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On Ramsey Properties, Function Spaces, and Topological Games

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Abstract. An open question of Gruenhage asks if all strategically selectively separable spaces are Markov selectively separable, a game-theoretic statement known to hold for countable spaces. As a corollary of a result by Berner and Juhász, we note that the "strong" version of this statement, where the second player is restricted to selecting single points rather than finite subsets, holds for all T_3 spaces without isolated points. Continuing this investigation, we also consider games related to selective sequential separability, and demonstrate results analogous to those for selective separability. In particular, strong selective sequential separability in the presence of the Ramsey property may be reduced to a weaker condition on a countable sequentially dense subset. Additionally, γ- and ω-covering properties on X are shown to be equivalent to corresponding sequential properties on $C_p(X)$. A strengthening of the Ramsey property is also introduced, which is still equivalent to α_2 and α_4 in the context of $C_p(X)$.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X. Then $S_1(\mathcal{A}, \mathcal{B})$ denotes a *selection principle*: for each sequence $(A_n : n \in \omega)$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \omega)$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \omega\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is another *selection principle*: for each sequence $(A_n : n \in \omega)$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \omega)$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\omega} B_n \in \mathcal{B}$.

In this paper, by a cover we mean a nontrivial one; that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

A cover $\mathcal{U}$ of a space X is:

- an *ω-cover* if every finite subset of X is contained in a member of $\mathcal{U}$.
- a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

Note that every infinite subset (in particular, every countably infinite subset) of a γ-cover is a γ-cover, and every γ-cover is also an ω-cover.

For a topological space X we denote:

- Ω — the family of all open ω-covers of X;

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- Γ — the family of all open γ-covers of X.

Let X be a Hausdorff topological space, and $x \in X$. A subset A of X *converges* to a unique $x = \lim A$ if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite; We also assume $x = \lim\{x\}$. (That is, we want $x = \lim A$ to mean that A is a set that unambiguously represents a sequence converging to x.) We may then consider the following collections:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A \text{ or } A = \{x\}\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

As was noted earlier, $\Gamma \subseteq \Omega$; likewise, $\Gamma_x \subseteq \Omega_x$. Also as before, if $A \in \Gamma_x$, then every infinite subset of A belongs to Γ_x.

Given these definitions, we may describe the following well-known selection principles.

- A space X has Arhangel'skii's *countable fan tightness* if X satisfies $S_{fin}(\Omega_x, \Omega_x)$ for every $x \in X$ [2].
- A space X has Sakai's *countable strong fan tightness* if X satisfies $S_1(\Omega_x, \Omega_x)$ for every $x \in X$ [26].
- A space X has Arhangel'skii's *property* α_4 if X satisfies $S_{fin}(\Gamma_x, \Gamma_x)$ for every $x \in X$ [1].
- A space X has Arhangel'skii's *property* α_2 if X satisfies $S_1(\Gamma_x, \Gamma_x)$ for every $x \in X$ [1].
- A space X is *strictly Fréchet-Urysohn* if X satisfies $S_1(\Omega_x, \Gamma_x)$ for every $x \in X$ [28].
- A space X is *strongly Fréchet-Urysohn* if X satisfies $S_{fin}(\Omega_x, \Gamma_x)$ for every $x \in X$ [19, 34].

Definition 1.1. ([21]) A space X has the *Ramsey property* if for any choices $x_{i,j} \in X$ for $i, j \in \omega$ such that $\lim\{\lim\{x_{i,j} : j \in \omega\} : i \in \omega\} = x$ for some point $x \in X$, there exists an infinite set $M \subseteq \omega$ such that for every open neighborhood U of x, $x_{m,n} \in U$ for sufficiently large $m, n \in M$ with $m < n$.

In particular, note that $x = \lim\{x_{m,m^+} : m \in M\}$ where $m^+ = \min(\{k \in M : k > M\})$, and thus Ramsey $\Rightarrow \alpha_4$ (and furthermore α_3; see [21]). But the relation between α_2 and the Ramsey property remains open, even for topological groups (Question 3.15 in [33]).

We also will use the following strengthening of Ramsey:

Definition 1.2. A space X has the Ω-*Ramsey property* if and only if for any choices $T_{i,j} \in [X]^{<\omega}$ for $i, j \in \omega$ such that $\lim\{\lim \bigcup_{j\in\omega} T_{i,j} : i \in \omega\} = x$ for some point $x \in X$, there exists an infinite set $M \subseteq \omega$ such that for every open neighborhood U of x, $T_{m,n} \subseteq U$ for sufficiently large $m < n \in M$.

The following implications follow for any topological space X since $\Gamma_x \subseteq \Omega_x$:

$$\begin{array}{ccccccc}
S_1(\Gamma_x, \Gamma_x) & \Rightarrow & S_{fin}(\Gamma_x, \Gamma_x) & \Rightarrow & S_{fin}(\Gamma_x, \Omega_x) & \Leftarrow & S_1(\Gamma_x, \Omega_x) \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
S_1(\Omega_x, \Gamma_x) & \Rightarrow & S_{fin}(\Omega_x, \Gamma_x) & \Rightarrow & S_{fin}(\Omega_x, \Omega_x) & \Leftarrow & S_1(\Omega_x, \Omega_x)
\end{array}$$

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space X is called sequentially separable if it has a countable sequentially dense set.

For a topological space X we denote:

- $\mathcal{D}$ is the family of all dense subsets of X;
- $\mathcal{S}$ is the family of all sequentially dense subsets of X.

Let Π represent S_1 or S_{fin}. When we write $\Pi(\mathcal{A}, \mathcal{B}_x)$ without specifying x, we mean $(\forall x)\Pi(\mathcal{A}, \mathcal{B}_x)$.

As above, the following implications hold on any topological space X since $\mathcal{S} \subseteq \mathcal{D}$:

$$\begin{array}{ccccccc} S_1(\mathcal{S},\Gamma_x) & \Rightarrow & S_{fin}(\mathcal{S},\Gamma_x) & \Rightarrow & S_{fin}(\mathcal{S},\Omega_x) & \Leftarrow & S_1(\mathcal{S},\Omega_x) \\ \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\ S_1(\mathcal{D},\Gamma_x) & \Rightarrow & S_{fin}(\mathcal{D},\Gamma_x) & \Rightarrow & S_{fin}(\mathcal{D},\Omega_x) & \Leftarrow & S_1(\mathcal{D},\Omega_x) \end{array}$$

Some of these selection principles are known by name.

- A space X is *R-separable*, if X satisfies $S_1(\mathcal{D},\mathcal{D})$ (Def. 47, [6]).
- A space X is *M-separable* (or selectively separable), if X satisfies $S_{fin}(\mathcal{D},\mathcal{D})$ (Def 2.1, [7]).
- A space X is *selectively sequentially separable*, if X satisfies $S_{fin}(\mathcal{S},\mathcal{S})$ (Def. 1.2, [8]).

Proposition 1.3. ([8, Proposition 1.3]) *Every sequentially dense subspace of a selectively sequentially separable space is sequentially separable. In particular, every selectively sequentially separable space is sequentially separable.*

And similarly the following implications hold on any topological space X:

$$\begin{array}{ccccccc} S_1(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{D}) & \Leftarrow & S_1(\mathcal{S},\mathcal{D}) \\ \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\ S_1(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{D}) & \Leftarrow & S_1(\mathcal{D},\mathcal{D}) \end{array}$$

We now have three types of topological properties described as selection principles:

- local properties of the form $S_*(\Phi_x,\Psi_x)$;
- semi-local properties of the form $S_*(\Phi,\Psi_x)$.
- global properties of the form $S_*(\Phi,\Psi)$;

There is a game, denoted by $G_{fin}(\mathcal{A},\mathcal{B})$, corresponding to $S_{fin}(\mathcal{A},\mathcal{B})$. In this game two players, ONE and TWO, play a round for each natural number n. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$ and TWO responds with a finite subset B_n of A_n. A play $A_1, B_1; ...; A_n, B_n; ...$ is won by TWO if $\bigcup_{n\in\omega} B_n \in \mathcal{B}$; otherwise, ONE wins. Similarly, one defines the game $G_1(\mathcal{A},\mathcal{B})$, associated with $S_1(\mathcal{A},\mathcal{B})$.

A strategy of a player is a function σ from the set of all finite sequences of moves of the opponent into the set of (legal) moves of the strategy owner. Formally:

Definition 1.4. A *strategy* for TWO in the game $G_{fin}(\mathcal{A},\mathcal{B})$ is a function σ satisfying $\sigma(\langle A_0, ..., A_n\rangle) \in [A_n]^{<\omega}$ for $\langle A_0, ..., A_n\rangle \in \mathcal{A}^{n+1}$. We say this strategy is *winning* if whenever ONE plays $A_n \in \mathcal{A}$ during each round $n < \omega$, TWO wins the game by playing $\sigma(\langle A_0, ..., A_n\rangle)$ during each round $n < \omega$. If a winning strategy exists, then we write TWO $\uparrow G_{fin}(\mathcal{A},\mathcal{B})$.

Strategies for ONE may be defined similarly. It then follows that the selection principle $S_*(\mathcal{A},\mathcal{B})$ is equivalent to player ONE lacking a winning *predetermined* strategy for $G_*(\mathcal{A},\mathcal{B})$ that is defined solely on the current round number n (ignoring the moves of TWO) [12]. Even when ONE lacks such a predetermined winning strategy, it is still possible for ONE to have a winning strategy that uses perfect information.

As such, we now have three types of topological games on a topological space X:

- local games of the form $G_*(\Phi_x,\Psi_x)$;
- semi-local games of the form $G_*(\Phi,\Psi_x)$.
- global games of the form $G_*(\Phi,\Psi)$;

We will also be interested in strategies that use limited information; specifically, those that only use the current round number n and the most recent move of the opponent.

Definition 1.5. A *Markov strategy* for TWO in the game $G_{fin}(\mathcal{A},\mathcal{B})$ is a function σ satisfying $\sigma(A,n)\in[A]^{<\omega}$ for $A\in\mathcal{A}$ and $n\in\omega$. We say this Markov strategy is *winning* if whenever ONE plays $A_n\in\mathcal{A}$ during each round $n<\omega$, TWO wins the game by playing $\sigma(A_n,n)$ during each round $n<\omega$. If a winning Markov strategy exists, then we write TWO $\underset{mark}{\uparrow} G_{fin}(\mathcal{A},\mathcal{B})$.

Both definitions may be naturally modified for the game $G_1(\mathcal{A},\mathcal{B})$ instead. It is then easily seen that

$$\text{TWO} \underset{mark}{\uparrow} G_*(\mathcal{A},\mathcal{B}) \Rightarrow \text{TWO} \uparrow G_*(\mathcal{A},\mathcal{B}) \Rightarrow S_*(\mathcal{A},\mathcal{B})$$

where $*\in\{1,fin\}$.

2. Main Results

Barman and Dow showed ([4], Theorem 2.9) that every separable Fréchet-Urysohn T_2-space is selectively separable. By definition of Fréchet-Urysohn, closure is equivalent to sequential closure in such spaces, so we immediately have:

Proposition 2.1. ([8, Proposition 2.2]) *Every Fréchet-Urysohn separable T_2-space is selectively sequentially separable.*

Let $\Gamma'_x=\{A\subseteq X:\exists B\in\Gamma_x(B\subseteq A)\}$, and note that $\mathcal{S}\subseteq\Gamma'_x$ (while $\mathcal{S}\not\subseteq\Gamma_x$). These may be considered the sequences which cluster at x (although we do not restrict to countable sets).

In particular, we have that $S_*(\Phi,\mathcal{S})\Rightarrow S_*(\Phi,\Gamma'_x)$ (with similar game-theoretic results). We now turn to the following theorem:

Theorem 2.2. *Let $*\in\{1,fin\}$; if $*=1$ assume X is Ramsey, and otherwise assume X is Ω-Ramsey. Then for any non-empty set Φ, the following are equivalent:*

1. *X satisfies $S_*(\Phi,\mathcal{S})$ (resp. TWO $\uparrow G_*(\Phi,\mathcal{S})$, TWO $\underset{mark}{\uparrow} G_*(\Phi,\mathcal{S})$);*
2. *X is sequentially separable and satisfies $S_*(\Phi,\Gamma'_x)$ (resp. TWO $\uparrow G_*(\Phi,\Gamma'_x)$, TWO $\underset{mark}{\uparrow} G_*(\Phi,\Gamma'_x)$);*
3. *X has a countable sequentially dense subset D where $S_*(\Phi,\Gamma'_x)$ (resp. TWO $\uparrow G_*(\Phi,\Gamma'_x)$, TWO $\underset{mark}{\uparrow} G_*(\Phi,\Gamma'_x)$) holds for all $x\in D$.*

Proof. Let $P\in\Phi$ and $P_n=P$ for $n<\omega$. Then for the countable set $\{P_n:n<\omega\}=\{P\}$, we may apply (any variant of) the first condition to obtain $T_i\in[P]^{<\omega}$ for $i\in\omega$ with $\bigcup\{T_i:i\in\omega\}\in\mathcal{S}$, demonstrating the respective second condition, which trivially implies the third. As such, we only need prove that the final condition implies the first; let $D=\{d_i:i\in\omega\}$ witness that final condition.

a) Assume $S_*(\Phi,\Gamma'_x)$ for $x\in D$. Let $P_{i,m}\in\Phi$ for all $i,m\in\omega$. For each $i\in\omega$, $S_*(\Phi,\Gamma'_{d_i})$ allows us to choose $T_{i,m}\in[P_{i,m}]^*$ and $m_t\in\omega$ for $t\in\omega$ such that $d_i=\lim\bigcup\{T_{i,m_t}:t\in\omega\}$. We claim that $\bigcup\{T_{i,m}:i,m\in\omega\}$ is sequentially dense. To see this, let $x\in X$, and choose $i_s\in\omega$ for $s\in\omega$ such that $x=\lim\{d_{i_s}:s\in\omega\}$. We then choose $M\subseteq\omega$ witnessing the appropriate Ramsey property for $\{T_{i_s,m_t}:s,t\in\omega\}$ and x; it follows that $x=\lim\bigcup\{T_{i_s,m_{s^+}}:s\in M\}$. Thus for any countable collection of sets $P_{i,m}\in\Phi$, we have $T_{i,m}\in[P_{i,m}]^*$ with $\bigcup\{T_{i,m}:i,m\in\omega\}$ sequentially dense, witnessing $S_1(\Phi,\mathcal{S})$.

b) Now assume *TWO* $\uparrow G_*(\Phi,\Gamma'_{d_i})$ is witnessed by the strategy σ_i for each $i\in\omega$. Let $p:\omega\to\omega$ be a function such that $p^{\leftarrow}(i)$ is infinite for all $i\in\omega$. For a nonempty finite sequence t, let t' be its subsequence removing all terms of index n such that $p(n)\neq p(|t|-1)$. We define the strategy σ for the game $G_*(\Phi,\mathcal{S})$ by $\sigma(t)=\sigma_{p(|t|-1)}(t')$; that is, σ partitions any counterplay by ONE into countably many subplays according to p, and uses a different σ_i for each subplay.

Let $\alpha\in\Phi^\omega$, and let α_i be its subsequence removing all terms of index n such that $p(n)\neq i$. Then $\bigcup\{\sigma_i(\alpha_i\restriction(n+1)):n\in\omega\}\in\Gamma'_{d_i}$ since σ_i is a winning strategy for TWO, so choose $n_{i,t}\in\omega$ for $t\in\omega$ where $d_i=\lim\bigcup\{\sigma_i(\alpha_i\restriction(n_{i,t}+1)):t\in\omega\}$.

We claim that $\bigcup\{\sigma(\alpha \restriction (n+1)) : n \in \omega\} \in \mathcal{S}$, so let $x \in X$. Then there exists $\{d_{i_s} : s \in \omega\}$ such that $x = \lim\{d_{i_s} : s \in \omega\}$. We then apply the appropriate Ramsey property to $\{\sigma_{i_s}(\alpha_{i_s} \restriction (n_{i_s,t}+1)) : s,t \in \omega\}$ to obtain an $M \subseteq \omega$ with $x = \lim\{\sigma_{i_s}(\alpha_{i_s} \restriction (n_{i_s,s^+}+1)) : s \in M\}$. Since each $\sigma_{i_s}(\alpha_{i_s} \restriction (n_{i_s,s^+}+1)) = \sigma(\alpha \restriction (n+1))$ for some $n \in \omega$, the result follows.

c) Finally let TWO $\underset{mark}{\uparrow} G_1(\mathcal{S}, \Gamma_{d_i})$ for each $i \in \omega$ be witnessed by σ_i. Let $p : \omega \to \omega$ be a function such that $p^{\leftarrow}(i)$ is infinite for all $i \in \omega$. We then define the Markov strategy σ by

$$\sigma(P, n) = \sigma_{p(n)}(P, |\{m < n : p(m) = p(n)\}|)$$

so that as in the previous case, σ partitions any counterplay by ONE into countably many subplays according to p, and uses a different σ_i for each subplay.

Let $\alpha \in \Phi^\omega$, and let α_i be its subsequence removing all terms of index n such that $p(n) \neq i$. Then $\{\sigma_i(\alpha_i(n), n) : n \in \omega\} \in \Gamma'_{d_i}$ since σ_i is a winning strategy for TWO, so choose $n_{i,t} \in \omega$ for $t \in \omega$ where $d_i = \lim\{\sigma_i(\alpha_i(n_{i,t}), n_{i,t}) : t \in \omega\}$.

We claim that $\{\sigma(\alpha(n), n) : n \in \omega\} \in \mathcal{S}$, so let $x \in X$. Then there exists $\{d_{i_s} : s \in \omega\}$ such that $x = \lim\{d_{i_s} : s \in \omega\}$. We then apply the appropriate Ramsey property to $\{\sigma_{i_s}(\alpha_{i_s}(n_{i_s,t}), n_{i_s,t}) : s,t \in \omega\}$ to obtain an $M \subseteq \omega$ with $x = \lim\{\sigma_{i_s}(\alpha_{i_s}(n_{i_s,s^+}), n_{i_s,s^+}) : s \in M\}$. Since each $\sigma_{i_s}(\alpha_{i_s}(n_{i_s,s^+}), n_{i_s,s^+}) = \sigma(\alpha(n), n)$ for some $n \in \omega$, the result follows. □

The previous result mirrors the following slight generalization of Theorems 16 and 41 of [10].

Theorem 2.3. ([10]) *For a topological space X, nonempty set Φ, and $* \in \{1, fin\}$, the following are equivalent:*

1. *X satisfies $S_*(\Phi, \mathcal{D})$ (resp. TWO $\uparrow G_*(\Phi, \mathcal{D})$, TWO $\underset{mark}{\uparrow} G_*(\Phi, \mathcal{D})$);*
2. *X is separable and satisfies $S_*(\Phi, \Omega_x)$ (resp. TWO $\uparrow G_*(\Phi, \Omega_x)$, TWO $\underset{mark}{\uparrow} G_*(\Phi, \Omega_x)$);*
3. *X has a countable dense subset D where $S_*(\Phi, \Omega_x)$ (resp. TWO $\uparrow G_*(\Phi, \Omega_x)$, TWO $\underset{mark}{\uparrow} G_*(\Phi, \Omega_x)$) holds for all $x \in D$.*

Proof. In [10], $\Phi = \mathcal{D}$ was an additional assumption, but was never required in the proofs, since $S_*(\Phi, \mathcal{D})$ implies separability for any non-empty Φ. □

Recall that a π-base for a space X is a family $\mathcal{U}$ of nonempty open subsets of X such that for each nonempty open set $V \subseteq X$ there is a $U \in \mathcal{U}$ with $U \subseteq V$. Then the π-weight of a space X, denoted $\pi(X)$, is the minimal cardinality of a π-base for X.

Corollary 2.4. *Let X be a T_3-space with no isolated points. Then the following are equivalent:*

1. $\pi(X) = \aleph_0$;
2. *TWO $\uparrow G_1(\mathcal{D}, \mathcal{D})$;*
3. *TWO $\underset{mark}{\uparrow} G_1(\mathcal{D}, \mathcal{D})$;*
4. *X is separable and TWO $\uparrow G_1(\mathcal{D}, \Omega_x)$;*
5. *X is separable and TWO $\underset{mark}{\uparrow} G_1(\mathcal{D}, \Omega_x)$;*
6. *X has a countable dense subset D where TWO $\uparrow G_1(\mathcal{D}, \Omega_x)$ for all $x \in D$.*
7. *X has a countable dense subset D where TWO $\underset{mark}{\uparrow} G_1(\mathcal{D}, \Omega_x)$ for all $x \in D$.*

Proof. The equivalence of (1) and (2) is [9, Theorem 2.1].

Assuming (1), let $\{P_n : n \in \omega\}$ be a countable π-base. We may then define $\sigma(D, n) \in D \cap P_n$ arbitrarily, and it's easy to see that this is winning for TWO, implying (3) and therefore (2).

All other equivalences follow from Theorem 2.3. □

The equivalence (2) ⇔ (3) is similar to the following open question of Gruenhage, first shown to be true when X is countable by Barman and Dow in [5, Theorem 2.11]; see [10, Lemma 37] for a general sufficient condition which guarantees that a winning strategy may be improved to a Markov winning strategy.

Question 2.5. *When does TWO $\uparrow G_{fin}(\mathcal{D}, \mathcal{D})$ imply TWO $\underset{mark}{\uparrow} G_{fin}(\mathcal{D}, \mathcal{D})$?*

3. Ω-Ramsey in Topological Groups

We now adapt techniques of Sakai [27] to obtain the following lemma giving a useful recharacterization of the Ω-Ramsey property for topological groups, which we require in the following section.

Lemma 3.1. *Let $\langle G, \cdot\rangle$ be a topological group with unit e. Then the Ω-Ramsey property is equivalent to the following: if $T_{n,m} \in [G]^{<\omega}$ and $e = \lim \bigcup\{T_{n,m} : m \in \omega\}$ for each $n \in \omega$, then there exists an infinite $M \subseteq \omega$ such that $e = \lim \bigcup\{T_{n,m} : n, m \in M, n < m\}$.*

Proof. The forward direction follows by noting that $e = \lim\{e\}$ and thus applying the Ω-Ramsey property to $\{T_{n,m} : n, m \in \omega\}$.

For the converse, let $x_n = \lim \bigcup\{T_{n,m} : m \in \omega\}$ for each $n \in \omega$, and $e = \lim\{x_n : n \in \omega\}$ (since G is homogeneous). If $S_{n,m} = x_n^{-1} \cdot T_{n,m}$, it follows that $\lim \bigcup\{S_{n,m} : m \in \omega\} = x_n^{-1} \cdot x_n = e$. We apply the assumption to obtain an infinite $M \subseteq \omega$ where $e = \lim \bigcup\{S_{n,m} : n, m \in M, n < m\}$, and claim that M witnesses Ω-Ramsey.

Let U be a neighborhood of e, which must contain $\{x_n : n \geq k'\}$ for some $k' \in \omega$. By applying [27, Lemma 2.3], we may choose an open neighborhood V of e where $\{x_n : n \geq k'\} \cdot V \subseteq U$. Since $e = \lim \bigcup\{S_{n,m} : n, m \in M, n < m\}$, we may choose $k \geq k'$ where $\bigcup\{S_{n,m} : n, m \in M, k \leq n < m\} \subseteq V$. So for $k \leq n < m$,

$$S_{n,m} \subseteq V \Rightarrow T_{n,m} = x_n \cdot S_{n,m} \subseteq x_n \cdot V \subseteq U.$$

□

4. Applications in C_p-Theory

For a Tychonoff space X, we denote by $C_p(X)$ the topological additive group of all real-valued continuous functions on X with the topology of pointwise convergence. The symbol $\mathbf{0}$ stands for the constant function to 0.

Basic open sets of $C_p(X)$ are of the form $[x_1, ..., x_k; U_1, ..., U_k] = \{f \in C_p(X) : f(x_i) \in U_i, i = 1, ..., k\}$, where each $x_i \in X$ and each U_i is a non-empty open subset of $\mathbb{R}$. When $U_i = U$ for all $i \leq k$, we simply write $[x_1, \ldots, x_k; U]$.

Consider the following result of Sakai.

Theorem 4.1. ([27, Theorem 2.5]) *The Ramsey property is equivalent to α_2 and α_4 for $C_p(X)$.*

By using the previous Lemma 3.1, we may show the following.

Theorem 4.2. *The Ω-Ramsey property is equivalent to the Ramsey, α_2, and α_4 properties for $C_p(X)$.*

Proof. Let $T_{n,m} \in [C_p(X)]^{<\omega}$ and $\mathbf{0} = \lim \bigcup\{T_{n,m} : m \in \omega\}$ for each $n \in \omega$. We let $g_{n,m}(x) = \max\{|f(x)| : f \in \bigcup_{i\leq n} T_{i,m}\}$, noting $\mathbf{0} = \lim\{g_{n,m} : m \in \omega\}$ for each $n \in \omega$. We apply α_2, that is, $S_1(\Gamma_\mathbf{0}, \Gamma_\mathbf{0})$ to $\{g_{n,m} : n < m \in \omega\}$ to obtain an increasing mapping $\phi : \omega \to \omega$ with $\mathbf{0} = \lim\{g_{m,\phi(m)} : m \in \omega\}$.

Now let $\phi^0(n) = n$ and $\phi^{i+1}(n) = \phi(\phi^i(n))$ and set $M = \{\phi^i(0) : i \in \omega\}$. We will demonstrate that $\mathbf{0} = \lim\{T_{n,m} : n, m \in M, n < m\}$. For $x \in X$ and $\epsilon > 0$, pick $k \in \omega$ where $|g_{m,\phi(m)}(x)| < \epsilon$ for $k < m, m \in M$.

Let $f \in T_{n,m}$ for $k < n < m$ and $n, m \in M$. We may choose $m' \in M$ such that $n \leq m'$ and $m = \phi(m')$. Then $|f(x)| \leq |g_{n,m}(x)| \leq |g_{m',m}(x)| = |g_{m',\phi(m')}(x)| < \epsilon$. Thus $C_p(X)$ is Ω-Ramsey.

Since Ω-Ramsey implies Ramsey, the result follows from the previous theorem. □

Recall that the *i*-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of weight not greater than τ.

Theorem 4.3. ([20]) *Let X be a space. A space $C_p(X)$ is separable if and only if $iw(X) = \aleph_0$.*

Note that if X is itself Tychonoff and $iw(X) = \aleph_0$, then the image of X under a witnessing one-to-one continuous mapping yields a coarser topology for X which is separable and metrizable; this is the characterization given in [18].

In the papers [2, 3, 6, 16, 22–26, 32] various selection principles for a Tychonoff space X were related to the selection principles for $C_p(X)$. Likewise, in [10, 17, 26, 30, 31] various selection games for X and $C_p(X)$ and a bitopological space $(C(X), \tau_k, \tau_p)$ were related.

So we have the following applications in C_p-theory.

Theorem 4.4. ([10, Theorems 22 and 43]) *For a Tychonoff space X and $* \in \{1, fin\}$, the following are equivalent:*

1. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\Omega, \Omega)$ *on* X*;*
2. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$ *on* $C_p(X)$*;*
3. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\mathcal{D}, \Omega_{\mathbf{0}})$ *on* $C_p(X)$*.*

Corollary 4.5. *Let X be a Tychonoff space with a coarser second-countable topology (that is, $iw(X) = \aleph_0$) and $* \in \{1, fin\}$. The following assertions are equivalent:*

1. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\Omega, \Omega)$ *on* X*;*
2. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\Omega_{\mathbf{0}}, \Omega_{\mathbf{0}})$ *on* $C_p(X)$*;*
3. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\mathcal{D}, \Omega_{\mathbf{0}})$ *on* $C_p(X)$*.*
4. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\mathcal{D}, \mathcal{D})$ *on* $C_p(X)$*;*

Proof. By Theorems 2.3, 4.3 and 4.4, items (1-4) are equivalent. □

Corollary 4.6. *Let X be a Tychonoff space with a coarser second-countable topology. The following assertions are equivalent:*

1. $\pi(C_p(X)) = \aleph_0$*;*
2. *TWO* $\uparrow G_1(\mathcal{D}, \mathcal{D})$ *for* $C_p(X)$*;*
3. *TWO* $\uparrow G_1(\mathcal{D}, \Omega_{\mathbf{0}})$ *for* $C_p(X)$*;*
4. *TWO* $\uparrow G_1(\Omega, \Omega)$ *for* X*;*
5. *TWO* $\underset{mark}{\uparrow} G_1(\mathcal{D}, \mathcal{D})$ *for* $C_p(X)$*;*
6. *TWO* $\underset{mark}{\uparrow} G_1(\mathcal{D}, \Omega_{\mathbf{0}})$ *for* $C_p(X)$*;*
7. *TWO* $\underset{mark}{\uparrow} G_1(\Omega, \Omega)$ *for* X*;*
8. *X is countable.*

Proof. Items (1-7) follow from Corollary 2.4 and Corollary 4.5. The fact that (8) is equivalent to (6) and (7) doesn't require $iw(X) = \aleph_0$ and may be found in [13, Theorem 17] along with several other equivalences. □

We now turn to the case where TWO may choose finite sets each round.

Corollary 4.7. *Let X be a separable metrizable space. Then the following are equivalent:*

1. *TWO* $\uparrow G_{fin}(\mathcal{D}, \mathcal{D})$ *for* $C_p(X)$*;*
2. *TWO* $\uparrow G_{fin}(\mathcal{D}, \Omega_{\mathbf{0}})$ *for* $C_p(X)$*;*
3. *TWO* $\uparrow G_{fin}(\Omega, \Omega)$ *for* X*;*
4. *TWO* $\underset{mark}{\uparrow} G_{fin}(\mathcal{D}, \mathcal{D})$ *for* $C_p(X)$*;*
5. *TWO* $\underset{mark}{\uparrow} G_{fin}(\mathcal{D}, \Omega_{\mathbf{0}})$ *for* $C_p(X)$*;*
6. *TWO* $\underset{mark}{\uparrow} G_{fin}(\Omega, \Omega)$ *for* X*;*

7. *X is σ-compact.*

Proof. Second-countability allows us to apply Corollary 4.5 to show (1-3) are mutually equivalent, as are (4-6). By [10, Corollary 39], (3) is equivalent to (6), and by [10, Lemma 24], (6) equivalent to (7). □

We now demonstrate analogous results, replacing $\mathcal{D}$ and $\Omega_{\mathbf{0}}$ with $\mathcal{S}$ and $\Gamma_{\mathbf{0}}$.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set.

A γ-cover $\mathcal{U}$ of co-zero sets of X is *γ_F-shrinkable* if there exists a γ-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X such that $F(U) \subseteq U$ for some $U \in \mathcal{U}$ ([23]).

For a topological space X we let $\Gamma_F \subseteq \Gamma$ denote the family of γ_F-shrinkable covers of X.

Theorem 4.8. *For a Tychonoff space X with $* \in \{1, fin\}$, the following are equivalent:*

1. *TWO $\uparrow$ (resp. $\underset{mark}{\uparrow}$) $G_*(\Gamma_F, \Omega)$ on X;*
2. *TWO $\uparrow$ (resp. $\underset{mark}{\uparrow}$) $G_*(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$ on $C_p(X)$;*
3. *TWO $\uparrow$ (resp. $\underset{mark}{\uparrow}$) $G_*(\mathcal{S}, \Omega_{\mathbf{0}})$ on $C_p(X)$.*

Proof. (1) ⇒ (2). For each $B \in \Gamma_{\mathbf{0}}$ we define $\mathcal{U}_n(B) = \{f^{\leftarrow}[(-\frac{1}{2^n}, \frac{1}{2^n})] : f \in B\}$. To see that $\mathcal{U}_n(B) \in \Gamma_F$, let $x \in X$. Since $B \in \Gamma_{\mathbf{0}}$, $B \setminus [x; (-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})]$ is finite. It follows that for $f \in B \cap [x; (-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})]$,

$$x \in f^{\leftarrow}\left[(-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})\right] \subseteq f^{\leftarrow}\left[[-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}]\right] \subseteq f^{\leftarrow}\left[(-\frac{1}{2^n}, \frac{1}{2^n})\right]$$

and we have shown that $\{f^{\leftarrow}[[-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}]] : f \in B\}$ is a γ-cover by zero sets; therefore $\mathcal{U}_n(B) \in \Gamma_F$.

For convenience in the rest of this proof, whenever $B_n \in \Gamma_{\mathbf{0}}$ is known for some $n < \omega$, fix $f_{U,n} \in B_n$ for each $U \in \mathcal{U}_n(B_n)$ such that $U = f^{\leftarrow}_{U,n}[(-\frac{1}{2^n}, \frac{1}{2^n})]$.

If TWO $\uparrow G_*(\Gamma_F, \Omega)$ holds, then we may find a winning strategy σ that not only produces ω covers, but produces covers such that every cofinite subset is an ω cover. To see this, partition any play by ONE into infinitely many subplays and consider the strategy that applies the known winning strategy to each subplay (the beginnings of which are cofinal in ω).

Now let $\tau(\langle B_0, \dots, B_n\rangle) = \{f_{U,n} : U \in \sigma(\langle \mathcal{U}_0(B_0), \dots, \mathcal{U}_n(B_n)\rangle)\}$. (Note here that the cardinalities of moves made by τ are no greater than the cardinalities produced by σ, so this proof applies to both G_1 and G_{fin}.) We claim that $\mathbf{0} \in \overline{\bigcup_{n<\omega} \tau(\langle B_0, ..., B_n\rangle)}$.

To see this, let $G \in [X]^{<\omega}$ and $\epsilon > 0$. Then choose $n < \omega$ such that $\frac{1}{2^n} < \epsilon$ and $G \subseteq U$ for some $U \in \sigma(\langle \mathcal{U}_0(B_0), \dots, \mathcal{U}_n(B_n)\rangle)$. Then

$$G \subseteq f^{\leftarrow}_{U,n}\left[(-\frac{1}{2^n}, \frac{1}{2^n})\right] \subseteq f^{\leftarrow}_{U,n}[(-\epsilon, \epsilon)]$$

demonstrates that $f_{U,n} \in \tau(\langle B_0, \dots, B_n\rangle) \cap [G; (-\epsilon, \epsilon)]$, verifying our claim.

If TWO $\underset{mark}{\uparrow} G_*(\Gamma_F, \Omega)$ holds, then we may again assume we have a witnessing strategy σ producing omega covers such that every cofinite subset is an ω-cover, for the same reason as above.

Now let $\tau(B_n, n) = \{f_{U,n} : U \in \sigma(\mathcal{U}_n(B_n), n)\}$. (Note again here that the cardinality of σ matches the cardinality of τ, so this proof applies to both G_1 and G_{fin}.) We claim that $\mathbf{0} \in \overline{\bigcup_{n<\omega} \tau(B_n, n)}$.

To see this, let $G \in [X]^{<\omega}$ and $\epsilon > 0$. Then choose $n < \omega$ such that $\frac{1}{2^n} < \epsilon$ and $G \subseteq U$ for some $U \in \sigma(\mathcal{U}_n(B_n), n)$. Then

$$G \subseteq f^{\leftarrow}_{U,n}\left[(-\frac{1}{2^n}, \frac{1}{2^n})\right] \subseteq f^{\leftarrow}_{U,n}[(-\epsilon, \epsilon)]$$

demonstrates that $f_{U,n} \in \tau(B_n, n) \cap [G; (-\epsilon, \epsilon)]$, verifying our claim.

(2) ⇒ (3). For each $S \in \mathcal{S}$, select $G_S \subseteq S$ such that $\lim G = \mathbf{0}$. Given a strategy for TWO in $G_*(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$, TWO's strategy for $G_*(\mathcal{S}, \Omega_{\mathbf{0}})$ simply substitutes each $S \in \mathcal{S}$ with G_S.

(3) ⇒ (1). For each $\mathcal{U} \in \Gamma_F$ define $S(\mathcal{U}) = \{f \in C(X) : f \upharpoonright (X \setminus U) \equiv 1 \text{ for some } U \in \mathcal{U}\}$. By [23, Lemma 6.5], $S(\mathcal{U})$ is sequentially dense in $C_p(X)$. Whenever $\mathcal{U}_n \in \Gamma_F$ is known for some $n < \omega$, choose $U_{f,n} \in \mathcal{U}_n$ for each $f \in S(\mathcal{U}_n)$ such that $f \upharpoonright (X \setminus U_{f,n}) \equiv 1$.

So let σ witness TWO $\uparrow G_*(\mathcal{S}, \Omega_{\mathbf{0}})$, so $\mathbf{0} \in \overline{\bigcup_{n<\omega} \sigma(\langle S(\mathcal{U}_0), ..., S(\mathcal{U}_n)\rangle)}$. We then define $\tau(\langle \mathcal{U}_0, ..., \mathcal{U}_n\rangle) = \{U_{f,n} : f \in \sigma(\langle S(\mathcal{U}_0), ..., S(\mathcal{U}_n)\rangle)\}$. Let $F \in [X]^{<\omega}$, so we may choose $n \in \omega$ such that there exists $f \in \sigma(\langle S(\mathcal{U}_0), \ldots, S(\mathcal{U}_n)\rangle) \cap [F; (-1/2, 1/2)]$. Then as $f \upharpoonright F$ cannot map to 1, $F \subseteq U_{f,n}$. Therefore τ produces ω-covers.

Finally, let σ witness TWO $\underset{mark}{\uparrow} G_*(\mathcal{S}, \Omega_{\mathbf{0}})$, so $\mathbf{0} \in \overline{\bigcup_{n<\omega} \sigma(S(\mathcal{U}_n), n)}$. We then define $\tau(\mathcal{U}_n, n) = \{U_{f,n} : f \in \sigma(S(\mathcal{U}_n), n)\}$. Let $F \in [X]^{<\omega}$, so we may choose $n \in \omega$ such that there exists $f \in \sigma(S(\mathcal{U}_n), n) \cap [F; (-1/2, 1/2)]$. Then as $f \upharpoonright F$ cannot map to 1, $F \subseteq U_{f,n}$. Therefore τ produces ω-covers. □

Corollary 4.9. *Let X be a Tychonoff space with a coarser second countable topology and* $* \in \{1, fin\}$. *The following assertions are equivalent:*

1. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\Gamma_F, \Omega)$ *on X;*
2. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$ *on* $C_p(X)$*;*
3. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\mathcal{S}, \Omega_{\mathbf{0}})$ *on* $C_p(X)$*.*
4. *TWO* $\uparrow$ *(resp.* $\underset{mark}{\uparrow}$ *)* $G_*(\mathcal{S}, \mathcal{D})$ *on* $C_p(X)$*;*

Proof. By Theorems 2.3 and 4.3, items (3) and (4) are equivalent. □

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