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On the Equality of Triple Derivations and Derivations of Lie Algebras

Mohammad Hossein Jafari[a], Ali Reza Madadi[a]

[a]*Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran*

Abstract. Let L be a Lie algebra over a commutative ring with identity. In the present paper under some mild conditions on L, it is proved that every triple derivation of L is a derivation. In particular, we show that in perfect Lie algebras and free Lie algebras every triple derivation is a derivation. Finally we apply our results to show that every triple derivation of the Lie algebra of block upper triangular matrices is a derivation.

1. Introduction

Let R denote a commutative ring with identity and L denote a Lie algebra over R. An R-linear map $D : L \to L$ is called a derivation of L if

$$D([x, y]) = [D(x), y] + [x, D(y)],$$

for any $x, y \in L$, and is called a triple derivation of L if

$$D([x, [y, z]]) = [D(x), [y, z]] + [x, [D(y), z]] + [x, [y, D(z)]],$$

for any $x, y, z \in L$.

The set of all derivations of L and all triple derivations of L is denoted by $\mathrm{Der}(L)$ and $\mathrm{TDer}(L)$, respectively. One can easily see that $\mathrm{TDer}(L)$ is a Lie algebra which contains $\mathrm{Der}(L)$ as a subalgebra.

For a subset X of L, the centralizer of X in L is denoted by $C_L(X)$ and is defined as follows:

$$C_L(X) = \{y \in L | [x, y] = 0, \forall x \in X\}.$$

In particular, $Z(L) = C_L(L)$ is called the center of L. Now let $\{L^n\}_{n\geq 1}$ and $\{Z_n(L)\}_{n\geq 0}$ denote respectively the lower central series of L and the upper central series of L, that is,

$$L^1 = L, \quad L^{n+1} = [L, L^n],$$

$$Z_0(L) = 0, \quad Z_{n+1}(L)/Z_n(L) = Z(L/Z_n(L)).$$

A Lie algebra L is said to be perfect if $L = L^2$. Also L is called nilpotent if $L^n = 0$ for some $n \geq 1$ or equivalently $Z_n(L) = L$ for some $n \geq 0$.

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Email addresses: jafari@tabrizu.ac.ir (Mohammad Hossein Jafari), a-madadi@tabrizu.ac.ir (Ali Reza Madadi)

It is obvious that every derivation maps $Z(L)$ into itself and L^2 into itself. Also every triple derivation clearly maps $Z_2(L)$ into itself and L^3 into itself, the point which will be used later. It should be mentioned that, in general, it is not true that $Z(L)$ and L^2 are invariant under every triple derivation. For example, let L be the Heisenberg Lie algebra, i.e. the Lie algebra of all 3×3 strictly upper triangular matrices over R with the standard basis $\{e_{12}, e_{13}, e_{23}\}$. Then one can easily see that $Z(L) = L^2 = Re_{13}$, $L^3 = 0$, $Z_2(L) = L$, and every R-linear map on L is a triple derivation of L. Now if D is the R-linear map on L defined by $D(e_{12}) = D(e_{13}) = D(e_{23}) = e_{12}$, then it is a triple derivation of L but not a derivation of L and it does not map $Z(L) = L^2$ into itself.

Algebraic systems with derivations and their generalizations are a popular object of study nowadays. In particular, the algebras of derivations and generalized derivations are important in the study of algebraic systems of Lie type. Triple derivations, which are sometimes called prederivations, and their generalizations, Leibniz-derivations of order n, were used to study nilpotent Lie algebras, see [1], [3], [4], [7], [8]. For instance, Bajo in [1] shows that a finite-dimensional Lie algebra over a field of characteristic zero admitting a non-singular triple derivation is necessarily nilpotent. For the converse, Burde in [3] proves that if L is a finite-dimensional nilpotent Lie algebra with $L^5 = 0$ over a field of characteristic zero, then L possesses a non-singular triple derivation. In [4], the authors show that any finite-dimensional Lie algebra L over the field of complex numbers admitting a periodic triple derivation (a non-singular triple derivation which has finite multiplicative order) is necessarily nilpotent. They also prove that L admits a periodic triple derivation of odd order iff $L^3 = 0$. Moens in [8] proves that a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero is nilpotent iff it has an invertible Leibniz-derivation. The analogous results to the ones obtained by Moens are proved in [7] for other finite-dimensional nonassociative algebras.

As we have seen earlier in the Heisenberg Lie algebra, $\mathrm{Der}(L)$ may be a proper subset of $\mathrm{TDer}(L)$. Now it is natural to ask under what assumptions on L one obtains $\mathrm{TDer}(L) = \mathrm{Der}(L)$. This is true for abelian Lie algebras, for any R-linear map is a derivation. But if L is a nonabelian Lie algebra and R is of characteristic 2, then the identity map on L is a triple derivation of L but not a derivation of L. So the inclusion in $\mathrm{Der}(L) \subseteq \mathrm{TDer}(L)$ is strict in this case. In [12], the author proved that if L is a perfect Lie algebra with trivial center and $2 \in R$ is a unit, then every triple derivation of L is a derivation. The same result was proved in [8] if L is a finite-dimensional perfect Lie algebra over an algebraically closed field of characteristic zero. Also, by combining the results from [6] and [11], one can conclude that every triple derivation of the Lie algebra of block upper triangular matrices is a derivation. In this paper, first we give some sufficient conditions for the equality of triple derivations and derivations of Lie algebras. In particular, it is shown that every triple derivation of a free Lie algebra over a field of characteristic different from 2 is a derivation. We also prove that $\mathrm{TDer}(L) = \mathrm{Der}(L)$ if L is a perfect Lie algebra and $2 \in R$ is a unit, which generalizes the result in [12]. In the sequel, a necessary condition will be given for the equality of triple derivations and derivations of Lie algebras. Finally we obtain, as an application of our results, that if L is the Lie algebra of block upper triangular matrices, then $\mathrm{TDer}(L) = \mathrm{Der}(L)$.

2. Main Results

Throughout this section R denotes a commutative ring with identity and L denotes a Lie algebra over R.

We begin this section with a theorem which gives a simple criterion for a linear map to be a triple derivation if L is 3-torsion free, i.e. if $0 \neq x \in L$, then $3x \neq 0$.

Theorem 2.1. *Let L be 3-torsion free. Then an R-linear map $D : L \to L$ is a triple derivation iff*

$$D([x,[x,y]]) = [D(x),[x,y]] + [x,[D(x),y]] + [x,[x,D(y)]],$$

for any $x, y \in L$.

Proof. "The only if" part is trivial. For the "if" part, we have for any $x, y, z \in L$

$$D([x,[x,z]]) = [D(x),[x,z]] + [x,[D(x),z]] + [x,[x,D(z)]],$$

$$D([y,[y,z]]) = [D(y),[y,z]] + [y,[D(y),z]] + [y,[y,D(z)]],$$
$$D([x+y,[x+y,z]]) = [D(x+y),[x+y,z]] + [x+y,[D(x+y),z]] + [x+y,[x+y,D(z)]],$$

so

$$\begin{aligned} D([x,[y,z]]) + D([y,[x,z]]) &= [D(x),[y,z]] + [D(y),[x,z]] \\ &+ [x,[D(y),z]] + [y,[D(x),z]] \\ &+ [x,[y,D(z)]] + [y,[x,D(z)]]. \quad (*) \end{aligned}$$

Using the Jacobi identity, one has

$$\begin{aligned} D([z,[y,x]]) + 2D([y,[x,z]]) &= [z,[y,D(x)]] + 2[D(y),[x,z]] \\ &+ [z,[D(y),x]] + 2[y,[D(x),z]] \\ &+ [D(z),[y,x]] + 2[y,[x,D(z)]]. \end{aligned}$$

Changing the role of x and z in the above relation gives

$$\begin{aligned} D([x,[y,z]]) + 2D([y,[z,x]]) &= [x,[y,D(z)]] + 2[D(y),[z,x]] \\ &+ [x,[D(y),z]] + 2[y,[D(z),x]] \\ &+ [D(x),[y,z]] + 2[y,[z,D(x)]]. \quad (**) \end{aligned}$$

Now adding 2 times $(*)$ to $(**)$ we get

$$3D([x,[y,z]]) = 3[D(x),[y,z]] + 3[x,[D(y),z]] + 3[x,[y,D(z)]],$$

which implies that D is a triple derivation, for L is 3-torsion free. $\square$

We show that the assumption "L is 3-torsion free" is essential in Theorem 2.1. First we need a lemma about 2-Engel Lie algebras. Recall that a Lie algebra L is said to be 2-Engel if $[x,[x,y]] = 0$, for any $x, y \in L$.

Lemma 2.2. *Let L be a 2-Engel Lie algebra over a field $\mathbb{F}$. Then*

i) $[x,[y,z]] = [y,[z,x]] = [z,[x,y]]$ *and* $3[x,[y,z]] = 0$ *for any* $x,y,z \in L$.

ii) $L^4 = 0$ *and if* $\mathrm{char}\mathbb{F} \neq 3$, *then* $L^3 = 0$.

iii) if $L^3 \neq 0$, *then* $\dim L \geq 7$.

Proof. i) and ii) For a proof the reader is referred to either Theorem 3.1.1 of [10] or Lemma 2.1 and Theorem 2.3 of [5].
iii) Let $L^3 \neq 0$. So $\mathbb{F}$ is of characteristic 3 and there exist three elements $x,y,z \in L$ such that $[x,[y,z]] \neq 0$. Using part (i) and the fact that L is 2-Engel and $L^4 = 0$, one can easily check that the set

$$\{x, y, z, [x,y], [y,z], [z,x], [x,[y,z]]\}$$

is linearly independent over $\mathbb{F}$ and so $\dim L \geq 7$. $\square$

Example 2.3. *First we construct a 2-Engel Lie algebra L with $L^3 \neq 0$ over any field $\mathbb{F}$ of characteristic 3. Let L be the 7-dimensional vector space over $\mathbb{F}$ with the basis $\mathcal{B} = \{x,y,z,a,b,c,u\}$. Define the non-zero brackets as follows:*

$$[x,a] = [y,b] = [z,c] = u, [y,z] = a, [z,x] = b, [x,y] = c.$$

Since $u \in Z(L)$, $a,b,c \in Z_2(L)$, and $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 3u = 0$, the elements of $\mathcal{B}$ satisfy the Jacobi identity and so L is a Lie algebra. Also, for any $\alpha,\beta,\gamma \in \mathcal{B}$ we have

$$[\alpha,[\alpha,\gamma]] = 0, \ [\alpha,[\beta,\gamma]] + [\beta,[\alpha,\gamma]] = 0,$$

which means that L is 2-Engel.
Now if we let D be the linear map on L sending all elements of $\mathcal{B}$ to u, then D obviously is not a triple derivation but it satisfies the condition in Theorem 2.1.

In the sequel L denotes a 2-torsion free Lie algebra over R, i.e. $2x \neq 0$ for every nonzero $x \in L$, unless otherwise stated.

The following theorem plays a crucial role in the next results.

Theorem 2.4. *Let $L^2 \bigcap Z(L) = 0$ and $D \in \mathrm{TDer}(L)$. Then the map $\delta_D : L^2 \to L^2$ defined by*

$$\delta_D(x) = \Sigma_{i=1}^{n}[D(a_i), b_i] + [a_i, D(b_i)]$$

is well-defined and a derivation of L^2, where $x = \sum_{i=1}^{n}[a_i, b_i]$. Furthermore, the map $\varphi = D - \delta_D : L^2 \to L$ has the following properties:

i) $\varphi([x, [y, z]]) = -[x, \varphi([y, z])]$, *for any $x, y, z \in L$.*

ii) $\varphi(L^2) \subseteq C_L(L^2)$. *Equivalently, for any $x, y \in L$*

$$D([x, y]) - [D(x), y] - [x, D(y)] \in C_L(L^2).$$

iii) $\varphi(L^3) \subseteq Z(L^2)$. *In particular, if $Z(L^2) = 0$, then, for any $x, y, z \in L$*

$$D([x, [y, z]]) = [D(x), [y, z]] + [x, D([y, z])].$$

iv) $\varphi([L^2, L^2]) = 0$. *Equivalently, for any $x, y \in L^2$*

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

Proof. Let $x = \sum_{i=1}^{n}[a_i, b_i] = \sum_{i=1}^{m}[c_i, d_i]$ be two expressions for x, and let

$$\alpha = \sum_{i=1}^{n}[D(a_i), b_i] + [a_i, D(b_i)]$$

and

$$\beta = \sum_{i=1}^{m}[D(c_i), d_i] + [c_i, D(d_i)].$$

Obviously, $\alpha, \beta \in L^2$. Since $D \in \mathrm{TDer}(L)$, for any $y \in L$, one has

$$\begin{aligned}
[y, \alpha] &= \sum_{i=1}^{n}[y, [D(a_i), b_i] + [a_i, D(b_i)]] \\
&= \sum_{i=1}^{n} D([y, [a_i, b_i]]) - [D(y), [a_i, b_i]] \\
&= D([y, x]) - [D(y), x] \\
&= [\mathrm{ad}_x, D](y).
\end{aligned}$$

Similarly, $[y, \beta] = [\mathrm{ad}_x, D](y)$. Therefore, $\alpha - \beta \in L^2 \bigcap Z(L)$. Now by hypothesis $\alpha = \beta$, that is, $\delta_D(x)$ is independent of an expression for x. Now it suffices to show that $\delta_D \in \mathrm{Der}(L^2)$. The above relation shows that

$$\mathrm{ad}_{\delta_D(x)}(y) = -[y, \delta_D(x)] = -[\mathrm{ad}_x, D](y) = [D, \mathrm{ad}_x](y).$$

Hence for any $x, y \in L^2$, we have

$$\begin{aligned} \mathrm{ad}_{\delta_D([x,y])} &= [D, \mathrm{ad}_{[x,y]}] \\ &= [D, [\mathrm{ad}_x, \mathrm{ad}_y]] \\ &= [\mathrm{ad}_x, [D, \mathrm{ad}_y]] + [[D, \mathrm{ad}_x], \mathrm{ad}_y] \\ &= [\mathrm{ad}_x, \mathrm{ad}_{\delta_D(y)}] + [\mathrm{ad}_{\delta_D(x)}, \mathrm{ad}_y] \\ &= \mathrm{ad}_{[x,\delta_D(y)]+[\delta_D(x),y]}. \end{aligned}$$

This implies that $\delta_D([x, y]) - [x, \delta_D(y)] - [\delta_D(x), y] \in L^2 \bigcap Z(L) = 0$, showing that $\delta_D \in \mathrm{Der}(L^2)$.

It remains to show that φ satisfies (i)-(iv).

i) It is clear from the definitions of D and δ_D.

ii) and iv) Suppose that $x, y \in L^2$ are arbitrary. On the one hand, $\delta_D \in Der(L^2)$ yields that

$$\begin{aligned} \varphi([x, y]) - [x, \varphi(y)] &= [\varphi, \mathrm{ad}_x](y) \\ &= [D, \mathrm{ad}_x](y) - [\delta_D, \mathrm{ad}_x](y) \\ &= [\delta_D(x), y] - \delta_D([x, y]) + [x, \delta_D(y)] \\ &= 0, \end{aligned}$$

implying that $\varphi([x, y]) = [x, \varphi(y)]$. On the other hand, $\varphi([x, y]) = -[x, \varphi(y)]$ by part (i). Since L is 2-torsion free, $\varphi([x, y]) = [x, \varphi(y)] = 0$ and the result follows.

iii) It comes from part (ii) and the fact that $\varphi(L^3) \subseteq L^2$. □

The following theorem gives some sufficient conditions for the equality of triple derivations and derivations of Lie algebras.

Theorem 2.5. *Either of the following conditions implies that* $\mathrm{TDer}(L) = \mathrm{Der}(L)$*:*

i) L^2 *is perfect and* $L^2 \bigcap Z(L) = 0$.

ii) $Z(L^2) = 0$ *and* $L^2 = L^3$.

iii) $Z(L) = Z(L^m) = 0$ *for some natural number* $m \geq 2$.

Proof. To prove that $\mathrm{TDer}(L) = \mathrm{Der}(L)$, it is sufficient to show that the restriction of D on L^2 is δ_D for any $D \in \mathrm{TDer}(L)$. If L^2 is perfect and $L^2 \bigcap Z(L) = 0$, then $\varphi(L^2) = \varphi([L^2, L^2]) = 0$, by part (iv) of Theorem 2.4. If $L^2 = L^3$ and $Z(L^2) = 0$, then $\varphi(L^2) = 0$, by part (iii) of Theorem 2.4. We may now assume that $Z(L) = Z(L^m) = 0$ for some natural number $m \geq 2$. By part (ii) of Theorem 2.4, we have $[\varphi(L^{m+1}), L^m] \subseteq [\varphi(L^2), L^2] = 0$, and since $\varphi(L^{m+1}) \subseteq L^m$, hence $\varphi(L^{m+1}) \subseteq Z(L^m)$. It then follows that $\varphi(L^{m+1}) = 0$. Using part (i) of Theorem 2.4, for any $x_1, \ldots, x_{m+1} \in L$, one has

$$\begin{aligned} 0 &= \varphi([x_{m+1}, [x_m, \ldots, [x_3, [x_2, x_1]] \ldots]]) \\ &= (-1)^{m-1}[x_{m+1}, [x_m, \ldots, [x_3, \varphi([x_2, x_1])] \ldots]]. \end{aligned}$$

Now using the hypothesis $Z(L) = 0$, one obtains $\varphi([x_2, x_1]) = 0$, and the proof is complete. □

The first consequence of the above theorem is the following.

Corollary 2.6. *If* $Z(L^2) = 0$ *and* $L = L^2 + A$*, for some abelian subalgebra* A *of* L*, then* $\mathrm{TDer}(L) = \mathrm{Der}(L)$.

Proof. Since A is abelian, we have

$$L^2 = [L^2, L^2] + [L^2, A] + [A, A] = [L^2, L^2] + [L^2, A] = L^3,$$

and the result follows from part (ii) of Theorem 2.5. □

The next result, which is the main theorem of [12], is an immediate corollary of Theorem 2.5.

Corollary 2.7. *If $L = L^2$ and $Z(L) = 0$, then* $\mathrm{TDer}(L) = \mathrm{Der}(L)$.

The third interesting consequence is about free Lie algebras.

Corollary 2.8. *If L is a free Lie algebra over a field of characteristic not* 2*, then* $\mathrm{TDer}(L) = \mathrm{Der}(L)$.

Proof. Since each nonabelian free Lie algebra has trivial center, see page 186 of [2], and each subalgebra of a free Lie algebra is free, see [9], we obtain $Z(L) = Z(L^2) = 0$ if L is nonableian. Now the result follows from Theorem 2.5, part (iii). □

The next theorem tells us that if we weaken the assumptions of Theorem 2.5, then a triple derivation could be close to a derivation.

Theorem 2.9. *Let $D \in \mathrm{TDer}(L)$ and $Z_2(L^2) \subseteq Z_2(L)$. If either $L^2 = L^3$ or $Z_2(L) = Z(L)$, then, for any $x, y, z \in L$,*

$$D([x,[y,z]]) = [D(x),[y,z]] + [x, D([y,z])].$$

Proof. Let $\overline{L} = L/Z_2(L)$ and so $\overline{L}^2 = (L^2 + Z_2(L))/Z_2(L)$. First we show that $Z(\overline{L}^2) = \overline{0}$. Suppose $\overline{x} \in Z(\overline{L}^2)$, where $x \in L^2$. Then $[L^2, x] \subseteq Z_2(L)$ and hence $[L,[L,[L^2,x]]] = 0$. So, by the Jacobi identity, $[L^2,[L^2,x]] = 0$. It follows that $x \in Z_2(L^2)$ and therefore $x \in Z_2(L)$ by hypothesis, i.e. $\overline{x} = \overline{0}$. Obviously, D maps $Z_2(L)$ into itself and so the map $\overline{D} : \overline{L} \to \overline{L}$ defined by $\overline{D}(\overline{x}) = \overline{D(x)}$ is a well-defined triple derivation of $\overline{L}$.

First assume that $L^2 = L^3$. Then $\overline{L}^2 = \overline{L}^3$ and hence $\overline{D} \in \mathrm{Der}(\overline{L})$ by part (ii) of Theorem 2.5. This implies that

$$D([a,b]) - [D(a),b] - [a,D(b)] \in Z_2(L),$$

for any $a, b \in L$. Thus, for any $x, y, a, b \in L$

$$[x,[y,D([a,b]]]) = [x,[y,[D(a),b]]] + [x,[y,[a,D(b)]]].$$

Now we obtain

$$\begin{aligned} D([x,[y,[a,b]]]) &= [D(x),[y,[a,b]]] + [x,[D(y),[a,b]]] + [x,[y,D([a,b])]] \\ &= [D(x),[y,[a,b]]] + [x,[D(y),[a,b]]] + [x,[y,[D(a),b]]] \\ &+ [x,[y,[a,D(b)]]] \\ &= [D(x),[y,[a,b]]] + [x,D([y,[a,b]])]. \end{aligned}$$

Since $L^2 = L^3$, hence, for any $x, y, z \in L$, one obtains

$$D([x,[y,z]]) = [D(x),[y,z]] + [x,D([y,z])],$$

completing the proof in the first case.
Assume now that $Z_2(L) = Z(L)$. Then $\overline{L} = L/Z(L)$ and $Z(\overline{L}) = \overline{0}$. Therefore, $\overline{D} \in \mathrm{Der}(\overline{L})$ by part (iii) of Theorem 2.5. This means that

$$D([y,z]) - [D(y),z] - [y,D(z)] \in Z(L),$$

for any $y, z \in L$. Thus, for any $x, y, z \in L$

$$[x,D([y,z])] = [x,[D(y),z]] + [x,[y,D(z)]] = D([x,[y,z]]) - [D(x),[y,z]],$$

completing the proof in the second case. □

Obviously every derivation satisfies the conclusion of Theorem 2.9 but the converse is not true. Also there exists a triple derivation which does not satisfy the conclusion of Theorem 2.9 and there exists a linear map which satisfies the conclusion of Theorem 2.9 but it is not a triple derivation. The following example clarifies these claims.

Example 2.10. *Let L be the Lie algebra of all 3×3 strictly upper triangular matrices over R with the standard basis $\{e_{12}, e_{13}, e_{23}\}$ and let D_1, D_2 be two linear maps on L as follows:*

$$D_1(e_{12}) = D_1(e_{13}) = D_1(e_{23}) = e_{13},$$

$$D_2(e_{12}) = D_2(e_{13}) = D_2(e_{23}) = e_{23}.$$

Since $[e_{12}, e_{23}] = e_{13}$, $L^2 = Z(L) = Re_{13}$, and $L^3 = 0$, D_1 is not a derivation but it satisfies the conclusion of Theorem 2.9 and D_2 is a triple derivation but it does not satisfy the conclusion of Theorem 2.9.

Also if L is the Lie algebra of all 4×4 strictly upper triangular matrices over R with the standard basis $\mathcal{B} = \{e_{ij} | 1 \leq i < j \leq 4\}$, then it can be easily verified that $L^2 = Re_{13} + Re_{14} + Re_{24}$. Now consider the linear map D defined on L via

$$D(e_{23}) = e_{23}, D(\mathcal{B} \setminus \{e_{23}\}) = 0.$$

Since $D(L^2) = 0$ and $e_{23} \in C_L(L^2)$, D satisfies the conclusion of Theorem 2.9 but it is not a triple derivation because

$$D([e_{12}, [e_{23}, e_{34}]]) = D(e_{14}) = 0,$$

$$[D(e_{12}), [e_{23}, e_{34}]] + [e_{12}, [D(e_{23}), e_{34}]] + [e_{12}, [e_{23}, D(e_{34})]] = [e_{12}, [e_{23}, e_{34}]] = e_{14}.$$

It should be remarked that Grün's lemma in group theory says that in any perfect group the second center of the group coincides with the first center of the group. The same statement is true for Lie algebras, see Theorem 2.2 in [5]. Also, Theorem 4.9 in [5] generalizes Grün's lemma as follows: If $L = L^2 + Z_m(L)$ for natural number m, then $Z_m(L) = Z_{m+1}(L)$. Before stating the next corollary, we have to prove another generalization of Grün's lemma.

Lemma 2.11. *Let $L = L^2 + A$, for some abelian subalgebra A of L with $[A, L] \bigcap Z(L) = 0$. Then $L^2 = L^3$ and $Z_2(L) = Z(L)$. In particular, if $L = L^2 + Z(L)$, then $L^2 = L^3$ and $Z_2(L^2) \subseteq Z_2(L) = Z(L)$.*

Proof. The proof of Corollary 2.6 shows that $L^2 = L^3$. Let $x \in Z_2(L)$ be arbitrary. So $[L, [L, x]] = 0$. It then follows by the Jacobi identity that $[L^2, x] = 0$. Also $[A, x] \subseteq [L, x] \subseteq Z(L)$ and $[A, x] \subseteq [A, L]$. Hence by hypothesis $[A, x] = 0$. This implies that $[L, x] = [L^2, x] + [A, x] = 0$, i.e. $x \in Z(L)$. It remains to show that $Z_2(L^2) \subseteq Z_2(L)$ provided that $L = L^2 + Z(L)$. This is clear for if $x \in Z_2(L^2)$, then $[L, [L, x]] = [L^2, [L^2, x]] = 0$, and the proof is complete. □

The following result can be viewed as a generalization of Corollary 2.7.

Corollary 2.12. *Either of the following conditions implies that* $\mathrm{TDer}(L) = \mathrm{Der}(L)$*:*

i) $L = L^2 + A$, *for some abelian subalgebra A of L with $[A, L] \bigcap Z(L) = 0$ and $Z_2(L^2) \subseteq Z_2(L)$.*

ii) $L = L^2 + Z(L)$.

iii) $L = L^2 \oplus Z_m(L)$ *for some natural number $m \geq 2$.*

iv) $L = L^2$.

v) $L = A \oplus B$, *where A is a perfect Lie algebra and B is an abelian Lie algebra.*

Proof. i) Using Lemma 2.11 and Theorem 2.9, we see that, for any $x, y, z \in L$

$$D([x,[y,z]]) = [D(x),[y,z]] + [x, D([y,z])],$$

which is equivalent to

$$D([y,z]) - [D(y),z] - [y, D(z)] \in Z(L).$$

Hence for any $a \in L$ and $b \in L^2$

$$D([a,b]) = [D(a),b] + [a, D(b)].$$

Since $L = L^2 + A$, it suffices to show that

$$D([a,b]) = [D(a),b] + [a, D(b)],$$

for any $a, b \in A$. But this is clear because A is abelian and

$$D([a,b]) - [D(a),b] - [a, D(b)] = -[D(a),b] - [a, D(b)] \in [A, L] \bigcap Z(L) = 0,$$

which completes the proof.
ii) It is a combination of Lemma 2.11 and part (i).
iii) Obviously, if $x \in Z_m(L)$, then $[x, L] \subseteq L^2 \bigcap Z_{m-1}(L) = 0$. Thus $x \in Z(L)$, i.e. $L = L^2 + Z(L)$. Now the result follows by part (ii).
iv) This follows at once from part (ii).
v) Since $L^2 = A$ and $B \subseteq Z(L)$, the result follows from part (ii). □

We remark that the above corollary is not true if we only assume that $L = L^2 + Z_2(L)$, as the Heisenberg Lie algebra shows.

Finally a necessary condition for the equality of triple derivations and derivations of a Lie algebra is given, which is closely related to the sufficient conditions given in parts (ii) and (iii) of Theorem 2.5. We have to work on Lie algebras over a field of arbitrary characteristic instead of over a commutative ring with identity.

Theorem 2.13. *Let L be a Lie algebra over a field and let* $\mathrm{TDer}(L) = \mathrm{Der}(L)$*. Then either $L^2 = L^3$ or $Z(L) = 0$.*

Proof. Assume that $L^2 \neq L^3$ and let $z \in Z(L)$ be arbitrary. Hence there exist two elements $a, b \in L$ so that $[a,b] \in L^2 \setminus L^3$. Suppose now that $\mathcal{B}_1$ is a basis of L^3, $\mathcal{B}_2$ is a basis of L^2 containing $\mathcal{B}_1$ and $[a,b]$, and $\mathcal{B}$ is a basis of L containing $\mathcal{B}_2$. Consider the linear map D which maps $[a,b]$ to z and other elements of $\mathcal{B}$ to zero. Clearly, $D \in \mathrm{TDer}(L)$, and so $D \in \mathrm{Der}(L)$. Since $D(L) \subseteq Z(L)$, one obtains

$$z = D([a,b]) = [D(a),b] + [a, D(b)] = 0,$$

which completes the proof. □

The following corollary gives a characterization of nilpotent Lie algebras whose triple derivations are derivations.

Corollary 2.14. *Let L be a Lie algebra over a field. If L is nilpotent with* $\mathrm{TDer}(L) = \mathrm{Der}(L)$*, then L is abelian.*

Proof. We may assume that L is a nonzero nilpotent Lie algebra. It is well known that any nonzero nilpotent Lie algebra has nonzero center, so $Z(L) \neq 0$. Now Theorem 2.13 implies that $L^2 = L^3$. Therefore $L^2 = 0$, for L is nilpotent. □

3. Application

Throughout this section, R is a commutative ring with identity which is also 2-torsion free. Applying our results we show that every triple derivation of the Lie algebra of block upper triangular matrices is a derivation. To be precise, we denote by $M_{p\times q}(R)$ and $M_p(R)$ the set of all $p\times q$ matrices over R and the set of all $p\times p$ matrices over R, respectively. As usual, the standard basis of the free R-module $M_{p\times q}(R)$ is denoted by $\{e_{ij}|\ 1\le i\le p, 1\le j\le q\}$. Also let $gl_p(R)$ and $sl_p(R)$ be the general linear Lie algebra and the special linear Lie algebra, respectively. Now let $m,n\in\mathbb{N}$ with $m\le n$ and let $(n_1,\dots,n_m)\in\mathbb{N}^m$ be a fixed partition of n, i.e. $n=n_1+\cdots+n_m$. The block upper triangular matrix Lie algebra $B_n((n_1,\dots,n_m),R)$ is a subalgebra of $gl_n(R)$ of the form

$$\begin{bmatrix} M_{n_1}(R) & \cdots & M_{n_1\times n_i}(R) & \cdots & M_{n_1\times n_m}(R) \\ & \ddots & \vdots & & \vdots \\ & & M_{n_i}(R) & \cdots & M_{n_i\times n_m}(R) \\ & O & & \ddots & \vdots \\ & & & & M_{n_m}(R) \end{bmatrix}.$$

It should be noted that if $m=n$, then $B_n((1,\dots,1),R)$ is $T_n(R)$, the upper triangular matrix Lie algebra, and if $m=1$, then $B_n((n),R)$ is $gl_n(R)$. Note also that there is a unique basis $\mathcal{B}$ of the free R-module $B_n((n_1,\dots,n_m),R)$ contained in the standard basis of $M_n(R)$.

Some facts regarding the block upper triangular matrix Lie algebra which we need are given in the following lemma.

Lemma 3.1. *Let $L=B_n((n_1,\dots,n_m),R)$. Then*

i)

$$L^2=L^3=\begin{bmatrix} sl_{n_1}(R) & \cdots & M_{n_1\times n_i}(R) & \cdots & M_{n_1\times n_m}(R) \\ & \ddots & \vdots & & \vdots \\ & & sl_{n_i}(R) & \cdots & M_{n_i\times n_m}(R) \\ & O & & \ddots & \vdots \\ & & & & sl_{n_m}(R) \end{bmatrix}.$$

ii) $L=L^2+A$, where A is the abelian subalgebra of L generated by the set $\{e_{r_ir_i}|\ 1\le i\le m\}$ and $r_i=1+n_1+\cdots+n_{i-1}$.

iii) $Z(L)=Z_2(L)=RI_n$ and $[A,L]\bigcap Z(L)=0$.

iv) if either $n_1>1$ or $n_m>1$, then

$$Z(L^2)=Z_2(L^2)=\{\lambda I_n|\ \lambda\in R,\ n_i\lambda=0,\ \forall 1\le i\le m\}.$$

v) if $n\ge 2$ and $n_1=n_m=1$, then $Z(L^2)=Re_{1n}$.

Proof. i) For any $e_{ii},e_{jj},e_{ij},e_{ji},e_{kl}\in\mathcal{B}, i\ne j$, the following relations hold:

$$[e_{kl},e_{ij}]=\delta_{li}e_{kj}-\delta_{jk}e_{il},$$

$$e_{ij}=[e_{ii},e_{ij}]=[e_{ii},[e_{ii},e_{ij}]],$$

$$e_{ii}-e_{jj}=[e_{ij},e_{ji}]=[e_{ij},[e_{jj},e_{ji}]],$$

where δ_{ij} is the Kronecker delta. This completes the proof of part (i).
ii) It comes from part (i).
iii) We may assume that $x = (a_{ij}) \in L$ and $e_{kl} \in \mathcal{B}$ are arbitrary with $n \geq 2$. Then clearly

$$[x, e_{kl}] = (a_{kk} - a_{ll})e_{kl} + a_{lk}e_{ll} - a_{lk}e_{kk} + \sum_{i \neq k,l} a_{ik}e_{il} - \sum_{i \neq k,l} a_{li}e_{ki}. \quad (*)$$

In particular, $[x, e_{kk}]$ is a matrix all of whose diagonal entries are zero. Now, if $x \in Z(L)$, then the above relation shows that x is a scalar matrix. Therefore, $[A, L] \bigcap Z(L) = 0$. Now Lemma 2.11 implies that $Z_2(L) = Z(L)$, as desired.
iv) We may assume that $x = (a_{ij}) \in Z(L^2)$ and $e_{kl} \in \mathcal{B}$ are arbitrary with $n \geq 2$ and $k \neq l$. Then $[x, e_{kl}] = 0$, and we deduce using $(*)$ that there exists some $\lambda \in R$ so that $x = \lambda I_n + a_{1n}e_{1n}$. But $x \in L^2$ and hence, by part (i), $n_i\lambda = 0$, for any $1 \leq i \leq m$.

By assumption we have either $n_1 > 1$ or $n_m > 1$, hence either $0 = [x, e_{21}] = -a_{1n}e_{2n}$ or $0 = [x, e_{n\,n-1}] = a_{1n}e_{1\,n-1}$, which implies that $a_{1n} = 0$. Therefore,

$$Z(L^2) = \{\lambda I_n |\ \lambda \in R,\ n_i\lambda = 0,\ \forall 1 \leq i \leq m\}.$$

Now, if $x \in Z_2(L^2)$, then $[x, e_{kl}] \in Z(L^2)$ for any $e_{kl} \in \mathcal{B}$ with $k \neq l$ and so $[x, e_{kl}] = \lambda I_n$, for some $\lambda \in R$. In particular, $a_{lk} = -a_{lk} = \lambda$ by $(*)$ and so $2\lambda = 0$. This implies that $\lambda = 0$, for R is 2-torsion free. Hence $[x, e_{kl}] = 0$ and by the above paragraph $x = \mu I_n + a_{1n}e_{1n}$ for some $\mu \in R$. Suppose first that $n = 2$. So $m = 1$ and $n_1 = 2$. Then one has $0 = [e_{12}, [e_{21}, x]] = 2a_{12}e_{12}$, i.e. $2a_{12} = 0$. But R is 2-torsion free, so $a_{12} = 0$ and $x = \mu I_2 \in Z(L^2)$, as wanted. Suppose now that $n > 2$. Depending on either $n_1 > 1$ or $n_m > 1$, one has either $0 = [e_{12}, [e_{21}, x]] = a_{1n}e_{1n}$ or $0 = [e_{n-1\,n}, [e_{n\,n-1}, x]] = a_{1n}e_{1\,n}$, i.e. $a_{1n} = 0$. Hence $x = \mu I_n \in Z(L^2)$, as required.
v) Assume that $n_1 = n_m = 1$. Similar to part (iv), one can show that if $x = (a_{ij}) \in Z(L^2)$, then $x = \lambda I_n + a_{1n}e_{1n}$ with $n_i\lambda = 0$, for any $1 \leq i \leq m$. Therefore, $\lambda = 0$ and $x = a_{1n}e_{1n}$. It can also be easily seen that $e_{1n} \in Z(L^2)$ and hence $Z(L^2) = Re_{1n}$. □

We are now ready to state and prove our final result.

Theorem 3.2. *Let $L = B_n((n_1, \ldots, n_m), R)$. Then* TDer$(L)$ = Der(L).

Proof. The case $n = 1$ is clear since $L = R$ is abelian. So we may assume that $n \geq 2$. If either $n_1 > 1$ or $n_m > 1$, then the result follows from Lemma 3.1 and part (i) of Corollary 2.12. Assume now that $n_1 = n_m = 1$. Again, by Lemma 3.1, $L^2 \bigcap Z(L) = Z(L^2) \bigcap Z(L) = 0$. Therefore, we can apply Theorem 2.4. By part (iv), for any $x, y \in L^2$ we have

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

By part (iii) of Theorem 2.4, we also have $\varphi(L^2) = \varphi(L^3) \subseteq Z(L^2)$, for $L^2 = L^3$. By Lemma 3.1 this is equivalent to

$$D([x, y]) - [D(x), y] - [x, D(y)] \in Re_{1n}, \quad (**)$$

for any $x, y \in L$. But we know that $L = L^2 + A$, so to finish the proof, it is sufficient to show that

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for any $x \in A$ and $y \in L$. To do this, we work with basis elements $\{e_{r_ir_i} | \ 1 \leq i \leq m\}$ of A and $\mathcal{B}$ of L. Suppose that $e_{jj} \in \{e_{r_ir_i} | \ 1 \leq i \leq m\}$ and $e_{kl} \in \mathcal{B}$ are arbitrary. If $[e_{jj}, e_{kl}] = 0$, then we obtain by definition of δ_D that

$$D([e_{jj}, e_{kl}]) = 0 = \delta_D([e_{jj}, e_{kl}]) = [D(e_{jj}), e_{kl}] + [e_{jj}, D(e_{kl})].$$

Now we consider the case $[e_{jj}, e_{kl}] \neq 0$. This means that either $j = k \neq l$ or $j = l \neq k$. We give a proof for the former case. The proof of the latter case is similar. By $(**)$

$$D([e_{jj}, e_{jl}]) = [D(e_{jj}), e_{jl}] + [e_{jj}, D(e_{jl})] + \lambda e_{1n},$$

for some $\lambda \in R$. The result will follow if we show that $\lambda = 0$. Using the above relation and that $D \in \mathrm{TDer}(L)$, one obtains

$$\begin{aligned} D([e_{jj}, e_{jl}]) &= D([e_{jj}, [e_{jj}, e_{jl}]]) \\ &= [D(e_{jj}), [e_{jj}, e_{jl}]] + [e_{jj}, [D(e_{jj}), e_{jl}]] + [e_{jj}, [e_{jj}, D(e_{jl})]] \\ &= [D(e_{jj}), [e_{jj}, e_{jl}]] + [e_{jj}, D([e_{jj}, e_{jl}]) - \lambda e_{1n}] \\ &= [D(e_{jj}), e_{jl}] + [e_{jj}, D(e_{jl})] - \lambda [e_{jj}, e_{1n}] \\ &= D([e_{jj}, e_{jl}]) - \lambda e_{1n} - \lambda [e_{jj}, e_{1n}], \end{aligned}$$

hence $\lambda(e_{1n} + [e_{jj}, e_{1n}]) = 0$. Since $n_m = 1$, $[e_{jj}, e_{jl}] \neq 0$, and $e_{jl} \in \mathcal{B}$, hence $j < n$. The two distinct cases $j = 1$ and $j \neq 1$ will result in $2\lambda e_{1n} = 0$ and $\lambda e_{1n} = 0$, respectively. Now one concludes that $\lambda = 0$, since R is 2-torsion free. □

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