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(m,q)-Isometric and (m,∞)-Isometric Tuples of Commutative Mappings on a Metric Space

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Abstract. In this paper, we introduce new concepts of (m,q)-isometries and (m,∞)-isometries tuples of commutative mappings on metrics spaces. We discuss the most interesting results concerning this class of mappings obtained form the idea of generalizing the (m,q)-isometries and (m,∞)-isometries for single mappings. In particular, we prove that if $\mathbf{T} = (T_1,\cdots,T_n)$ is an (m,q)-isometric commutative and power bounded tuple, then $\mathbf{T}$ is a $(1,q)$-isometric tuple. Moreover, we show that if $\mathbf{T} = (T_1,\cdots,T_d)$ is an (m,∞)-isometric commutative tuple of mappings on a metric space (E,d), then there exists a metric d_∞ on E such that $\mathbf{T}$ is a $(1,\infty)$-isometric tuple on (E,d_∞).

1. Introduction and preliminaries

Let $\mathcal{X}$ (resp. $\mathcal{H}$) denote a complex Banach (resp. Hilbert) space and let $\mathcal{B}(\mathcal{X})$ (resp. $\mathcal{B}(\mathcal{H})$) be the algebra of all bounded linear operators on $\mathcal{X}$ (resp. on $\mathcal{H}$). An operator T acting on a Hilbert space $\mathcal{H}$ is said to be *m-isometric* if

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k = 0$$

or equivalently if

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\|T^kx\|^2 = 0, \quad \forall \ x\in\mathcal{H}.$$

An operator $T\in\mathcal{B}(\mathcal{H})$ is said to be a *strict m-isometric* operator if T is m-isometric but it is not $(m-1)$-isometric. Such m-isometric operators were introduced by J. Agler back in the early nineties and were studied in great detail by J. Agler and M. Stankus in a series of three papers, including analytic and spectral

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properties (see [1]). Several algebraic properties have been studied products, tensor products and nilpotent perturbations of such operators (see [2] and [3] for more details).

In recent work, T. Bermúdez, A. Martinôn and V. Müller introduced and studied the concept of (m,q)-isometric maps on metric spaces (see [4]). Let E be a metric space and $m \geq 1$ be integer and $q > 0$. A map $T : E \to E$ is called an *(m,q)-isometry* if for all $x, y \in E$,

$$\sum_{0\leq k\leq m} (-1)^k \binom{m}{k} d\Big(T^{m-k}x, T^{m-k}y\Big)^q = 0. \tag{1}$$

For $m \geq 2$, a mapping T is a *strict (m,q)-isometry* if it is an (m,q)-isometry, but is not an $(m-1,q)$-isometry.

For any $q > 0$, $(1,q)$-isometry coincides with isometry, that is, $d\Big(Tx, Ty\Big) = d\Big(x, y\Big)$ for all $x, y \in E$. Every isometry is an (m,q)-isometry for all $m \geq 1$ and $q > 0$. An (m,q)-isometry is an $(m+1,q)$-isometry and any power of (m,q)-isometry is again an (m,q)-isometry (see [4]).

Let $T : E \to E$ be an $(m,\ q)$-isometry. In [4], the authors defined $f_T(h,\ q;\ x,y)$ for a positive integer h, a positive real number q, and $x, y \in E$ by :

$$f_T(h,q;\ x,y) = \sum_{0\leq k\leq h} (-1)^{h-k} \binom{h}{k} d\Big(T^k x, T^k y\Big)^q. \tag{2}$$

We derive from (2) that

$$d\Big(T^n x, T^n y\Big)^q = \sum_{0\leq k\leq n} \binom{n}{k} f_T(k,q;\ x,y) \tag{3}$$

for all $n \geq 1$ and $x, y \in E$ (see [4, Lemma 2.4]).

Very recently, the author in [9] has introduced the notion of (m,∞)-isometric mapping on a metric space as follows: for $m \geq 1$, a mapping T acting on a metric space E is called an *(m,∞)-isometry* if for all $x, y \in E$

$$\max_{\substack{0\leq k\leq m \\ k \text{ even}}} d\Big(T^k x, T^k y\Big) = \max_{\substack{0\leq k\leq m \\ k \text{ odd}}} d\Big(T^k x, T^k y\Big). \tag{4}$$

It is natural to seek extensions of (m,q)-isometric mappings and (m,∞)-isometric mappings classes to multivariable mappings in a metric space.

Let E be a metric space and $T_1, T_2, \cdots, T_n$ be commutative mappings from E to E, i.e., $T_j : E \to E$ and for all $x \in E$, $T_jT_ix = T_iT_jx$ for $1 \leq i, j \leq n$. By an n-tuple of commutative mappings, it means the n-component $\mathbf{T} = (T_1, \cdots, T_n)$.

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{N}_0^n := \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ (n-times). For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ and $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n$, we write

$$|\alpha| := \alpha_1 + \cdots + \alpha_n = \sum_{1\leq k\leq n} \alpha_k \quad \text{and} \quad \alpha \leq \beta \ \text{ if } \ \alpha_k \leq \beta_k \ \text{ for } \ k = 1, \cdots, n.$$

For $\alpha \leq \beta$, we let

$$\binom{\beta}{\alpha} := \prod_{1\leq k\leq n} \binom{\beta_k}{\alpha_k}.$$

Let $\mathbf{T} = (T_1, \cdots, T_n)$ be an n-tuple of commutative mappings on a metric space E. Define

$$\mathbf{T}^\alpha := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n}, \quad \text{where } \ T_k^{\alpha_k} = \underbrace{T_k \circ T_k \circ \cdots \circ T_k}_{\alpha_k - times} = \underbrace{T_k \cdot T_k \cdots T_k}_{\alpha_k - times}.$$

In higher dimensions ($n \geq 1$), J. Gleason and S. Richter in [7] extended the notion of m-isometric operators to the case of commuting n-tuples of bounded linear operators on a Hilbert space. A tuple $\mathbf{T} = (T_1, \cdots, T_n) \in \mathcal{B}(\mathcal{H})^n$ is said to be an *m-isometric tuple* if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{T}^{*\alpha} \mathbf{T}^{\alpha} \right) = 0, \tag{5}$$

or equivalently,

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{T}^{\alpha} x\|^2 \right) = 0 \text{ for all } x \in \mathcal{H}. \tag{6}$$

More recently, P. H. W. Hoffmann and M. Mackey ([6]) introduced the concept of (m, p)-isometric tuples on a normed space. A tuple of commuting linear operators $\mathbf{T} := (T_1, \cdots, T_n)$ with $T_j : \mathcal{X} \to \mathcal{X}$ (normed space) is called an *(m, p)-isometry* (or an *(m, p)-isometric tuple*) if and only if for given $m \in \mathbb{N}$ and $p \in (0, \infty)$,

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{T}^{\alpha} x\|^p \right) = 0 \text{ for all } \quad x \in \mathcal{X}. \tag{7}$$

An extension of (7) to include the case $p = \infty$ was introduced in [6] as the following; For $m \in \mathbb{N}$, a tuple $\mathbf{T} = (T_1, \cdots, T_d) \in \mathcal{B}(\mathcal{X})^d$ of commuting operators is called *(m, ∞)-isometry* (or *(m, ∞)-isometric tuple*) if and only if

$$\max_{\substack{|\beta| \in \{0, \cdots, m\} \\ |\beta| \text{ even}}} \|\mathbf{T}^{\beta} x\| = \max_{\substack{|\beta| \in \{0, \cdots, m\} \\ |\beta| \text{ odd}}} \|\mathbf{T}^{\beta} x\|. \tag{8}$$

The goal of our paper is to introduce and study parallel results for commutative tuples of mappings in a metric space. Specifically, we seek analogous characterizations of the classes of mappings introduced in section 2 and section 3, respectively.

2. (m, q)-isometric commutative mappings

In this section, we give definitions of certain classes of mappings and investigate their basic properties of such tuples. Let (E, d) be a metric space and let $\mathbf{T} = (T_1, \cdots, T_n)$ be a commutative mappings where $T_j : E \to E$ for $j = 1, \cdots, n$. For $x, y \in E$, we write

$$\mathcal{H}_l^{(q)}(\mathbf{T}; x, y) := \sum_{0 \leq k \leq l} (-1)^{l-k} \binom{l}{k} \Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!} d\Big(\mathbf{T}^{\alpha} x, \mathbf{T}^{\alpha} y\Big)^q \Big). \tag{9}$$

Definition 2.1. *A tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of commutative mappings on a metric space E is said to be an (m, q)-isometric commutative tuple of mappings if for all $x, y \in E$,*

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!} d\Big(\mathbf{T}^{\alpha} x, \mathbf{T}^{\alpha} y\Big)^q \Big) = 0. \tag{10}$$

Remark 2.2. (i) When $n = 1$, this definition coincides with the definition of an (m, q)-isometry in a single variable mapping introduced by T. Bermúdez et al in ([4]).

(ii) Observe that if E is a Banach space and each T_j is a bounded linear operator on E, then (10) is equivalent to (7).

Remark 2.3. (i) Let $\mathbf{T} = (T_1, T_2)$ be a commuting 2-tuple of mappings on a metric space E. Then $\mathbf{T}$ is a $(1, q)$-isometric pair if

$$d\Big(T_1x, T_1y\Big)^q + d\Big(T_2x, T_2y\Big)^q = d\Big(x, y\Big)^q \text{ for all } x, y \in E$$

and $\mathbf{T}$ is a $(2, q)$-isometric pair if

$$d\Big(T_1^2x, T_1^2y\Big)^q + d\Big(T_2^2x, T_2^2y\Big)^q + 2d\Big(T_1T_2x, T_1T_2y\Big)^q - 2d\Big(T_1x, T_1y\Big)^q - 2d\Big(T_2x, T_2y\Big)^q + d\Big(x, y\Big)^q = 0$$

for all $x, y \in E$.
(ii) Let $\mathbf{T} = (T_1, \cdots, T_n)$ be a tuple of commutative mappings on a metric space E. Then $\mathbf{T}$ is a $(1, q)$-isometric commutative mappings if

$$\sum_{1\leq k\leq n} d(T_kx, T_ky)^q - d(x, y)^q = 0 \text{ for all } x, y \in E$$

and $\mathbf{T}$ is a $(2, q)$-isometric commutative mappings if for all $x, y \in E$,

$$\sum_{1\leq k\leq n} d\Big(T_k^2x, T_k^2y\Big)^q + 2\sum_{1\leq i<k\leq n} d\Big(T_iT_kx, T_iT_ky\Big)^q - 2\sum_{1\leq k\leq n} d\Big(T_kx, T_ky\Big)^q + d\Big(x, y\Big)^q = 0.$$

(iii) $\mathbf{T} = (T_1, \cdots, T_n)$ is a $(3, q)$-isometric commutative tuple of mappings of a metric space E if

$$\begin{aligned}
&\sum_{1\leq j\leq n} d\Big(T_j^3x, T_j^3y\Big)^q + 3\sum_{1\leq i\neq j\leq n} d\Big(T_iT_j^2x, T_iT_j^2y\Big)^q \\
&+6\sum_{1\leq i\neq j\neq r\leq n} d\Big(T_iT_jT_rx, T_iT_jT_ry\Big)^q \\
&-3\sum_{1\leq j\leq n} d\Big(T_j^2x, T_j^2y\Big)^q - 6\sum_{1\leq i\neq j\leq n} d\Big(T_jT_ix, T_jT_iy\Big)^q \\
&+3\sum_{1\leq j\leq n} d\Big(T_jx, T_jy\Big)^q - d\Big(x, y\Big)^q = 0 \text{ for all } x, y \in E.
\end{aligned}$$

Example 2.4. *Let $(\mathbb{R}, d_0)$ be the metric space where d_0 is the Euclidean metric and let $T_0 : \mathbb{R} \to \mathbb{R}$ be an (m, q)-isometric on a metric space (X, d_0) such that the distance d_0 satisfies $d_0(tx, ty) = td_0(x, y)$ for all $x, y \in X$ and for all $t > 0$. Then $\mathbf{T} = (T_1, \cdots, T_n)$, where $T_j = \dfrac{1}{\sqrt[q]{n}}T_0$ $(j = 1, \cdots, n)$ is an (m, q)-isometric commutative tuple of mappings. Indeed, clearly $T_iT_j = T_jT_i$ for all $i, j = 1, \cdots, n$ and*

$$\begin{aligned}
\sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}d_0\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)^q &= \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}d_0\Big(T^{|\alpha|}x, T^{|\alpha|}y\Big)^q \\
&= \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k}n^k d_0\Big(T^kx, T^ky\Big)^q \\
&= \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k}n^k|T^kx - T^ky|^q \\
&= \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k}d_0\Big(T_0^kx, T_0^ky\Big)^q = 0.
\end{aligned}$$

In [4, Proposition 1.4], it was proved that if T is a bijective (m, q)-isometric mapping on a metric space E, then T^{-1} is also an (m, q)-isometric mapping. However, this result is not true for commutative tuple of mappings as show in the following example.

Example 2.5. *Let $T_j : (\mathbb{R}, d) \longrightarrow (\mathbb{R}, d)$ be defined by $T_j x = \frac{1}{\sqrt[q]{n}} x$ for $q > 0$, $j = 1, \cdots, n$, where $d(x, y) = |x - y|$. Clearly, each T_j $(j = 1, \cdots, n)$ is bijective and*

$$\sum_{1\leq j\leq n} d(T_j x, T_j y)^q = \sum_{1\leq j\leq n} \Big(\frac{1}{\sqrt[q]{n}}\Big)^q |x - y|^q = |x - y|^q = d(x, y)^q.$$

Thus $\mathbf{T} = (T_1, \cdots, T_n)$ is a $(1, q)$-isometric commutative tuple. Furthermore, $\mathbf{T}^{-1} := (T_1^{-1} \cdots, T_n^{-1})$ where $T_j^{-1} x = \sqrt[q]{n}\, x$ $(j = 1, \cdots, n)$. A simple computation shows that

$$\sum_{1\leq j\leq n} d(T_j^{-1} x, T_j^{-1} y)^q = \sum_{1\leq j\leq n} \Big(\sqrt[q]{n}\Big)^q |x - y|^q = n^2 |x - y|^q \neq d(x, y)^q.$$

Consequently, $\mathbf{T}^{-1} := (T_1^{-1} \cdots, T_n^{-1})$ is not a $(1, q)$-isometric tuple.

Proposition 2.6. *Let $\mathbf{T} = (T_1, \cdots, T_n)$ be a tuple of commutative mappings on a metric space E. Then for all positive integer m, a positive real number q and $x, y \in (E, d)$, we have*

$$\mathcal{H}_{m+1}^{(q)}(\mathbf{T}; x, y) = \sum_{1\leq k\leq n} \mathcal{H}_m^{(q)}\Big(\mathbf{T}; T_k x, T_k y\Big) - \mathcal{H}_m^{(q)}(\mathbf{T}; x, y). \tag{11}$$

In particular, if $\mathbf{T}$ is an (m, q)-isometric commutative tuple, then $\mathbf{T}$ is a (k, q)-isometric commutative tuple for all $k \geq m$.

Proof. By taking into account Equation (9), a straightforward calculation shows that

$$\begin{aligned}
&\mathcal{H}_{m+1}^{(q)}(\mathbf{T}; x, y) \\
= \;& \sum_{0\leq k\leq m+1} (-1)^{m+1-k} \binom{m+1}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)^q \\
= \;& (-1)^{m+1} d\Big(x, y\Big)^q - \sum_{1\leq k\leq m} (-1)^{m-k} \left[\binom{m}{k} + \binom{m}{k-1}\right] \sum_{|\alpha|=k} \frac{k!}{\alpha!} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)^q + \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)^q \\
= \;& -\mathcal{H}_m^{(q)}(\mathbf{T}; x, y) + \sum_{0\leq k\leq m-1} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)^q + \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)^q \\
= \;& -\mathcal{H}_m^{(q)}(\mathbf{T}; x, y) + \sum_{0\leq k\leq m-1} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k+1} \frac{k!(\alpha_1 + \cdots + \alpha_n)}{\alpha_1! \cdot \alpha_2! \cdots \alpha_n!} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)^q \\
& + \sum_{|\alpha|=m+1} \frac{m!(\alpha_1 + \cdots + \alpha_n)}{\alpha_1! \cdot \alpha_2! \cdots \alpha_n!} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)^q
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\mathcal{H}_m^{(q)}(\mathbf{T};x,y) \\
& + \sum_{1\le j\le n}\ \sum_{0\le k\le m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k+1}\frac{k!\alpha_j}{\alpha_1!\cdot\alpha_2!\cdots\alpha_n!} \\
& \cdot d\Big(T_j\Big(T_1^{\alpha_1}\cdots T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}\cdots T_n^{\alpha_n}x\Big),\Big(T_1^{\alpha_1}\cdots T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}\cdots T_n^{\alpha_n}\Big)T_jy\Big)^q \\
& + \sum_{1\le j\le n}\ \sum_{|\alpha|=m+1}\frac{m!\alpha_j}{\alpha_1!\cdot\alpha_2!\cdots\alpha_n!} \\
& \cdot d\Big(T_jT_1^{\alpha_1}\cdots T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}\cdots T_n^{\alpha_n}x,T_1^{\alpha_1}\cdots T_j^{\alpha_j-1}T_{j+1}^{\alpha_{j+1}}\cdots T_n^{\alpha_n}T_jy\Big)^q \\
= \quad & -\mathcal{H}_m^{(q)}(\mathbf{T};x,y) \\
& + \sum_{1\le j\le n}\ \sum_{0\le k\le m-1}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha T_jx,\ \mathbf{T}^\alpha T_jy\Big)^q + \sum_{1\le j\le n}\ \sum_{|\alpha|=m}\frac{m!}{\alpha!}d\Big(\mathbf{T}^\alpha T_jx,\ \mathbf{T}^\alpha T_jy\Big)^q \\
= \quad & -\mathcal{H}_m^{(q)}(\mathbf{T}x,y) + \sum_{1\le j\le n}\Big(\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha T_jx,\ \mathbf{T}^\alpha T_jy\Big)^q\Big) \\
= \quad & -\mathcal{H}_m^{(q)}(\mathbf{T};x,y) + \sum_{1\le j\le n}\mathcal{H}_m^{(q)}(\mathbf{T};T_jx,T_jy),
\end{aligned}
$$

and so Equality (11) is satisfied on a metric space (E,d). From the fact that $\mathbf{T}$ is a (k,q)-isometric tuple of mappings when $\mathbf{T}$ is an (m,q)-isometric tuple of mappings for $k\ge m$, the proof follows immediately from Equation (11). □

The following theorem is a generalization of [8, Theorem 2.1] into the structure of metric.

Theorem 2.7. *Let $\mathbf{T}=(T_1,\cdots,T_n)$ be a tuple of commutative mappings on a metric space E. Then the following statements hold:*

(i)

$$\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q=\sum_{0\le j\le k}\binom{k}{j}\mathcal{H}_j^{(q)}(\mathbf{T};x,y) \tag{12}$$

for every integer $k\ge 1, q>0$ and $\forall\ x,y\in E$.

(ii) $\mathbf{T}$ *is an (m,q)-isometric tuple of commutative mappings if and only if*

$$\sum_{|\alpha|=n}\frac{n!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q=\sum_{0\le j\le m-1}\binom{n}{j}\mathcal{H}_j^{(q)}(\mathbf{T};x,y) \tag{13}$$

for all $n\in\mathbb{N}, q>0$ and $\forall\ x,y\in E$.

(iii) *If* $\mathbf{T}$ *is an (m,q)-isometric tuple of mappings, then*

$$\mathcal{H}_{m-1}^{(q)}(\mathbf{T};x,y)=\lim_{k\to\infty}\frac{1}{\binom{k}{m-1}}\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q. \tag{14}$$

Proof. (i) We prove (12) by using mathematical induction. The result is true for $k=0,1$. Now we assume that the result is true for k and let us prove it for $k+1$.

In this way, applying (9) and (12), we have

$$\begin{aligned}
&\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}d\Big(\mathbf{T}^{\alpha}x,\mathbf{T}^{\alpha}y\Big)^{q}\\
=\;&\mathcal{H}^{(q)}_{k+1}(\mathbf{T};x,y)-\sum_{0\le j\le k}(-1)^{k+1-j}\binom{k+1}{j}\sum_{|\alpha|=j}\frac{j!}{\alpha!}d\Big(\mathbf{T}^{\alpha}x,\mathbf{T}^{\alpha}y\Big)\\
=\;&\mathcal{H}^{(q)}_{k+1}(\mathbf{T};x,y)-\sum_{0\le j\le k}(-1)^{k+1-j}\binom{k+1}{j}\sum_{0\le r\le j}\binom{j}{r}\mathcal{H}^{(q)}_{r}(\mathbf{T};x,y)\\
=\;&\mathcal{H}^{(q)}_{k+1}(\mathbf{T};x,y)-\sum_{0\le r\le k}\mathcal{H}^{(q)}_{r}(\mathbf{T};x,y)\sum_{r\le j\le k}(-1)^{k+1-j}\binom{k+1}{j}\binom{j}{r}\\
=\;&\mathcal{H}^{(q)}_{k+1}(\mathbf{T};x,y)-\sum_{0\le r\le k}\mathcal{H}^{(q)}_{r}(\mathbf{T};x,y)\Big(\sum_{r\le j\le k}(-1)^{k+1-j}\binom{k+1}{r}\binom{k+1-r}{j-r}\Big)\\
=\;&\mathcal{H}^{(q)}_{k+1}(\mathbf{T};x,y)-\sum_{0\le r\le k}\binom{k+1}{r}\mathcal{H}^{(q)}_{r}(\mathbf{T};x,y)\Big(\sum_{r\le j\le k}(-1)^{k+1-j}\binom{k+1-r}{j-r}\Big)\\
=\;&\mathcal{H}^{(q)}_{k+1}(\mathbf{T};x,y)-\sum_{0\le r\le k}\binom{k+1}{r}\mathcal{H}^{(q)}_{r}(\mathbf{T};x,y)\Big(-1+\underbrace{\sum_{0\le r\le k+1-j}(-1)^{k+1-j-r}\binom{k+1-r}{r}}_{=0}\Big)\\
=\;&\sum_{0\le r\le k+1}\binom{k+1}{r}\mathcal{H}^{(q)}_{r}(\mathbf{T};x,y).
\end{aligned}$$

(ii) The only if part of the statement (ii) follows from (12) since an (m,q)-isometric tuple of mappings it is a (k,q)-isometric tuple of mappings for $k\ge m$.

(iii) Under the statement (ii), it is straightforward to see that

$$\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^{\alpha}x,\mathbf{T}^{\alpha}y\Big)^{q}=\sum_{0\le j\le m-2}\binom{k}{j}\mathcal{H}^{(q)}_{j}(\mathbf{T};x,y)+\binom{k}{m-1}\mathcal{H}^{(q)}_{m-1}(\mathbf{T};x,y).$$

Dividing both sides by $\binom{k}{m-1}$ and using that $\dfrac{\binom{k}{j}}{\binom{k}{m-1}}\to 0$ for $0\le j\le m-2$ (as $k\to\infty$), we obtain the required identity and completes the proof. □

In the following proposition, we generalize [8, Corollary 1 and Corollary 2] into the structure of metric. Since the proof is very similar, we omit it.

Proposition 2.8. *Let $\mathbf{T}=(T_1,\cdots,T_n)$ be a commutative tuple of mappings on a metric space E. Then the following statements hold.*

(i) $\mathbf{T}$ *is an (m,q)-isometric tuple of commutative mappings if and only if*

$$\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^{\alpha}x,\mathbf{T}^{\alpha}y\Big)^{q}=\sum_{0\le j\le m-1}\Big(\sum_{j\le p\le m-1}(-1)^{p-j}\binom{k}{p}\binom{p}{j}\Big)\sum_{|\alpha|=j}\frac{j!}{\alpha!}d\Big(\mathbf{T}^{\alpha}x,\mathbf{T}^{\alpha}y\Big)^{q}.\tag{15}$$

(ii) *If* $\mathbf{T}$ *is an* (m,q)*-isometric tuple of commutative mappings and* $k \in \mathbb{N}$*, then the following identities hold for* $k \geq m$*,*

$$\sum_{0\leq j\leq k}(-1)^j\binom{k}{j}j^p\Big(\sum_{|\alpha|=k-j}\frac{(k-j)!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q\Big)=0 \tag{16}$$

for each $p \in \{0,1,\cdots,k-m\}$*.*

Corollary 2.9. *Let* $\mathbf{T} = (T_1,\cdots,T_n)$ *be a tuple of commuting mappings in a metric space* (E,d)*. We have:*

(i) $\mathbf{T}$ *is a* $(2,q)$*-isometric commutative tuple if and only if* $\mathbf{T}$ *satisfies*

$$\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q=(1-k)d\Big(x,y\Big)^q+k\sum_{1\leq j\leq n}d\Big(T_jx,T_jy\Big)^q \tag{17}$$

for all $k \in \mathbb{N}$ *and* $x,y \in E$*.*

(ii) *If* $\mathbf{T}$ *is a* $(2,q)$*-isometric commutative tuple of mappings, then the following identities hold:*

$$\sum_{1\leq j\leq n}d\Big(T_jx,T_jy\Big)^q\geq\frac{k-1}{k}d\Big(x,y\Big)^q\quad(\forall\, k\in\mathbb{N}\text{ and }\forall\, x,y\in E), \tag{18}$$

$$\sum_{1\leq j\leq n}d\Big(T_jx,T_jy\Big)^q\geq d\Big(x,y\Big)^q\quad(\forall\, x,y\in E), \tag{19}$$

$$\lim_{k\to\infty}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q\Big)^{\frac{1}{k}}=1\qquad(\forall\, x,y\in E,\ \ x\neq y). \tag{20}$$

Proof. (*i*) Using the statement (ii) of Proposition 2.9 and the statement (ii) of Remark 2.3, we get the desired equivalence.

(*ii*) Inequality (18) follows from (17) and Inequality (19) follows from (18) by taking $k \to \infty$.

To prove (20), take $x,y \in E$ with $x \neq y$. It follows from (17) that

$$\limsup_{k\to\infty}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q\Big)^{\frac{1}{k}}\leq 1.$$

However, according to (19), the sequence $\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q\Big)_{k\in\mathbb{N}}$ is monotonically increasing, so

$$\liminf_{k\to\infty}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q\Big)^{\frac{1}{k}}\geq\lim_{k\to\infty}\Big(d(x,y)^q\Big)^{\frac{1}{k}}=1.$$

□

Definition 2.10. *Let* $\mathbf{T} = (T_1,\cdots,T_n)$ *be a tuple of commutative mappings on a metric space* (E,d)*. Then* $\mathbf{T}$ *is said to be a power bounded tuple if*

$$\sup\Big\{\sum_{|\alpha|=k}\frac{k!}{\alpha!}d\Big(\mathbf{T}^\alpha x,\mathbf{T}^\alpha y\Big)^q\ ,\ \forall\, k\in\mathbb{N}\Big\}<\infty$$

for all $x,y \in E$*.*

Theorem 2.11. *Let* $\mathbf{T} = (T_1, \cdots, T_n)$ *be an* (m, q)*-isometric commutative and power bounded tuple. Then*

$$\Big(\sum_{1\leq i\leq n} d(T_i x, T_i y)^q\Big)^{\frac{1}{q}} = d(x, y),$$

i.e., $\mathbf{T}$ *is a* $(1, q)$*-isometric tuple.*

Proof. Since $\mathbf{T}$ is (m, q)-isometric, by Equation (13), for every $k \in \mathbb{N}$ it holds

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} d\Big(\mathbf{T}^\alpha x,\ \mathbf{T}^\alpha y\Big)^q = \sum_{0\leq j\leq m-1} \binom{k}{j} \mathcal{H}_j^{(q)}(\mathbf{T}; x, y).$$

Hence, there exist real numbers $\delta_0(x, y), \delta_1(x, y), ..., \delta_{m-1}(x, y)$ such that

$$\sum_{1\leq i\leq n} d\Big(T_i^k x,\ T_i^k y\Big)^q = \sum_{0\leq j\leq m-1} \delta_j(x, y)\, k^j. \tag{21}$$

Since $\mathbf{T}$ is power bounded, we put

$$M = \sup\Big\{\sum_{|\alpha|=k} \frac{k!}{\alpha!} d\Big(\mathbf{T}^\alpha x,\ \mathbf{T}^\alpha y\Big)^q,\ \ k = 0, 1, 2, ...\Big\} < \infty$$

for $x, y \in E$. Then we have

$$0 \leq \sup\Big\{\sum_{0\leq j\leq m-1} \delta_j(x, y) k^j\ :\ k = 0, 1, 2, ...\Big\} \leq M^q.$$

Since k is arbitrary, we have $\delta_1(x, y) = \delta_2(x, y) = \cdots = \delta_{m-1}(x, y) = 0$. Hence

$$\sum_{0\leq i\leq n} d(T_i^k x,\ T_i^k y)^q = d(x, y)^q.$$

Since k is arbitrary, letting $k = 1$ we have a desired equality. □

3. (m, ∞)-Isometric commutative mappings

In this section, we focus on (m, ∞)-Isometric commutative mappings.

Definition 3.1. *A tuple* $\mathbf{T} = (T_1, \cdots, T_n)$ *of commutative mappings on a metric space* E *is said to be an* (m, ∞)*-isometric commutative tuple of mappings if for all* $x, y \in E$

$$\max_{\begin{pmatrix} |\alpha| \in \{1, \cdots, m\} \\ |\alpha|\ even \end{pmatrix}} \{d\Big(\mathbf{T}^\alpha x,\ \mathbf{T}^\alpha y\Big)\} = \max_{\begin{pmatrix} |\alpha| \in \{1, \cdots, m\} \\ |\alpha|\ odd \end{pmatrix}} \{d\Big(\mathbf{T}^\alpha x,\ \mathbf{T}^\alpha y\Big)\}. \tag{22}$$

Remark 3.2. (i) For $n = 1$, this definition coincides with the definition of an (m, ∞)- isometry in a single variable mapping introduced in [9].

(ii) If $n = 2$ and $\mathbf{T} = (T_1, T_2)$ is a pair of commutative mappings of a metric space E, then $\mathbf{T}$ is a $(2, \infty)$-isometric tuple if

$$\max\Big\{d\Big(T_1 x, T_1 y\Big), d\Big(T_2 x, T_2 y\Big)\Big\}$$

$$= \max\Big\{d\big(x,y\big), d\big(T_1^2x, T_1^2y\big), d\big(T_2^2x, T_2^2y\big), d\big(T_1T_2x, T_1T_2y\big)\Big\}.$$

(iii) A tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of commutative mappings of a metric space E is a $(1,\infty)$- isometric tuple if

$$\max_{|\alpha|=1}\Big\{d\big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\big)\Big\} := \max\Big\{d\big(T_jx, T_jy\big); \quad j = 1, \cdots, n\Big\} = d\big(x,y\big), \ \forall \ x, y \in E.$$

Example 3.3. *Let T_0 be a mapping on a metric space (E,d) which is (m,∞)-isometric mapping. Then the n-tuple of mappings $\mathbf{T} = (T_0, \cdots, T_0)$ is an (m,∞)-isometric tuple of commutative mappings on E. In fact, since $\mathbf{T} = (T_0, \cdots, T_0)$ and T_0 is an (m,∞)-isometry, it follows that*

$$\begin{aligned}\max_{\begin{pmatrix}|\alpha| \in \{1,\cdots,m\}\\ |\alpha| \text{ even}\end{pmatrix}} \{d\big(\mathbf{T}^\alpha x, \ \mathbf{T}^\alpha y\big)\} &= \max_{\begin{pmatrix}|\alpha| \in \{1,\cdots,m\}\\ |\alpha| \text{ even}\end{pmatrix}} \{d\big(T_0^{|\alpha|}x, \ T_0^{|\alpha|}y\big)\}\\ &= \max_{\begin{pmatrix}|\alpha| \in \{1,\cdots,m\}\\ |\alpha| \text{ odd}\end{pmatrix}} \{d\big(T_0^{|\alpha|}x, \ T_0^{|\alpha|}y\big)\}\\ &= \max_{\begin{pmatrix}|\alpha| \in \{1,\cdots,m\}\\ |\alpha| \text{ odd}\end{pmatrix}} \{d\big(\mathbf{T}^\alpha x, \ \mathbf{T}^\alpha y\big)\}.\end{aligned}$$

Example 3.4. *Let (E,d) be a metric space where $E = \{0,1,2\}$ and $d(x,y) = |x-y|$ for $x, y \in E$. Define mappings T_1 and $T_2 : E \to E$ by*

$$T_1(0) = 2, \ T_1(1) = T_1(2) = 0 \quad \textit{and} \quad T_2(0) = 0, \ T_2(1) = T_2(2) = 2.$$

Then a straightforward calculation shows that $T_1T_2 = T_2T_1$ and

$$\begin{aligned}&\max\Big\{d\big(x,y\big), d\big(T_1^2x, T_1^2y\big), d\big(T_2^2x, T_2^2y\big), d\big(T_1T_2x, T_1T_2y\big)\Big\}\\ = \ & 2\\ = \ & \max\Big\{d\big(T_1x, T_1y\big), d\big(T_2x, T_2y\big)\Big\}.\end{aligned}$$

Therefore, we conclude that the pair $\mathbf{T} = (T_1, T_2)$ is a $(2,\infty)$-isometric pair of mappings.

Example 3.5. *Let $E := \mathbb{R}^2$ and $\{e_j\}_{j=1,2}$ be the natural base. For $x = (x_1, x_2), y = (y_1, y_2) \in E$, let $d(x,y) := \max_{j=1,2}|x_j - y_j|$. Define mappings T_1 and $T_2 : E \to E$ by*

$$T_1e_1 = e_1, \ T_1e_2 = 0 \quad \textit{and} \quad T_2e_1 = 0, \ T_2e_2 = e_2,$$

that is, T_1, T_2 are projections. Then $T_1T_2 = T_2T_1 = 0$ and a straightforward calculation shows that

$$\begin{aligned}&\max\Big\{d\big(x,y\big), d\big(T_1^2x, T_1^2y\big), d\big(T_2^2x, T_2^2y\big), d\big(T_1T_2x, T_1T_2y\big)\Big\}\\ = \ & d(x,y)\\ = \ & \max\Big\{d\big(T_1x, T_1y\big), d\big(T_2x, T_2y\big)\Big\}.\end{aligned}$$

Therefore, we conclude that the pair $\mathbf{T} = (T_1, T_2)$ is a $(2,\infty)$-isometric pair of mappings.

Proposition 3.6. *Let $\mathbf{T} = (T_1, \cdots, T_n)$ be a commutative tuple of mappings acting on a metric space E. Then $\mathbf{T}$ is an (m,∞)-isometric tuple if and only if for all $l \in \mathbb{N}$ and for all $x, y \in E$,*

$$\max_{\begin{pmatrix}|\alpha| \in \{l,\cdots,m+l\}\\ |\alpha| \text{ even}\end{pmatrix}} \{d\big(\mathbf{T}^\alpha x, \ \mathbf{T}^\alpha y\big)\} = \max_{\begin{pmatrix}|\alpha| \in \{l,\cdots,m+l\}\\ |\alpha| \text{ odd}\end{pmatrix}} \{d\big(\mathbf{T}^\alpha x, \ \mathbf{T}^\alpha y\big)\}. \tag{23}$$

Proof. Assume that $\mathbf{T}$ is an (m,∞)-isometric tuple and $l \in \mathbb{N}$ is an even integer. Then we have

$$\begin{aligned}
\max_{\substack{|\alpha| \in \{l,\cdots,m+l\} \\ |\alpha| \text{ even}}} \{d\Big(\mathbf{T}^\alpha x,\ \mathbf{T}^\alpha y\Big)\} &= \max_{\substack{|\beta| \in \{0,\cdots,m\} \\ |\gamma| = l \\ |\beta|+|\gamma| \text{ even}}} \{d\Big(\mathbf{T}^\beta\mathbf{T}^\gamma x,\ \mathbf{T}^\alpha\mathbf{T}^\gamma y\Big)\} \\
&= \max_{|\gamma|=l}\Big(\max_{\substack{|\beta| \in \{0,\cdots,m\} \\ |\beta| \text{ even}}} d\Big(\mathbf{T}^\beta\mathbf{T}^\gamma x,\ \mathbf{T}^\alpha\mathbf{T}^\gamma y\Big)\Big) \\
&= \max_{|\gamma|=l}\Big(\max_{\substack{|\beta| \in \{0,\cdots,m\} \\ |\beta| \text{ odd}}} d\Big(\mathbf{T}^\beta\mathbf{T}^\gamma x,\ \mathbf{T}^\alpha\mathbf{T}^\gamma y\Big)\Big) \\
&= \max_{\substack{|\beta| \in \{0,\cdots,m\} \\ |\gamma| = l \\ |\beta|+|\gamma| \text{ odd}}} \{d\Big(\mathbf{T}^\beta\mathbf{T}^\gamma x,\ \mathbf{T}^\alpha\mathbf{T}^\gamma y\Big)\} \\
&= \max_{\substack{|\alpha| \in \{l,\cdots,m+l\} \\ |\alpha| \text{ odd}}} \{d\Big(\mathbf{T}^\alpha x,\ \mathbf{T}^\alpha y\Big)\}.
\end{aligned}$$

If l is an odd integer, we can repeat quite similar arguments as those above to prove that

$$\max_{\substack{|\alpha| \in \{l,\cdots,m+l\} \\ |\alpha| \text{ even}}} \{d\Big(\mathbf{T}^\alpha x,\ \mathbf{T}^\alpha y\Big)\} = \max_{\substack{|\alpha| \in \{l,\cdots,m+l\} \\ |\alpha| \text{ odd}}} \{d\Big(\mathbf{T}^\alpha x,\ \mathbf{T}^\alpha y\Big)\}.$$

This implies that (23) holds for all $l \in \mathbb{N}$. □

In the following corollary, we denote by $\pi(k) = k \pmod 2$ the parity of $k \in \mathbb{N}$.

Corollary 3.7. *Let $\mathbf{T} = (T_1,\cdots,T_n)$ be a commutative tuple of mappings of a metric space E and $m \in \mathbb{N}$. Then $\mathbf{T}$ is an (m,∞)-isometric commuting tuple if and only if*

$$\max_{\alpha\in\mathbb{N}_0^n} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big) = \max_{\substack{|\beta| \in \{l,\cdots,m-1+l\} \\ \pi(|\beta|) = \pi(m-1+l)}} d\Big(\mathbf{T}^\beta x, \mathbf{T}^\beta y\Big)$$

for all $x, y \in E$ and for all $l \in \mathbb{N}_0$.

Proof. For $x, y \in E$, we can consider the sequence $a_\alpha := d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)$ for $\alpha \in \mathbb{N}_0^n$. The result follows from [6, Lemma 5.6] by choosing $(a_\alpha)_{\alpha\in\mathbb{N}_0^n}$. □

Corollary 3.8. *Let $\mathbf{T} = (T_1,\cdots,T_n)$ be a commutative tuple of mappings of a metric space E which is an (m,∞)-isometric tuple. Then for all $\alpha \in \mathbb{N}^n$ and for all $x, y \in E$,*

$$d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big) \le \max_{|\gamma|\in\{0,\cdots,m-1\}} d\Big(\mathbf{T}^\gamma x, \mathbf{T}^\gamma y\Big) \quad (\forall\ x, y \in E).$$

In particular, $\mathbf{T}$ is power bounded.

Proof. It is obvious that $d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big) \leq \max\limits_{\alpha_0 \in \mathbb{N}_0^n} d\Big(\mathbf{T}^{\alpha_0} x, \mathbf{T}^{\alpha_0} y\Big)$ for $x, y \in E$. Hence, taking into account Corollary 3.7, we obtain

$$d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big) \leq \max_{|\gamma| \in \{0, \cdots, m-1\}} d\Big(\mathbf{T}^\gamma x, \mathbf{T}^\gamma y\Big) \quad (\forall\, x, y \in E).$$

□

Theorem 3.9. *Let $\mathbf{T} = (T_1, \cdots, T_d)$ be an (m, ∞)-isometric commutative tuple of mappings on a metric space (E, d). Then there exists a metric d_∞ on E such that $\mathbf{T}$ is a $(1, \infty)$-isometric tuple on (E, d_∞). Moreover, d_∞ is given by*

$$d_\infty(x, y) = \max_{|\alpha| \in \{0, \cdots, m-1\}} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big), \quad \forall \;\; x, y \in E.$$

Proof. Working under the assumption described above that $\mathbf{T}$ is an (m, ∞)-isometric tuple, then as an application of Corollary 3.8 we obtain that

$$\max_{\alpha \in \mathbb{N}_0^n} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big) = \max_{|\alpha| \in \{0, \cdots, m-1\}} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big), \;\; \forall\, x, y \in E.$$

Define the map $d_\infty : E \times E \to \mathbb{R}_+$ given by

$$d_\infty\Big(x, y\Big) := \max_{|\alpha| \in \{0, \cdots, m-1\}} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big), \;\; \forall\, x, y \in E.$$

One can easily see that the map d_∞ is a metric on E. On the other hand, we have

$$\begin{aligned} d_\infty(x, y) &= \max_{|\alpha| \in \{0, \cdots, m-1\}} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y) \\ &= \max_{\alpha \in \mathbb{N}_0^n} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y) \\ &= \max_{|\alpha| \in \{l, \cdots, m-1+l\}} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y) \\ &= \max_{\substack{|\beta| \in \{0, \cdots, m-1\} \\ |\gamma| = l}} d\Big(\mathbf{T}^{\beta+\gamma} x, \mathbf{T}^{\beta+\gamma} y\Big) \\ &= \max_{|\gamma| = l} \max_{|\beta| \in \{0, \cdots, m-1\}} d\Big(\mathbf{T}^{\beta+\gamma} x, \mathbf{T}^{\beta+\gamma} y\Big), \; \forall\, l \in \mathbb{N}_0. \end{aligned}$$

In particular, we get that

$$\begin{aligned} d_\infty(x, y) &= \max_{|\gamma|=1}\Big(\max_{|\beta| \in \{0, \cdots, m-1\}} d\Big(\mathbf{T}^\beta \mathbf{T}^\gamma x, \mathbf{T}^\beta \mathbf{T}^\gamma y\Big)\Big) \\ &= \max_{|\gamma|=1}\Big(d_\infty\Big(\mathbf{T}^\gamma x, \mathbf{T}^\gamma y\Big)\Big). \end{aligned}$$

Consequently, T is a $(1, \infty)$-isometric tuple on (E, d_∞) and the proof is complete. □

Proposition 3.10. *Let $\mathbf{T} = (T_1, \cdots, T_n)$ be a tuple of commutative mappings on a metric space E. If $\mathbf{T}$ is an (m, ∞)-isometric tuple, then $\mathbf{T}$ is an $(m+1, \infty)$-isometric tuple.*

Proof. Under the assumption that $\mathbf{T}$ is an (m, ∞)-isometric tuple, it follows that

$$\max_{k \in \mathbb{N}_0^n} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big) = \max_{\substack{|\alpha| \in \{l, \cdots, m-1+l\} \\ \pi(|\alpha|) = \pi(m-1+l)}} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y) \;\; \forall\, x, y \in E, \; \forall\, l \in \mathbb{N}_0.$$

Hence we get, for all $x, y \in E$ and $\forall\ l \in \mathbb{N}_0$,

$$\begin{aligned}
\max_{|\alpha|\in\mathbb{N}_0^n}\{d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)\} &= \max_{\substack{|\alpha| \in \{l, \cdots, m-1+l\} \\ \pi(|\alpha|) = \pi(m-1+l)}} \{d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)\} \\
&\leq \max_{\substack{|\alpha| \in \{l, \cdots, m+l\} \\ \pi(|\alpha|) = \pi(m+l)}} \{d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)\} \\
&\leq \max_{\alpha\in\mathbb{N}_0^n}\{d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big)\}.
\end{aligned}$$

From which we obtain

$$\max_{\alpha\in\mathbb{N}_0^n} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big) = \max_{\substack{|\alpha| \in \{l, \cdots, m+l\} \\ \pi(|\alpha|) = \pi(m+l)}} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big).$$

We are able to deduce that $\mathbf{T}$ is an $(m+1, \infty)$-isometric tuple. □

Proposition 3.11. *Let* $\mathbf{T}$ *be a commutative mappings acting on a metric space* E*. Assume that there exists* $p = (p_1, \cdots, p_n) \in \mathbb{N}^n$ *with* $|p|$ *is an odd integer such that* $\mathbf{T}^p := T_1^{p_1} \cdots T_n^{p_n}$ *is an isometric. Then* $\mathbf{T}$ *is an* (m, ∞)*-isometric tuple for* $m \geq 2|p| - 1$.

Proof. Due to Proposition 3.10 it suffices to prove that $\mathbf{T}$ is a $(2|p|-1, \infty)$-isometric tuple. From the hypothesis, $\mathbf{T}^p$ is isometric and we have

$$d\Big(\mathbf{T}^{\alpha+p}x, \mathbf{T}^{\alpha+p}y\Big) = d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big), \quad \forall\ x, y \in E,\ \forall\ \alpha \in \mathbb{N}_0^n.$$

On the other hand, since $|p|$ is an odd integer, we observe that for all $\alpha \in \mathbb{N}_0^n$, $|\alpha|$ is even if and only if $|\alpha| + |p|$ is odd. Moreover, since $\mathbf{T}^p$ is isometric, it follows that

$$\Big\{d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big),\ |\alpha| \in \{0, \cdots, 2|p|-1\},\ |\alpha| \text{ even } \Big\} = \Big\{d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big),\ |\alpha| \in \{0, 1, \cdots, 2|p|-1\},\ |\alpha| \text{ odd } \Big\}.$$

Taking the max over $\{0, \cdots, 2|p|-1\ \}$, we obtain

$$\max_{\substack{|\alpha| \in \{0, 1, \cdots, 2|p|-1\} \\ |\alpha| \text{ odd}}} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big) = \max_{\substack{|\alpha| \in \{0, 1, \cdots, 2|p|-1\} \\ |\alpha| \text{ even}}} d\Big(\mathbf{T}^\alpha x, \mathbf{T}^\alpha y\Big).$$

Consequently, we are in a position to conclude that $\mathbf{T}$ is a $(2|p|-1, \infty)$-isometric tuple. □

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