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Geometry of Warped Product Pointwise Submanifolds of Sasakian Manifolds

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Abstract. Recently, Chen and Uddin introduced and studied warped product pointwise bi-slant submanifolds of Kähler manifolds in [13]. They have obtained many interesting results.

In the present paper, we investigate warped product pointwise bi-slant submanifolds in Sasakian manifolds and we derive contact version of results obtain in [13]. We give a non-trivial example to prove the existence of these submanifolds.

1. Introduction

Bishop and O'Neill introduced warped product manifolds as a natural and fruitful generalization of Riemannian product manifolds in 1969 [2]. The notion of warped product has many applications in physics as well as in differential geometry. By following this concept many researchers investigated geometric properties of the warped product manifolds.

On the other hand, the critical problem for a geometer is to establish some relationships between the main intrinsic and extrinsic invariants of a submanifold. After the statement of famous theorem of J. Nash, different type classes of submanifolds were defined and studied. One of them is slant submanifold which was introduced by Chen in [8]. Then this concept was studied extensively by many geometers and obtained interesting and fruitful results in ([8] and [9])

In 1998, Etayo extended the notion of slant submanifolds and defined pointwise slant submanifolds with a different name quasi-slant submanifold [14]. Then Chen and Garay studied and obtained new useful results for pointwise slant submanifolds in almost Hermitian manifolds. They also stated a new construction method to get pointwise slant submanifolds in [12].

Another natural and useful generalization of slant submanifolds was introduced as bi-slant submanifolds by Carriazo in [5]. Later Cabrerizo et al. studied bi-slant submanifolds of an almost contact manifolds [4]. Also, Uddin et al. investigated warped product bi-slant immersions of Kaehler manifolds in [22].

By combining the above notion warped products, pointwise slant submanifolds and bi-slant submanifolds, Chen and Uddin introduced warped product pointwise bi-slant submanifolds in a Kaehler manifold in [13] and they generalized some previous results derived in ([10], [16]-[24])

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In this paper, we consider warped product pointwise bi-slant submanifolds of Sasakian manifolds. We make a characterization for a warped product pointwise bi-slant submanifolds of the form $M = M_1 \times_f M_2$ with distinct slant function θ_1 and θ_2. We obtain some necessary and sufficient conditions for M to be locally warped product pointwise submanifold. Also, we construct some non-trivial example to prove the existence of warped product pointwise bi-slant submanifold in a Sasakian manifold.

2. Preliminaries

Now, let $\overline{M}$ be $(2m+1)$-dimensional differentiable manifold with the almost contact metric structure (φ, ξ, η, g) satisfying

$$\begin{aligned} &\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1 \quad \eta \circ \varphi = 0, \\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \end{aligned} \tag{1}$$

for any vector fields X and Y on $\overline{M}$, where φ is $(1, 1)$ tensor field, ξ is a vector field, η is 1-form and g is a compatible Riemannian metric. If $\overline{M}$ satisfies, for all vector fields X and Y on $\widetilde{M}$

$$(\nabla_X \varphi) Y = g(X, Y)\xi - \eta(Y) X, \quad \nabla_X \xi = -\varphi X, \tag{2}$$

then $\overline{M}$ is called a Sasakian manifold, where ∇ the Riemannian connection with respect to the Riemannian metric g [3].

Now we recall some basic facts on submanifolds from ([6]-[9]). Let M be an n-dimensional submanifold of a Sasakian manifold $\overline{M}$ such that the characteristic vector field ξ is tangent to M with induced metric g. Let $\Gamma(TM)$ be the Lie algebra of vector fields of M in $\widetilde{M}$ and $\Gamma\left(T^{\perp}M^{2m+1-n}\right)$ is set of all vector fields normal to M. Then, the Gauss and Weingarten formulas are defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{3}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{4}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $N \in \Gamma\left(T^{\perp}M^{2m+1-n}\right)$, where ∇ and $\nabla^{\perp}$ denote the induced connections on the tangent bundle and normal bundle of M, respectively and h is the second fundamental form, A is the shape operator of the submanifold and $\widetilde{\nabla}$ denotes the Levi-Civita connection defined on $\overline{M}$. There is a relation between the second fundamental form h and the shape operator A as

$$g(h(X, Y), N) = g(A_N X, Y) \tag{5}$$

for any vector fields $X, Y \in \Gamma(TM)$.

Let $p \in M$ and $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m+1}$ are orthonormal basis of the tangent space $\overline{M}$ such that restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M at p and hence $e_{n+1}, \ldots, e_{2m+1}$ are normal to M. Let $\left\{h^r_{ij}\right\}$, $i, j = 1, \ldots n$; $r = n+1, \ldots, 2m+1$ be the coefficients of the second fundamental form h with respect to the local frame field. Then we get

$$h^r_{ij} = g\left(h\left(e_i, e_j\right), e_r\right) = g\left(A_{e_r} e_i, e_j\right), \quad \|h\|^2 = \sum_{i, j=1}^{n} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right). \tag{6}$$

The mean curvature vector H is given by $H = \dfrac{1}{n} trace h = \dfrac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$. A submanifold is called totally geodesic, totally umbilical and minimal if $h = 0$, $h(X, Y) = g(X, Y) H$ and $H = 0$, respectively.

Furthermore, for any vector field X on M we can write

$$\varphi X = TX + FX \tag{7}$$

where TX is the tangential component of φX and FX is the normal component of φX. If $F = 0$ and $T = 0$, then M is an invariant and anti-invariant submanifold respectively. In a similar way, for any vector field N normal to M, we set

$$\varphi N = BN + CN \tag{8}$$

where BN and CN are the tangential and normal components of φN, respectively.

Now, let us recall some basic facts of pointwise slant submanifolds of almost contact manifolds from [15] and [27] before we start to study on our main parts of the present paper.

An n-dimensional submanifold M of an almost contact metric manifold $\overline{M}$ is called pointwise slant, if for a nonzero vector $X \in T_pM$ at any point $p \in M$, which is linearly independent to ξ_p, the angle $\theta(X)$ between φX and $T_pM - \{0\}$ is independent of the choice of nonzero vector $X \in \left\{T_pM - \{0\}\right\}$. So θ can be regarded as a function on M, which is called the slant function of the pointwise slant submanifold. On the other hand, a pointwise slant submanifold M is induced a slant submanifold if its slant function θ is globally constant on M. Furthermore, invariant and anti-invariant submanifolds are pointwise slant submanifolds with slant function $\theta = 0$ and $\theta = \dfrac{\pi}{2}$, respectively. If θ is different from 0 and $\dfrac{\pi}{2}$, then the pointwise slant submanifold is said to be proper.

Theorem 2.1. *Let M be an n-dimensional submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in \Gamma(TM)$. Then M is pointwise slant if and only if*

$$T^2 = \cos^2\theta\,(-Id + \eta \otimes \xi) \tag{9}$$

for some real valued function θ defined on the tangent bundle TM of M.

Also we have the following useful relations on a pointwise slant submanifold for any vector fields X and Y tangent to M :

$$g(TX, TY) = \cos^2\theta\,\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{10}$$

$$g(FX, FY) = \sin^2\theta\,\{g(X, Y) - \eta(X)\eta(Y)\} \tag{11}$$

and

$$BFX = \sin^2\theta\,(-X + \eta(X)\xi), \qquad CFX = -FTX. \tag{12}$$

3. Warped Product Pointwise Bi-slant Submanifolds in Sasakian Manifolds

Definition 3.1. *Let M be a submanifold of a Sasakian manifold $\overline{M}$. Then it is said that M is a pointwise bi-slant submanifold if there exist two orthogonal distributions D_1 and D_2 at any point $p \in M$ such that*

(*i*) $TM = D_1 \oplus D_2 \oplus \{\xi\}$,

(*ii*) $\varphi(D_1) \perp D_2 \oplus \{\xi\}$,

(*iii*) *the distributions D_1 and D_2 are pointwise slant with slant functions θ_1 and θ_2, respectively.*

The slant functions θ_1 and θ_2 are called the bi-slant functions if θ_1 and θ_2 are different from 0 and $\dfrac{\pi}{2}$ and both of them are not constant on M. Also M is called proper pointwise bi-slant submanifold.

Now, let M be a pointwise bi-slant submanifold of $\overline{M}$. Then for any vector field $X \in \Gamma(TM)$, then we have

$$X = P_1X + P_2X \tag{13}$$

where P_i is the projection from TM onto D_i. It is easily seen that P_iX is the component of X in D_i for each $i = 1,\ 2$.

By setting $T_i = P_i \circ T$ for each $i = 1,\ 2$, then from (13) we obtain

$$\varphi X = T_1X + T_2X + FX \tag{14}$$

for any vector field $X \in \Gamma(TM)$. By virtue of (9), then we get

$$T_i^2X = \cos^2\theta_i\left(-X + \eta(X)\xi\right) \tag{15}$$

for any vector field $X \in \Gamma(TM)$ and for each $i = 1,\ 2$.

Now we give the first result in the following lemma for pointwise bi-slant submanifolds of a Sasakian manifold.

Lemma 3.2. *Let M be a pointwise bi-slant submanifold of a Sasakian manifold $\overline{M}$ with pointwise bi-slant distributions D_1 and D_2 with distinct slant functions θ_1 and θ_2, respectively. Then for any vector fields $X,\ Y \in \Gamma(D_1 \oplus \{\xi\})$ and $Z,\ W \in \Gamma(D_2)$, we have*

(*i*) $\left(\sin^2\theta_1 - \sin^2\theta_2\right) g(\nabla_XY,\ Z) = g\left(A_{FT_1Y}Z - A_{FY}T_2Z,\ X\right) + g\left(A_{FT_2Z}Y - A_{FZ}T_1Y,\ X\right)$

and

(*ii*) $\left(\sin^2\theta_2 - \sin^2\theta_1\right) g(\nabla_ZW,\ X) = g\left(A_{FT_2W}X - A_{FW}T_1X,\ Z\right) + g\left(A_{FT_1X}W - A_{FX}T_2W,\ Z\right) + \eta(X)\, g(\varphi Z,\ W).$

Proof. For any vector fields X and $Y \in \Gamma(D_1 \oplus \{\xi\})$ and $Z \in \Gamma(D_2)$, then we have

$$\begin{aligned}
g(\nabla_XY,\ Z) &= g\left(\varphi\overline{\nabla}_XY,\ \varphi Z\right)\\
&= g\left(\overline{\nabla}_X\varphi Y,\ \varphi Z\right) - g\left(\left(\overline{\nabla}_X\varphi\right)Y,\ \varphi Z\right)\\
&= g\left(\overline{\nabla}_XT_1Y,\ \varphi Z\right) + g\left(\overline{\nabla}_XFY,\ \varphi Z\right) + \eta(Y)\, g(\varphi Z,\ X)\\
&= -g\left(\varphi\overline{\nabla}_XT_1Y,\ Z\right) + g\left(\overline{\nabla}_XFY,\ T_2Z\right) + g\left(\overline{\nabla}_XFY,\ FZ\right) + \eta(Y)\, g(\varphi Z,\ X)\\
&= -g\left(\overline{\nabla}_X\varphi T_1Y,\ Z\right) + g\left(\left(\overline{\nabla}_X\varphi\right)T_1Y,\ Z\right) + g\left(\overline{\nabla}_XFY,\ T_2Z\right) + g\left(\overline{\nabla}_XFY,\ FZ\right) + \eta(Y)\, g(\varphi Z,\ X)\\
&= -g\left(\overline{\nabla}_XT_1^2Y,\ Z\right) - g\left(\overline{\nabla}_XFT_1Y,\ Z\right) + g\left(\overline{\nabla}_XFY,\ T_2Z\right) - g\left(\overline{\nabla}_XFZ,\ \varphi Y\right) + g\left(\overline{\nabla}_XFZ,\ T_1Y\right)\\
&\quad + \eta(Y)\, g(\varphi Z,\ X).
\end{aligned}$$

By using (2), (4) and (12), then we obtain

$$\begin{aligned}
g(\nabla_XY,\ Z) &= \cos^2\theta_1 g\left(\overline{\nabla}_XY,\ Z\right) - \sin 2\theta_1 X(\theta_1)\, g(Y,\ Z) - g\left(\overline{\nabla}_XFT_1Y,\ Z\right) + g\left(\overline{\nabla}_XFY,\ T_2Z\right)\\
&\quad + g\left(\overline{\nabla}_XBFZ,\ Y\right) + g\left(\overline{\nabla}_XCFZ,\ Y\right) - g\left(\left(\overline{\nabla}_X\varphi\right)FZ,\ Y\right) + g\left(\overline{\nabla}_XFZ,\ T_1Y\right) + \eta(Y)\, g(\varphi Z,\ X).
\end{aligned}$$

By virtue of the orthogonality of two distributions and the symmetry of the shape operator, we find

$$\sin^2\theta_1 g(\nabla_XY,\ Z) = g\left(A_{FT_1Y}Z - A_{FY}T_2Z,\ X\right) + g\left(\overline{\nabla}_XBFZ,\ Y\right) + g\left(\overline{\nabla}_XCFZ,\ Y\right) + g\left(\overline{\nabla}_XFZ,\ T_1Y\right).$$

By taking into account of (12) in the last equation and using the orthogonality of vector fields, we arrive at

$$\sin^2\theta_1 g(\nabla_XY,\ Z) = g\left(A_{FT_1Y}Z - A_{FY}T_2Z,\ X\right) + \sin^2\theta_2 g\left(\overline{\nabla}_XY,\ Z\right) + g\left(A_{FT_2Z}Y - A_{FZ}T_1Y,\ X\right)$$

which gives the first part of the lemma. For the second part, given $X \in \Gamma(D_1 \oplus \{\xi\})$ and $Z,\ W \in \Gamma(D_2)$, then we have

$$\begin{aligned}
g(\nabla_Z W,\ X) &= g\left(\varphi\overline{\nabla}_Z W,\ \varphi X\right) + \eta(X)\, g\left(\overline{\nabla}_Z W,\ \xi\right)\\
&= g\left(\overline{\nabla}_Z \varphi W,\ \varphi X\right) - g\left(\left(\overline{\nabla}_Z \varphi\right) W,\ \varphi X\right) - \eta(X)\, g\left(W,\ \overline{\nabla}_Z \xi\right)\\
&= g\left(\overline{\nabla}_Z T_2 W,\ \varphi X\right) + g\left(\overline{\nabla}_Z F W,\ \varphi X\right) + \eta(X)\, g(\varphi Z,\ W)\\
&= -g\left(\varphi\overline{\nabla}_Z T_2 W,\ X\right) + g\left(\overline{\nabla}_Z F W,\ T_1 X\right) + g\left(\overline{\nabla}_Z F W,\ F X\right) + \eta(X)\, g(\varphi Z,\ W)\\
&= -g\left(\overline{\nabla}_Z T_2^2 W,\ X\right) - g\left(\overline{\nabla}_Z F T_2 W,\ X\right) + g\left(\left(\overline{\nabla}_Z \varphi\right) T_2 W,\ X\right) + g\left(\overline{\nabla}_Z F W,\ T_1 X\right) - g\left(\overline{\nabla}_Z F X,\ \varphi X\right)\\
&\quad + g\left(\overline{\nabla}_Z F X,\ T_2 W\right) + \eta(X)\, g(\varphi Z,\ W).
\end{aligned}$$

From (2), (4) and (12), the last equation takes the form

$$\begin{aligned}
g(\nabla_Z W,\ X) &= \cos^2\theta_2 g\left(\overline{\nabla}_Z W,\ X\right) - \sin 2\theta_2 Z(\theta_2)\, g(W,\ X) - g\left(\overline{\nabla}_Z F T_2 W,\ X\right) + g\left(\overline{\nabla}_Z F W,\ T_1 X\right)\\
&\quad + g\left(\overline{\nabla}_Z B F X,\ W\right) + g\left(\overline{\nabla}_Z C F X,\ W\right) + g\left(\overline{\nabla}_Z F X,\ T_2 W\right) + \eta(X)\, g(\varphi Z,\ W).
\end{aligned}$$

By using the orthogonality of two distributions and the symmetry of the shape operator

$$\begin{aligned}
g(\nabla_Z W,\ X) &= \cos^2\theta_2 g\left(\overline{\nabla}_Z W,\ X\right) + \sin^2\theta_1 g\left(\overline{\nabla}_Z W,\ X\right) + g\left(A_{FT_2W} X - A_{FW} T_1 X,\ Z\right)\\
&\quad + \eta(X)\, g(\varphi Z,\ W) + g\left(A_{FT_1X} W - A_{FX} T_2 W,\ Z\right)
\end{aligned}$$

which gives us the desired result. □

Thus from the above lemma, we can give the following corollaries without proof.

Corollary 3.3. *Let M be a pointwise bi-slant submanifold of a Sasakian manifold $\overline{M}$ with pointwise bi-slant distributions D_1 and D_2 with distinct slant functions θ_1 and θ_2, respectively. Then the distribution $D_1 \oplus \{\xi\}$ defines a totally geodesic foliation if and only if*

$$g\left(A_{FT_1Y} Z - A_{FY} T_2 Z + A_{FT_2Z} Y - A_{FZ} T_1 Y,\ X\right) = 0$$

for any vector fields X and $Y \in \Gamma(D_1 \oplus \{\xi\})$ and $Z \in \Gamma(D_2)$.

Corollary 3.4. *Let M be a pointwise bi-slant submanifold of a Sasakian manifold $\overline{M}$ with pointwise bi-slant distributions D_1 and D_2 with distinct slant functions θ_1 and θ_2, respectively. Then the distribution D_2 defines a totally geodesic foliation if and only if*

$$g\left(A_{FT_2W} X - A_{FW} T_1 X + A_{FT_1X} W - A_{FX} T_2 W,\ Z\right) = \eta(X)\, g(Z,\ \varphi W)$$

for any vector fields $X \in \Gamma(D_1 \oplus \{\xi\})$ and $Z,\ W \in \Gamma(D_2)$.

4. Warped Product Pointwise Bi-slant Submanifolds in Sasakian Manifolds

Let $(M_1,\ g_{M_1})$ and $(M_2,\ g_{M_2})$ be two Riemannian manifolds and f is a smooth function on M_1. The warped product of M_1 and M_2 is a Riemannian manifold

$$M = M_1 \times_f M_2$$

with the Riemannian metric

$$g = g_{M_1} + f^2 g_{M_2}.$$

If the warping function f is constant then the warped product $M_1 \times_f M_2$ is called trivial. In other words, the warped product is induced the Riemannian product.

Let X be any vector field on M_1 and Z a vector field tangent to M_2. Then we have

$$\nabla_X Z = \nabla_Z X = X(\ln f) Z \tag{16}$$

where ∇ denotes the Levi-Civita connection on M. For more details please see [2].

Now we investigate warped product pointwise bi-slant submanifolds in a Sasakian manifold $\overline{M}$. We begin following lemma.

Lemma 4.1. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Sasakian manifold $\overline{M}$, where M_1 and M_2 are pointwise slant submanifolds with distinct slant functions θ_1 and θ_2, respectively and $\xi \in \Gamma(TM_1)$. Then we have*

$$\sin 2\theta_2 X(\theta_2)\, g(Z, W) = g(h(X, W), FT_2 Z) - g(h(X, T_2 Z), FW) \tag{17}$$

and

$$2\tan\theta_2 X(\theta_2)\, g(T_2 Z, W) = g(h(X, Z), FW) - g(h(X, W), FZ) \tag{18}$$

for any vector fields X on M_1 and Z, W on M_2.

Proof. Given $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, then we get

$$g\left(\overline{\nabla}_X Z, W\right) = g(\nabla_X Z, W) = X(\ln f)\, g(Z, W). \tag{19}$$

Furthermore, we have

$$g\left(\overline{\nabla}_X Z, W\right) = g\left(\varphi\overline{\nabla}_X Z, \varphi W\right) + \eta(W)\, g\left(\overline{\nabla}_X Z, \xi\right) = g\left(\overline{\nabla}_X \varphi Z, \varphi W\right) - g\left(\left(\overline{\nabla}_X \varphi\right) Z, \varphi W\right)$$

for any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$. By using (2), (7) and (8) in the last equation, then we arrive at

$$\begin{aligned} g\left(\overline{\nabla}_X Z, W\right) &= g\left(\overline{\nabla}_X T_2 Z, T_2 W\right) + g\left(\overline{\nabla}_X T_2 Z, FW\right) + g\left(\overline{\nabla}_X FZ, \varphi W\right) \\ &= g\left(\overline{\nabla}_X T_2 Z, T_2 W\right) + g\left(\overline{\nabla}_X T_2 Z, FW\right) - g\left(\overline{\nabla}_X BFZ, W\right) - g\left(\overline{\nabla}_X CFZ, W\right) + g\left(\overline{\nabla}_X FT_2 Z, W\right). \end{aligned}$$

By taking into account of (2), (4) and (12), the above equation yields that

$$\begin{aligned} g\left(\overline{\nabla}_X Z, W\right) &= \cos^2\theta_2 X(\ln f)\, g(Z, W) + g(h(X, T_2 Z), FW) + \sin^2\theta_2 g\left(\overline{\nabla}_X Z, W\right) \\ &\quad + \sin 2\theta_2 X(\theta_2)\, g(Z, W) + g(h(X, W), FT_2 Z). \end{aligned} \tag{20}$$

Hence, the first part of the Lemma follows from (19) and (20).

For the second part, by replacing Z by $T_2 Z$ in the first part of this lemma and by using (9), we obtain the desired result. □

Lemma 4.2. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Sasakian manifold $\overline{M}$, where M_1 and M_2 are pointwise slant submanifolds with distinct slant functions θ_1 and θ_2, respectively and $\xi \in \Gamma(TM_1)$. Then the following holds.*

$$2X(\ln f)\cos^2\theta_2 g(Z, W) = g(h(X, W), FT_2 Z) - g(h(X, T_2 Z), FW) \tag{21}$$

for any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$.

Proof. For any $X \in \Gamma(TM_1)$ and $Z,\ W \in \Gamma(TM_2)$, then we get

$$\begin{aligned} g(h(X,\ Z),\ FW) &= g\left(\overline{\nabla}_Z X,\ FW\right) = g\left(\overline{\nabla}_Z X,\ \varphi W\right) - g\left(\overline{\nabla}_Z X,\ T_2 W\right) \\ &= -g\left(\overline{\nabla}_Z \varphi X,\ W\right) + g\left(\left(\overline{\nabla}_Z \varphi\right) X,\ W\right) - g\left(\overline{\nabla}_Z X,\ T_2 W\right). \end{aligned}$$

From (1), (2), (7) and (8), then we derive

$$g(h(X,\ Z),\ FW) = -g\left(\overline{\nabla}_Z T_1 X,\ W\right) - g\left(\overline{\nabla}_Z FX,\ W\right) - \eta(X)\, g(Z,\ W) - g\left(\overline{\nabla}_Z X,\ T_2 W\right).$$

By using again (1), (2), (4)-(8) and (16), we conclude that

$$g(h(X,\ Z),\ FW) = -T_1 X(\ln f)\, g(Z,\ W) + g(h(Z,\ W),\ FX) - \eta(X)\, g(Z,\ W) + X(\ln f)\, g(T_2 Z,\ W). \tag{22}$$

By virtue of polarization, we find

$$g(h(X,\ W),\ FZ) = -T_1 X(\ln f)\, g(Z,\ W) + g(h(W,\ Z),\ FX) - \eta(X)\, g(Z,\ W) + X(\ln f)\, g(Z,\ T_2 W). \tag{23}$$

By subtracting (23) from (22),

$$2X(\ln f)\, g(T_2 Z,\ W) = g(h(X,\ Z),\ FW) - g(h(X,\ W),\ FZ). \tag{24}$$

By interchanging Z by $T_2 Z$ in (24) and using (15), we obtain (21). □

Hence we can give the following theorem

Theorem 4.3. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Sasakian manifold $\overline{M}$, where M_1 and M_2 are pointwise slant submanifolds with distinct slant functions θ_1 and θ_2. Then M is a proper pointwise bi-slant submanifold if and only if* $\tan\theta_2 X(\theta_2) \neq 0$.

Proof. For any $X \in \Gamma(TM_1)$ and $Z,\ W \in \Gamma(TM_2)$, from (17) and (21), then we have

$$\sin 2\theta_2 X(\theta_2)\, g(Z,\ W) = 2X(\ln f)\cos^2\theta_2 g(Z,\ W). \tag{25}$$

By virtue of assumption, $\theta_2 \neq \dfrac{\pi}{2}$, from (25), we obtain

$$\{X(\ln f) - \tan\theta_2 X(\theta_2)\}\, g(Z,\ W) = 0$$

which means that $X(\ln f) = \tan\theta_2 X(\theta_2)$. □

Now, we note that a warped product submanifold $M = M_1 \times_f M_2$ of a Sasakian manifold $\overline{M}$ is mixed totally geodesic if $h(X,\ Z) = 0$, for $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$. As an application of the last lemma, we derive the following theorem.

Theorem 4.4. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Sasakian manifold $\overline{M}$, where M_1 and M_2 are pointwise slant submanifolds with distinct slant functions θ_1 and θ_2. If M is a mixed totally geodesic warped product submanifold, then we have one of the following cases:*
(*i*) *either M is a Riemannian product submanifold of M_1 and M_2*
(*ii*) *or $\theta_2 = \dfrac{\pi}{2}$, that is, M is a warped product submanifold of the form $M_1 \times_f M_\perp$, where $M_\perp$ is a totally real submanifold of $\overline{M}$.*

Proof. By virtue of (21), it yields that

$$2X(\ln f)\cos^2\theta_2 g(Z,\ W) = 0$$

which gives us either f is constant on M or $\cos^2\theta_2 = 0$. This completes the proof. □

Lemma 4.5. *Let $M = M_1 \times_f M_2$ be a warped product pointwise bi-slant submanifold of a Sasakian manifold $\overline{M}$, where M_1 and M_2 are pointwise slant submanifolds with distinct slant functions θ_1 and θ_2, respectively and $\xi \in \Gamma(TM_1)$. Then the following equalities hold:*

$$g(h(X, Y), FZ) = g(h(X, Z), FY), \tag{26}$$

$$\begin{aligned}\left\{\sin^2\theta_1 - \sin^2\theta_2\right\} X(\ln f) g(Z, W) &= g(A_{FT_1X}W - A_{FX}T_2W, Z) \\ &\quad + g(A_{FT_2W}X - A_{FW}T_1X, Z) + \eta(X) g(Z, T_2W).\end{aligned} \tag{27}$$

for any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$.

Proof. For $X, Y \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, then we get

$$\begin{aligned}g(h(X, Y), FZ) &= g\left(\overline{\nabla}_X Y, \varphi Z\right) - g\left(\overline{\nabla}_X Y, T_2 Z\right) \\ &= -g\left(\overline{\nabla}_X \varphi Y, Z\right) + g\left(\left(\overline{\nabla}_X \varphi\right) Y, Z\right) + g\left(Y, \overline{\nabla}_X T_2 Z\right).\end{aligned}$$

By using (2), (14) and (16), then we obtain

$$\begin{aligned}g(h(X, Y), FZ) &= -g\left(\overline{\nabla}_X T_1 Y, Z\right) - g\left(\overline{\nabla}_X FY, Z\right) + g\left(Y, \overline{\nabla}_X T_2 Z\right) \\ &= -g\left(\overline{\nabla}_X T_1 Y, Z\right) + g(h(X, Z), FY) + g\left(Y, \overline{\nabla}_X T_2 Z\right) \\ &= X(\ln f) g(T_1 Y, Z) + g(h(X, Z), FY) + X(\ln f) g(T_2 Y, Z)\end{aligned} \tag{28}$$

which gives us (26).

For the second part, it follows that

$$g\left(\overline{\nabla}_Z X, W\right) = g(\nabla_Z X, W) = X(\ln f) g(Z, W). \tag{29}$$

Also, we have

$$g\left(\overline{\nabla}_Z X, W\right) = g\left(\varphi\overline{\nabla}_Z X, \varphi W\right) = g\left(\overline{\nabla}_Z \varphi X, \varphi W\right) - g\left(\left(\overline{\nabla}_Z \varphi\right) X, \varphi W\right).$$

By using (2), (7) and (8), then we find

$$g\left(\overline{\nabla}_Z X, W\right) = g\left(\overline{\nabla}_Z T_1 X, \varphi W\right) + g\left(\overline{\nabla}_Z FX, T_2 W\right) + g\left(\overline{\nabla}_Z FX, FW\right) + \eta(X) g(Z, \varphi W)$$

From (1), (2) and (4), then we

$$g\left(\overline{\nabla}_Z X, W\right) = -g\left(\overline{\nabla}_Z \varphi T_1 X, W\right) + g\left(\left(\overline{\nabla}_Z \varphi\right) T_1 X, W\right) - g(A_{FX}Z, T_2 W) - g\left(FX, \overline{\nabla}_Z FW\right) + \eta(X) g(Z, \varphi W).$$

By taking into account of (7) and (8) in the last equation and by virtue of symmetry property of the shape operator, we get

$$\begin{aligned}g\left(\overline{\nabla}_Z X, W\right) &= -g\left(\overline{\nabla}_Z T_1^2 X, W\right) - g\left(\overline{\nabla}_Z FT_1 X, W\right) - g(A_{FX}T_2 W, Z) + g\left(\varphi\overline{\nabla}_Z FW, X\right) \\ &\quad + g\left(T_1 X, \overline{\nabla}_Z FW\right) + \eta(X) g(Z, \varphi W) \\ &= \cos^2\theta_1 g\left(\overline{\nabla}_Z X, W\right) - \sin 2\theta_1 Z(\theta_1) g(X, W) + g(A_{FT_1X}Z, W) - g(A_{FX}T_2 W, Z) \\ &\quad + g\left(\overline{\nabla}_Z \varphi FW, X\right) - g\left(\left(\overline{\nabla}_Z \varphi\right) FW, X\right) - g(A_{FW}Z, T_1 X) + \eta(X) g(Z, \varphi W).\end{aligned}$$

From (3), (7), (8), (16) and (29) and in view of the orthogonality of vector fields and symmetry of the shape operator,

$$\begin{aligned}\sin^2\theta_1 X(\ln f) g(Z, W) &= g(A_{FT_1X}W - A_{FX}T_2 W, Z) + g\left(\overline{\nabla}_Z BFW, X\right) + g\left(\overline{\nabla}_Z CFW, X\right) \\ &\quad - g(A_{FW}Z, T_1 X) + \eta(X) g(Z, \varphi W).\end{aligned}$$

Again from (12), it yields that

$$\sin^2\theta_1 X(\ln f)\, g(Z,\ W) = g\left(A_{FT_1X}W - A_{FX}T_2W,\ Z\right) - \sin^2\theta_2 g\left(\overline{\nabla}_Z W,\ X\right) - \sin 2\theta_2 Z(\theta_2)\, g(X,\ W)$$
$$- g\left(\overline{\nabla}_Z FT_2W,\ X\right) - g\left(A_{FW}Z,\ T_1X\right) + \eta(X)\, g(Z,\ \varphi W).$$

By using the orthogonality of vector fields and (3), (4) and (16),

$$\sin^2\theta_1 X(\ln f)\, g(Z,\ W) = g\left(A_{FT_1X}W - A_{FX}T_2W,\ Z\right) + \sin^2\theta_2 X(\ln f)\, g(Z,\ W) + g\left(A_{FT_2W}Z,\ X\right)$$
$$- g\left(A_{FW}T_1X,\ Z\right) + \eta(X)\, g(Z,\ \varphi W).$$

Finally using the symmetry of the shape operator and (14), we reach (27). □

Now, in consideration of Hiepko's Theorem, we can give the following main result.

Theorem 4.6. *Let M be a proper pointwise bi-slant submanifold of a Sasakian manifold $\overline{M}$ with pointwise slant distributions D_1 and D_2 such that ξ is tangent to D_2. Then M is locally a warped product submanifold of the form $M_1 \times_f M_2$ such that M_1 and M_2 are pointwise slant submanifolds with distinct slant functions θ_1 and θ_2 respectively of $\overline{M}$ if and only if the following identity holds*

$$A_{FT_1X}Z - A_{FX}T_2Z + A_{FT_2Z}X - A_{FZ}T_1X = \left\{\sin^2\theta_1 - \sin^2\theta_2\right\} X(\mu)\, Z \tag{30}$$

for any $X \in D_1$ and $Z \in D_2$ and for a function μ on M such that for any $W \in D_2$, $W(\mu) = 0$.

Proof. Let M be a pointwise bi-slant submanifold of the form $M_1 \times_f M_2$ of a Sasakian manifold $\overline{M}$. Then by virtue of (26), we derive that

$$g\left(A_{FY}Z - A_{FZ}Y,\ X\right) = 0 \tag{31}$$

for any $X,\ Y \in TM_1$ and $Z \in TM_2$. By replacing Y by T_1Y in (31), then we find

$$g\left(A_{FT_1Y}Z - A_{FZ}T_1Y,\ X\right) = 0. \tag{32}$$

By interchanging Z by T_2Z in (31), it yields that

$$g\left(A_{FY}T_2Z - A_{FT_2Z}Y,\ X\right) = 0. \tag{33}$$

By subtracting (33) from (32), we conclude

$$g\left(A_{FT_1Y}Z - A_{FZ}T_1Y + A_{FT_2Z}Y - A_{FY}T_2Z,\ X\right) = 0. \tag{34}$$

In view of (27) and using (34), we get the desired result.

Conversely, let M be a pointwise bi-slant submanifold with pointwise slant distributions D_1 and D_2 and we assume that (31) is satisfied. Then from Lemma 1 (*i*), it follows

$$\left(\sin^2\theta_1 - \sin^2\theta_2\right) g\left(\nabla_X Y,\ Z\right) = g\left(A_{FT_1Y}Z - A_{FZ}T_1Y + A_{FT_2Z}Y - A_{FY}T_2Z,\ X\right)$$

for any $X,\ Y \in TM_1$ and $Z \in TM_2$. Under the hypothesis, we derive

$$g\left(\nabla_X Y,\ Z\right) = X(\mu)\, g(X,\ Z) = 0$$

which means that the leaves of the distributions are totally geodesic in M. Moreover from Lemma 1 (*ii*)

$$\left(\sin^2\theta_2 - \sin^2\theta_1\right) g\left(\nabla_Z W,\ X\right) = g\left(A_{FT_2W}X - A_{FW}T_1X,\ Z\right) + g\left(A_{FT_1X}W - A_{FX}T_2W,\ Z\right).$$

Again in consideration of the hypothesis of the theorem, we get

$$g(\nabla_Z W,\ X) = -X(\mu)\, g(W,\ Z). \tag{35}$$

By using polarization, we also have

$$g(\nabla_W Z,\ X) = -X(\mu)\, g(W,\ Z). \tag{36}$$

By subtracting (36) from (35), we deduce that

$$g([Z,\ W],\ X) = 0$$

which gives us D_2 is integrable. On the other hand, from (35) we arrive at

$$g(h_2(Z,\ W),\ X) = g(\nabla_Z W,\ X) = -X(\mu)\, g(W,\ Z)$$

where h_2 denotes the second fundamental form of M_2. By virtue of the definition of the gradient we find

$$h_2(Z,\ W) = -\nabla\mu g(W,\ Z)$$

where $\nabla\mu$ is the gradient of μ. This means that he leaf M_2 is totally umbilical in M and its mean curvature vector $H_2 = -\nabla\mu$. Under the assumption, $W(\mu) = 0$ and this shows us the mean curvature is parallel. By Hiepko's Theorem, it is said that is locally a warped product submanifold. □

5. Some Remarks

In this section, we give some remarks as consequences of Theorem 2 and Theorem 4. Firstly, we begin Theorem 2. We have the following remarks as some consequences of Theorem 2.

1) If $\theta_1 = 0$ and $\theta_2 \neq \frac{\pi}{2}$ is a constant, then warped product is of the form $M_T \times_f M_\theta$ is a semi-slant warped product submanifold where $\theta = \theta_2$. Thus $X(\ln f) = 0$ which means f is constant.

2) If $\theta_2 = 0$, the warped product is of the form $M_\theta \times_f M_T$ such that M_T is a complex submanifold and M_θ is a pointwise slant submanifold with slant function $\theta = \theta_1$. Hence f is constant.

3) If $\theta_1 = \frac{\pi}{2}$ and θ_2 is a constant, the warped product pointwise bi-slant is of the from $M_\perp \times_f M_\theta$ becomes a hemi-slant warped product and so f is constant.

4) If $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = 0$, the warped product pointwise bi-slant submanifold turns to a warped product CR-submanifold of the form $M_\perp \times_f M_T$. Thus f is constant.

5) If θ_1 and θ_2 are constant, then the warped product $M = M_1 \times_f M_2$ is a warped product bi-slant submanifold and f is constant.

Under these considerations, we can give the final remark of Theorem 2.

6) It is easily seen that here exist no warped product pointwise bi-slant submanifolds of the forms $M_1 \times_f M_T$ or $M_1 \times_f M_\theta$. Here M_1 is a pointwise slant submanifold and M_T and M_θ are complex and proper slant submanifolds of $\overline{M}$, respectively.

From Theorem 4, then we have the followings:

1) If $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then M is induced contact CR-warped product and the equation (30) takes the form

$$A_{\varphi Z}\varphi X = -X(\mu)\, Z$$

for any $X \in D$ and $Z \in D^\perp$, where $D = D_1$ and $D^\perp = D_2$ such that D and $D^\perp$ are contact and totally real distributions, respectively.

2) If $\theta_1 = 0$ and $\theta_2 = \theta$ is a slant function then the M is induced a pointwise semi-slant submanifold and (30) is valid for for pointwise semi-slant warped product submanifolds and it will be

$$A_{FTZ}X - A_{FZ}\varphi X = -\left(\sin^2\theta\right)X(\mu)Z$$

for any $X \in D$ and $Z \in D_\theta$, where $D = D_1$ and $D_\theta = D_2$ such that D and D_θ are contact and proper pointwise slant distributions of M, respectively.

3) If $\theta_1 = \theta$ is a constant slant angle and $\theta_2 = \dfrac{\pi}{2}$, then M becomes hemi-slant warped product submanifold and (30) takes the form

$$A_{FTX}Z - A_{\varphi Z}TX = -\left(\cos^2\theta\right)X(\mu)Z$$

for any $X \in D_\theta$ and $Z \in D^\perp$, where $D_\theta = D_1$ and $D^\perp = D_2$ such that D_θ and $D^\perp$ are contact and totally real distributions, respectively.

4) If $\theta_1 = \dfrac{\pi}{2}$ and $\theta_2 = \theta$ is a pointwise slant function, M turns to pointwise hemi-slant warped product submanifold and (30) becomes

$$A_{FTZ}X - A_{\varphi X}TZ = \left(\cos^2\theta\right)X(\mu)Z$$

for any $X \in D^\perp$ and $Z \in D_\theta$, where $D^\perp = D_1$ and $D_\theta = D_2$ such that $D^\perp$ and D_θ are totally real and proper pointwise slant distributions of M, respectively.

Under the above considerations, our results generalize some previous results studied by different authors.

Example 5.1. *It is well-known that S^{2n+1} inherits a Sasakian structure. Let N be a pointwise slant submanifold of a 6-dimensional Kähler manifold. Consider the warped product $M = N \times_f S^3$. Then M becomes a warped product pointwise bi-slant submanifold of S^{2n+1} with the distinct slant functions θ_1 and θ_2.*

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