0, \infty)$ be a non-bounded strong pmp-function and let (X, p) be a partial metric space. The following assertions are equivalent:*

(1) *(X, p) is complete.*
(2) *(X, p_f) is complete.*

Proof. (1) ⇒ (2) First of all we show that every Cauchy sequence in (X, p_f) is a Cauchy sequence in (X, p). Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, p_f). Then there exists $b_0 \in [0, \infty)$ such that $\lim_{n,m\to\infty} p_f(x_n, x_m) = b_0$. Next we prove that there exists $a_0 \in [0, \infty)$ with $f(a_0) = b_0$. Indeed, the fact that $\lim_{n,m\to\infty} p_f(x_n, x_m) = b_0$ gives that, for each $\epsilon \in (0, \infty)$, we can find $n_0 \in \mathbb{N}$ satisfying $|p_f(x_n, x_m) - b_0| < \epsilon$ for all $n, m \geq n_0$. Thus $b_0 - \epsilon < p_f(x_n, x_m) < b_0 + \epsilon$ for all $n, m \geq n_0$. Whence we deduce that $f(0) \leq f(p(x_n, x_m)) = p_f(x_n, x_m) < b_0 + \epsilon$ for all $\epsilon \in (0, \infty)$. It follows that $f(0) \leq b_0$. By Lemma 4.1, there exists $a_0 \in [0, \infty)$ with $f(a_0) = b_0$. Since f is strictly monotone and continuous we have warranted the existence of the inverse f^{-1} of f on $[f(0), \infty)$ which is continuous. The continuity of f^{-1} and the fact that $\lim_{n,m\to\infty} p_f(x_n, x_m) = f(a_0)$ yield that $\lim_{n,m\to\infty} f^{-1}(f(p(x_n, x_m))) = f^{-1}(f(a_0))$. So $\lim_{n,m\to\infty} p(x_n, x_m) = a_0$ and, hence, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, p).

It remains to prove that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in X$ such that

$$p_f(x, x) = \lim_{n,m\to\infty} p_f(x_n, x_m) = \lim_{n\to\infty} p_f(x, x_n).$$

Since (X, p) is complete we have the existence of x such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x, x_n)$. The continuity of f gives immediately the desired conclusion.

(2) ⇒ (1) The proof runs following similar arguments to those given for (1) ⇒ (2). □

In view of the preceding result, one can pose the question whether the result remains true when the non-bounded character of the strong *pmp*-function is deleted from its statement. The next example shows that the answer to the posed question is negative

Example 4.3. Consider the complete partial metric space $([0, \infty), p_{\max})$ and the strong *pmp*-function $f : [0, \infty) \to [0, \infty)$ given by $f(a) = \frac{1+a}{2+a}$. Clearly f is bounded because $f([0, \infty)) = [\frac{1}{2}, 1)$. Take the sequence $(x_n)_{n\in\mathbb{N}}$ in $([0, \infty), p_{\max_f})$ with $x_n = n$ for all $n \in \mathbb{N}$. Then it is Cauchy in $([0, \infty), p_{\max_f})$. Indeed, $\lim_{n,m\to\infty} p_{\max_f}(x_n, x_m) = \lim_{n,m\to\infty} p_{\max_f}(n, m) = 1$ because, given $\epsilon \in (0, \infty)$, there exists $n_0 \in \mathbb{N}$ ($n_0 > \frac{1-2\epsilon}{\epsilon}$) such that for all $n, m \geq n_0$ we have that

$$1 - p_{\max_f}(x_n, x_m) = 1 - \frac{1 + \max\{n, m\}}{2 + \max\{n, m\}} \leq \frac{1}{2 + n_0} < \epsilon.$$

However, $(x_n)_{n\in\mathbb{N}}$ is not convergent with respect to $\tau(p_{\max_f})$ and, thus, $([0, \infty), p_{\max_f})$ is not complete. Notice that $1 \notin f([0, \infty))$ and, thus, by Lemma 4.1, f is not non-bounded.

According to [14], a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X, p) is said to be 0-convergent to $x \in X$ provided that it converges to x with respect to $\tau(p)$ and $p(x, x) = 0$. Besides, $(x_n)_{n\in\mathbb{N}}$ is called 0-Cauchy whenever $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Furthermore, a partial metric space (X, p) is said to be 0-complete if every 0-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ 0-converges, with respect to $\tau(p)$, to a point $x \in X$. Of course, every complete partial metric space is 0-complete but the converse is not true in general.

Following similar arguments to those given in the proof of Theorem 4.2 one can obtain the next surprising result.

Theorem 4.4. *Let $f : [0, \infty) \to [0, \infty)$ be an amenable strong pmp-function. The following assertions hold:*

(1) *If (X, p) is a partial metric space then (X, p) is 0-complete if and only if (X, p_f) is 0-complete.*
(2) *If (X, d) is a metric space then (X, d) is complete if and only if (X, d_f) is complete.*

4.2. Preserving Contractivity

To discern what conditions allow *pmp*-functions to keep contractivity of self-mappings, let us introduce the following pertinent notion.

Definition 4.5. Let $f : [0, \infty) \to [0, \infty)$ be a *pmp*-function. We will say that f is contraction-preserving provided that, for each partial metric space (X, p), every p-contraction is also a p_f-contraction.

Instance (4) in Example 3.5 is an example of *pmp*-function which is not contranction-preserving. Nonetheless, the function $f : [0, \infty) \to [0, \infty)$ defined by $f(a) = \alpha a$ for all $a \in [0, \infty)$ and with $\alpha \in (0, \infty)$ is an example of *pmp*-function which is, in addition, contraction-preserving.

The next result gives a characterization of those *pmp*-functions which preserve contractive mappings.

Theorem 4.6. *Let $f : [0, \infty) \to [0, \infty)$ be a pmp-function. The following assertions are equivalent:*

(1) *f is contraction-preserving.*

(2) *For each $k \in]0,1[$ there exists $c \in]0,1[$ such that $f(ka) \leq cf(a)$ for all $a \in [0,\infty)$.*

Proof. (1) ⇒ (2) For the purpose of contradiction, suppose that there exists $k_0 \in]0,1[$ such that for each $c \in]0,1[$ we can find $a_c \in [0,\infty)$ satisfying $f(k_0 a_c) > cf(a_c)$. Next we show that f is not contranction-preserving. Indeed, consider the partial metric space $([0,\infty), p_{\max})$ and define the mapping $T : [0,\infty) \to [0,\infty)$ by $T(x) = k_0 x$ for all $x \in [0,\infty)$. Then

$$p_{\max}(T(x), T(y)) = \max\{k_0 x, k_0 y\} = k_0 \max\{x, y\} = k_0 p_{\max}(x, y)$$

for all $x, y \in [0,\infty)$ and, hence, T is a $p_{\max}$-contraction. However,

$$p_{\max_f}(T(a_c), T(0)) = f(\max\{k_0 a_c, 0\}) > cf(a_c) = cf(\max\{a_c, 0\}) = cp_{\max_f}(a_c, 0)$$

and, thus, we have seen that for each $c \in]0,1[$ we can find $x, y \in [0,\infty)$ such that

$$p_{\max_f}(T(x), T(y)) > cp_{\max_f}(x, y),$$

which contradicts the fact that f is a contraction-pressrving.

(2) ⇒ (1) Let (X, p) be a partial metric space and let $T : X \to X$ be a p-contraction. Then there exists $k_0 \in]0,1[$ such that

$$p(T(x), T(y)) \leq k_0 p(x, y)$$

for all $x, y \in X$. Moreover, given k_0, there exists $c_0 \in]0,1[$ such that $f(k_0 a) \leq c_0 f(a)$ for all $a \in [0,\infty)$. Taking in mind that f is strictly monotone, it follows that

$$p_f(T(x), T(y)) = f(p(T(x), T(y))) \leq f(k_0 p(x, y)) \leq c_0 f(p(x, y)) = c_0 p_f(x, y)$$

for all $x, y \in [0,\infty)$. Therefore, T is a p_f-contraction and so f is contraction-preserving. □

The preceding result allows us to find non-trivial examples of *pmp*-functions which are contraction-preserving.

Example 4.7. Let $\alpha \in (0,\infty)$. Define the function $f : [0,\infty) \to [0,\infty)$ by $f(a) = \sqrt{a^2 + \alpha a}$ for all $a \in [0,\infty)$. It is not hard to check that f is strictly monotone and concave. Then, by Theorem 2.7, f is a *pmp*-function.

Now, let $k \in]0,1[$ and take $c = \sqrt{k}$. Then we have that

$$f(ka) = \sqrt{k^2 a^2 + \alpha k a} \leq \sqrt{c^2 a^2 + \alpha c^2 a} = c\sqrt{a^2 + \alpha a} = cf(a)$$

for all $a \in [0,\infty)$. Therefore, by Theorem 4.6, f is contraction-preserving.

It is worthy to stress that Example 4.7 clarifies that *pmp*-functions being contraction-preserving are not reduced to those that are homogeneous. Recall that, according to [12], a function $f : [0,\infty) \to [0,\infty)$ is homogeneous provided that $f(\alpha a) = \alpha f(a)$ for all $a, \alpha \in (0,\infty)$.

It seems natural to wonder if there exists any relationship between those *pmp*-functions that are contraction-preserving and strong. The next result makes clear such a question.

Corollary 4.8. *Let $f : [0,\infty) \to [0,\infty)$ be a pmp-function. If f is contraction-preserving, then the following assertions hold:*

(1) *f is amenable.*
(2) *f is a strong pmp-function.*
(3) *f is a strong metric-preserving function.*

Proof. (1) From Theorem 4.6 we deduce that $f(0) = 0$. Then the fact that f is strictly monotone gives that $f(a) = 0 \Leftrightarrow x = 0$.

(2) Next we show that f is continuous at 0. To this end, fix $k \in (0, 1)$. Then there exists $c \in (0, 1)$ such that $f(ka) \leq cf(a)$ for all $a \in [0, \infty)$. It follows that $f(k) \leq cf(1)$ and that $f(k^n) \leq c^n f(1)$ for all $n \in \mathbb{N}$. Suppose for the purpose of contradiction that f is not continuous at 0. By Corollary 2.10 in [3], every metric-preserving function g which is discontinuous at 0 satisfies that there exists $\epsilon \in (0, \infty)$ such that $\epsilon \leq g(a)$ for all $a \in (0, \infty)$. By Proposition 1.2 we have that f is metric preserving. Then there exists $\epsilon \in (0, \infty)$ such that $\epsilon \leq f(a)$ for all $a \in (0, \infty)$. Moreover, there exists $n_0 \in \mathbb{N}$ with $c^n f(1) < \epsilon$ for all $n \geq n_0$. Whence we obtain that $\epsilon \leq f(k^n) \leq c^n f(1) < \epsilon$ for all $n \geq n_0$, which is a contradiction. Therefore f is continuous at 0. By Theorem 3.4 we conclude that f is a strong *pmp*-function.

(3) Since f is an amenable a strong *pmp*-function Corollary 3.6 guarantees that f is a strong metric-preserving function. □

The function introduced in Example 4.3 is an instance of strong *pmp*-function which is not contraction-preserving.

In the light of Theorem 4.6 and the proof of Corollary 4.8 we derive the next result whose easy proof we omit.

Corollary 4.9. *Let $f : [0, \infty) \to [0, \infty)$ be a contraction-preserving pmp-function. If (X, p) is a partial metric space and T is a p-contraction with contractive constant k, then there exists $c_k \in (0, 1)$ such that T is a p_f-contraction with contractive constant c_k and, in addition, $f(k) \leq c_k f(1)$.*

In view of the preceding result, it seems natural to ask for those conditions that allow contraction-preserving functions preserve the contractive constant, that is, T is a p_f-contraction with contractive constant k whenever T is a p-contraction with contractive constant k. The next result clarifies this situation.

Corollary 4.10. *Let $f : [0, \infty) \to [0, \infty)$ be a contraction-preserving pmp-function. The following assertions are equivalent:*

(1) *f preserves the contractive constant of every p-contraction.*
(2) *$f(ka) = kf(a)$ for all $k \in (0, 1)$ and $a \in [0, \infty)$.*

Proof. (1) $\Rightarrow$ (2) Consider the partial metric space $([0, \infty), p_{\max})$ and $k \in (0, 1)$. Define the mapping $T_k : [0, \infty) \to [0, \infty)$ by $T_k(x) = kx$ for all $x \in [0, \infty)$. It is clear that T is a $p_{\max}$-contraction with k as contractive constant. Since f preserves the contractive constant we have that T_k is a $p_{\max_f}$-contraction with k as contractive constant. Then

$$f(ka) = f(\max\{ka, 0\}) = p_{\max_f}(T_k(a), T_k(0)) \leq k p_{\max_f}(a, 0) = kf(a)$$

for all $a \in [0, \infty)$. Moreover, by Theorem 2.7, we have that f is concave. Thus $kf(a) \leq f(ka + (1 - k)0) = f(ka)$. Whence we conclude that $f(ka) = kf(a)$ for all $k \in (0, 1)$ and $a \in [0, \infty)$.

(2) $\Rightarrow$ (1). It is obvious. □

It is clear that replacing, in Definition 4.5, *pmp*-functions and partial metric spaces by metric-preserving functions and metric spaces respectively, we obtain a contraction-preserving notion for the classical metric case. From now on, this type of functions will be called metric-contraction-preserving. The next result states a surprising relation between both type of functions.

Theorem 4.11. *Let $f : [0, \infty) \to [0, \infty)$ be a pmp-function. The following assertions are equivalent:*

(1) *f is contraction-preserving.*
(2) *f is metric-contraction-preserving.*

Proof. (1) $\Rightarrow$ (2) By Corollary 4.8 f is a strong metric-preserving function. Then, given a metric space (X, d), d_f is again a metric on X. By Theorem 4.6 we have that for each $k \in]0,1[$ there exists $c \in]0,1[$ such that $f(ka) \leq cf(a)$ for all $a \in [0, \infty)$. Next consider a d-contraction $T : X \to X$ with contractive constant k_0. Then, there exist $c_0 \in (0, 1)$ such that

$$d_f(T(x), T(y)) = f(d(T(x), T(y))) \leq f(k_0 p(x, y)) \leq c_0 f(d(x, y)) = c_0 d_f(x, y)$$

for all $x, y \in [0, \infty)$. Therefore, T is a d_f-contraction and so f is metric-contraction-preserving.

(2) $\Rightarrow$ (1). Assume that f is not contraction-preserving. By Theorem 4.6 we have that there exists $k_0 \in]0,1[$ such that for each $c \in]0,1[$ we can find $a_c \in [0, \infty)$ satisfying $f(k_0 a_c) > cf(a_c)$. Consider the partial metric space $([0, \infty), p_{\max})$ and define the mapping $T : [0, \infty) \to [0, \infty)$ by $T(x) = k_0 x$ for all $x \in [0, \infty)$. Then it is clear that T is a $p_{\max}$-contraction. Define the mapping $d_p : [0, \infty) \times [0, \infty) \to [0, \infty)$ by

$$d_{p_{\max}}(x, y) = \begin{cases} 0 & \text{if } x = y \\ p_{\max}(x, y) & \text{if } x \neq y \end{cases} .$$

It is easy to see that $d_{p_{\max}}$ is a metric on $[0, \infty)$ (compare [7]) and that T is a $d_{p_{\max}}$-contraction. Since f is a metric-contraction-preserving function we have that T is a $d_{{p_{\max}}_f}$-contraction. Hence, there exists $k_1 \in (0, 1)$ such that

$$d_{{p_{\max}}_f}(T(x), T(y)) \leq k_1 d_{{p_{\max}}_f}(x, y)$$

for all $x, y \in [0, \infty)$. Concretely, we have that

$$p_{\max_f}(T(x), T(y)) \leq k_1 p_{\max_f}(x, y)$$

for all $x, y \in [0, \infty)$ with $x \neq y$. It follows that

$$f(k_0 a_{k_1}) = p_{\max_f}(T(a_{k_1}), T(0))) \leq k_1 p_{\max_f}(a_{k_1}, 0) = k_1 f(a_{k_1}).$$

Nevertheless $k_1 f(a_{k_1}) < f(k_0 a_{k_1}) \leq k_1 f(a_{k_1})$, which is a contradiction. Consequently f is contraction-preserving. □

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