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A Note on the Power Graphs of Finite Nilpotent Groups

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Abstract.
The power graph $\mathcal{P}(G)$ of a group G is the graph with vertex set G and two distinct vertices are adjacent if one is a power of the other. Two finite groups are said to be conformal, if they contain the same number of elements of each order. Let Y be a family of all non-isomorphic odd order finite nilpotent groups of class two or p-groups of class less than p. In this paper, we prove that the power graph of each group in Y is isomorphic to the power graph of an abelian group and two groups in Y have isomorphic power graphs if they are conformal. We determine the number of maximal cyclic subgroups of a generalized extraspecial p-group (p odd) by determining the power graph of this group. We also determine the power graph of a p-group of order p^4 (p odd).

1. Introduction

Given a group, there are different methods to associate a graph with the group. Recently, the power graph associated with a group has deserved a lot of attention. The term "power graph" was first considered and introduced by Kelarev and Quinn [12]. Let G be a group. The undirected power graph $\mathcal{P}(G)$ has the vertex set G and two distinct vertices x and y are adjacent if $x = y^m$ or $y = x^m$ for some positive integer m. Because this paper deals only with undirected graphs, for convenience throughout we use the term "power graph" to refer to an undirected power graph defined as above, see also [1, Section 3].

Recently, a lot of interesting results on the power graphs have been obtained, see for examples [3–5, 8, 18]. A detailed list of open problems and results about power graphs can be found in [1]. Cameron and Ghosh [4] showed that for two finite abelian groups A_1 and A_2, $\mathcal{P}(A_1) \cong \mathcal{P}(A_2)$ if and only if $A_1 \cong A_2$. They also showed that two finite groups which have isomorphic power graphs are conformal [3, 4]. In general, converse of above result is false (see Remark 4.17). In Section 4 of this paper, we find a family of non-abelian groups in which converse holds, that is, if two finite groups are conformal, then they have isomorphic power graphs.

In [15], Mehranian, Gholami and Ashrafi gave the structure of the power graphs of cyclic groups, dicyclic groups, semidihedral groups and Mathieu group M_{11} or the Janko group J_1. In [10], Ghorbani and Barfaraz obtained the structure of power graphs of groups of order a product of three primes. The structure of the power graphs of elementary abelian p-groups and dihedral groups are also known [7, 18].

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In this paper, we find the structure of the power graphs of generalized extraspecial p-groups and p-groups of order p^4 (p odd) and as an application, we find the number of maximal cyclic subgroups (a cyclic subgroup that is not a proper subgroup of any another proper cyclic subgroup) of generalized extraspecial p-groups. Let $Z(G)$ denote the center of the group G. A finite p-group G is called extraspecial p-group if $Z(G)$ and $\gamma_2(G)$ coincide and have order p, where $\gamma_2(G)$ is the commutator subgroup of G. If $Z(G)$ of a finite p-group G is cyclic and $\gamma_2(G)$ has order p, then G is said to be a generalized extraspecial p-group. For more details see [19].

2. Notations and Basic Definitions

Throughout the paper all groups considered are finite and p denotes a prime. Let $C(G)$ denote the set of all distinct cyclic subgroups of the group G. Further, let $c_k(G)$ denote the number of cyclic subgroups of order p^k (k is a non-negative integer) in the group G. Cardinality of a set X is denoted by $|X|$, $o(x)$ denotes the order of the element x in the group G and identity element of the group G is denoted by 1.

Let Γ be a graph. A set of pairwise non-adjacent vertices of Γ is called an independent set. The independence number of a graph Γ is the cardinality of the largest independent set and is denoted by $\beta(\Gamma)$. Let Γ_1 and Γ_2 be the graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their union $\Gamma_1 \cup \Gamma_2$ is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The join of Γ_1 and Γ_2 is denoted by $\Gamma_1 + \Gamma_2$ and it consists of graph $\Gamma_1 \cup \Gamma_2$ and all edges joining V_1 with V_2. For any graph Γ, let $\cup_{i=1}^{s}\Gamma$ denote the graph obtain by union of s copies of Γ.

Definition 2.1. *Let G be a group. For elements u and v in G, define a relation R such that uRv if $\langle u\rangle = \langle v\rangle$. It is evident that R is an equivalence relation.*

Let $[u]$ denote the equivalence class containing $u \in G$ under the relation R and let $C'(G)$ denote the set of all equivalence classes G/R. Following [9], write

$$C'(G) = \{[u] \mid u \in G\} = \{[u_{00}], [u_{11}], \cdots, [u_{1i}], \cdots, [u_{21}], \cdots, [u_{mn}]\}, \tag{1}$$

where $[u_{00}] = \{1\}$ and $[u_{it}] = \{u_{it,1}, \cdots, u_{it,r_i}\}$.

Definition 2.2. *Let G be a group. For u and v in G, we say $u \prec v$ if one of the following holds.*

(i) for some i and t, $u = u_{it,l_1}, v = v_{it,l_2}$, and $l_1 < l_2$.
(ii) $\langle u\rangle \subsetneq \langle v\rangle$.

Define $u \preceq v$ if $u \prec v$ or $u = v$.

Definition 2.3. *[9] An ordered pair $(S, \preceq_S)$, where S is a finite set, is said to be a partially ordered set or poset if the binary relation $\preceq_S$ is reflexive, antisymmetric and transitive. For $u, v \in S$, if $u \preceq_S v$ or $v \preceq_s u$, then u and v are said to be comparable otherwise u and v are incomparable.*

Definition 2.4. *[9] Let $(S, \preceq_S)$ be a poset. Then the comparability graph of S is the graph with vertex set S, where two distinct elements are joined if they are comparable and it is denoted by $\mathcal{T}_S$.*

Let G be a group. It is immediate from Definition 2.2, $(G, \preceq)$ is a poset. For rest of this paper, let us denote this poset by $\mathcal{L}_G$. Clearly, the comparability graph of $\mathcal{L}_G$ is the power graph of a group G, that is, $\mathcal{P}(G) = \mathcal{T}_{\mathcal{L}_G}$ ([9, Example 1]).

Definition 2.5. *[9] A subset S' of S in a poset $(S, \preceq_S)$ is said to be chain, if all elements in S' are pairwise comparable. A subset W of S is said to be homogeneous if one of the following condition holds, for any $v \in S\backslash W$.*

- *for all $u \in W$, $u \preceq_S v$.*
- *for all $u \in W$, $v \preceq_S u$.*
- *for all $u \in W$, u and v are incomparable.*

Definition 2.6. *[9] A chain in a poset $(S, \preceq_S)$ that is also homogeneous is called a homogeneous chain.*

Remark 2.7. *[9, Example 2] Let G be group. Then each element $[x] \in C'(G)$ is a homogeneous chain in $\mathcal{L}_G$.*

3. Basic Results

In this section, we state some results that will be used later. Let $G \cong \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_s}} \cong \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_s \rangle$ such that $x_i^{p^{\alpha_i}} = 1$ for $i \in \{1, 2, \cdots, s\}$ and $\alpha_1, ..., \alpha_s \geq 1$. Then we have the following result.

Lemma 3.1. *If $1 \neq g \in G$, where $g = x_1^{p^{k_1}\beta_1} x_2^{p^{k_2}\beta_2} \cdots x_s^{p^{k_s}\beta_s}$ such that $0 < k_i$ and $p \nmid \beta_i$ $\forall$ i, then there are p^{s-1} cyclic subgroups of order $o(g)p$ containing $\langle g \rangle$. Further, if for some $i = i_o$, $k_{i_o} = 0$, $\beta_{i_o} \neq 0$, then there doesn't exist any cyclic subgroup of order $o(g)p$ containing $\langle g \rangle$.*

Proof. Let $g \in G$ such that $g = x_1^{p^{k_1}\beta_1} x_2^{p^{k_2}\beta_2} \cdots x_s^{p^{k_s}\beta_s}$ where $p \nmid \beta_i$. First, we count the number of elements $h \in G$ such that $h^p = g$. Consider $h = x_1^{r_1} x_2^{r_2} \cdots x_s^{r_s}$. Now, $h^p = g$ implies $x_1^{pr_1} x_2^{pr_2} \cdots x_s^{pr_s} = x_1^{p^{k_1}\beta_1} x_2^{p^{k_2}\beta_2} \cdots x_s^{p^{k_s}\beta_s}$. So $p^{k_i}\beta_i = pr_i \mod p^{\alpha_i}$ $\forall$ $i \in \{1, 2, \cdots, s\}$. For fixed i, latter equation has integer solution r_i if and only if $p \mid p^{k_i}\beta_i$. Thus, if for some $i = i_o$, $k_{i_o} = 0$ and $\beta_{i_o} \neq 0$, then there doesn't exist any $h \in G$ such that $h^p = g$.

Now, assume $k_i > 0$, $\forall$ i. So, if $p^{k_i}\beta_i \equiv pr_i \mod p^{\alpha_i}$, then $p^{k_i-1}\beta_i \equiv r_i \mod p^{\alpha_i-1}$. Thus, the latter equation has p distinct solutions for each fixed i and that are $r_i = p^{k_i-1}\beta_i + kp^{\alpha_i-1}$, where $0 \leq k \leq p-1$. Thus, for given $g = x_1^{p^{k_1}\beta_1} x_2^{p^{k_2}\beta_2} \cdots x_s^{p^{k_s}\beta_s}$, where $p \nmid \beta_i$ and $k_i > 0$, there are p^s elements $h \in G$ such that $h^p = g$ and $o(h) = o(g)p$.

Now, let $\langle h \rangle$ be a cyclic subgroup of order $o(g)p$ such that $\langle g \rangle \subset \langle h \rangle$ and $h^p = g$. Suppose $w \in \langle h \rangle$ such that $w^p = g$, then $w = h^r$ and $h^{rp} = w^p = g$. This implies that $rp \equiv p \mod o(h)$. Thus, $r = 1 + k\frac{o(h)}{p}$, where $1 \leq k \leq p$. Thus, each cyclic subgroup $\langle h \rangle$ of order $o(g)p$ contains p distinct elements $w \in \langle h \rangle$ such that $w^p = g$. Hence that, there are $\frac{p^s}{p} = p^{s-1}$ cyclic subgroups of order $o(g)p$ containing g for $k_i > 0$ $\forall$ i. This completes the proof. □

Lemma 3.2. *Let G be a finite abelian group such that $G \cong \mathbb{Z}_{p^m} \times \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n \text{ factors}}$. Then the number of elements of order p^t in G is*

$$\begin{cases} 1, & t = 0 \\ p^{n+1} - 1, & t = 1 \\ p^{n+t} - p^{n+t-1} & 2 \leq t \leq m. \end{cases}$$

Proof. Let $a_1, a_2, \ldots, a_n, a_{n+1}$ be the generators of G such that $a_1^{p^m} = 1, a_i^p = 1$, for $i = 2, 3, \ldots, n+1$. Then each element of G is uniquely written as $\prod_{i=1}^{n+1} a_i^{\beta_i}$ $, 0 \leq \beta_1 < p^m, 0 \leq \beta_i < p$ for $i = 2, 3, \cdots, n+1$.

Take $g = \prod_{i=1}^{n+1} a_i^{\beta_i}$. Now, for $1 \leq t \leq m$,

$$g^{p^t} = \left(\prod_{i=1}^{n+1} a_i^{\beta_i} \right)^{p^t} = a_1^{\beta_1 p^t}.$$

Thus, the number of the elements $g \in G$ such that $g^{p^t} = 1$ is p^{n+t}. Hence, the number of elements of order p^t of G is $p^{n+1} - 1$, for $t = 1$ and $p^{n+t} - p^{n+t-1}$, for $2 \leq t \leq m$. This completes the proof. □

Corollary 3.3. *Let G be a finite abelian group such that $G \cong \mathbb{Z}_{p^m} \times \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n \text{ factors}}$. Then*

$$c_t(G) = \begin{cases} 1, & t = 0 \\ \frac{p^{n+1}-1}{p-1}, & t = 1 \\ p^n & 2 \leq t \leq m. \end{cases}$$

Proof. The number of elements of order p^t is equal to $c_t(G)\phi(p^t)$. Thus, the result follows from the Lemma 3.2. □

Theorem 3.4. *[14] Let* $A \cong \underbrace{\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m} \times \cdots \times \mathbb{Z}_{p^m}}_{n \text{ factors}}$. *Then* $\mathcal{P}(A)$ *is isomorphic to*

$$K_1 + \cup_{i=1}^{l}\Big(K_{\phi(p)} + \cup_{i=1}^{p^{n-1}}\Big(K_{\phi(p^2)} + \cup_{i=1}^{p^{n-1}}\Big(\cdots + \cup_{i=1}^{p^{n-1}}\Big(K_{\phi(p^{m-1})} + \cup_{i=1}^{p^{n-1}} K_{\phi(p^m)}\Big)\cdots\Big)\Big)\Big),$$

where $l = \frac{p^n-1}{p-1}$.

Let G be a finite group. Recall that a cyclic subgroup of G that is not a proper subgroup of any other proper cyclic subgroup of G is called a maximal cyclic subgroup of G. Let $\mathcal{M}_G$ denote the set of all maximal cyclic subgroups of G.

Theorem 3.5. *[13, Corollary 2.14] Let G be a p-group. Then* $\beta(\mathcal{P}(G)) = |\mathcal{M}_G|$.

Following [16], two finite groups are said to be conformal if they have same number of elements of each order.

Theorem 3.6. *[16, Page 107] Two finite abelian groups are isomorphic if and only if they are conformal.*

4. Power Graph of a Nilpotent Group

In this section, we use Baer's trick to prove Theorem 4.1 and 4.2.

Let G be a group. Then we may define a binary operation $\circ$ on G by $x \circ y = w(x, y)$ where w is some fixed word in x and y. If the set G, with the binary operation $\circ$, define a group, then we say w to be a group-word for G, and we write the corresponding group by G_w, that is, as a set $G_w = G$ and operation of G_w is $\circ$.

Let $(H, \cdot)$ be an odd order nilpotent group of class two. Then we can define a group-word w as follows: for $x, y \in H$, $w(x, y) := xy[x, y]^n$ (by xy we mean $x \cdot y$). If $\gamma_2(H)$ the commutator subgroup of H, has finite exponent m and $n = \frac{m-1}{2}$, then corresponding group H_w is an abelian group. Indeed, $x \circ y = xy[x, y]^{\frac{m-1}{2}} = yx[x, y]^{\frac{m+1}{2}} = yx[y, x]^{\frac{m-1}{2}} = y \circ x$ (for more details see [11, p. 142]). This H_w is the corresponding abelian group to H. It is easy to observe that H and H_w are conformal.

Theorem 4.1. *Let H be an odd order nilpotent group of class two. Then* $\mathcal{P}(H) \cong \mathcal{P}(H_w)$.

Proof. The powers of elements in H and H_w are same. Thus, $\mathcal{P}(H) \cong \mathcal{P}(H_w)$. This completes the proof. □

Above result is false for an even ordered group. For example, D_8 the dihedral group of order 16, is a nilpotent group of class two but $\mathcal{P}(D_8)$ is not isomorphic to the power graph of any abelian group [17, Theorem 15].

Theorem 4.2. *Let* H^1 *and* H^2 *be two odd order nilpotent group of class two. If* H^1 *and* H^2 *are conformal, then* $\mathcal{P}(H^1) \cong \mathcal{P}(H^2)$.

Proof. By Theorem 4.1, $\mathcal{P}(H^1) \cong \mathcal{P}(H^1_w)$ and $\mathcal{P}(H^2) \cong \mathcal{P}(H^2_w)$. Also H^i is conformal to H^i_w, $i = 1$ or 2. Hence, H^1_w and H^2_w are conformal. So, by Theorem 3.6, $H^1_w \cong H^2_w$. Thus, $\mathcal{P}(H^1_w) \cong \mathcal{P}(H^2_w)$. Hence, $\mathcal{P}(H^1) \cong \mathcal{P}(H^2)$. This completes the proof. □

Two finite groups with isomorphic power graphs are conformal and two finite abelian groups have isomorphic power graphs if and only if they are isomorphic (see [3, 4]). Thus, we can easily deduce the following corollaries.

Corollary 4.3. *The power graphs of two odd order nilpotent groups of class at most two are isomorphic if and only if they are conformal.*

Corollary 4.4. *The number of non-isomorphic power graphs for the nilpotent groups of class at most two and order n (n is odd) is equal to the number of non-isomorphic abelian groups of order n.*

For finite p-groups, Theorems 4.1, 4.2 can be generalized for groups of larger class. If the class of a finite p-group G is less than p, then there exists a group-word w such that G_w is an abelian group [6, p. 446, Theorem 4.8]. In fact, following [6], group-word w which makes G_w abelian, can be obtained from Lazard's inversion of the Baker-Campbell-Hausdorff formula

$$x \circ y = xy[x, y]^{-1/2}[[x, y], x]^{1/12}[[x, y], y]^{-1/12} \cdots$$

Thus, in similar manner as above, we can easily deduce the following result.

Theorem 4.5. *Let X be a class of all finite non-isomorphic p-groups of class less than p. Then for $G \in X$, $\mathcal{P}(G) \cong \mathcal{P}(G_w)$, where G_w is the corresponding abelian group to G and two groups in X have isomorphic power graphs if they are conformal.*

Proposition 4.6. *Let G be p-group of class less than p with $|G| = p^{r_1+\cdots+r_s}$ such that*

$$G = \langle x_1, x_2, x_3, \cdots, x_s \mid x_1^{p^{r_1}} = x_2^{p^{r_2}} = \cdots = x_s^{p^{r_s}} = 1, R\rangle,$$

where R is a set of commutator relations. Then the corresponding abelian group G_w is given as

$$G_w \cong \mathbb{Z}_{p^{r_1}} \times \cdots \times \mathbb{Z}_{p^{r_s}}.$$

Proof. Let $K = \langle x_1, x_2, x_3, \cdots, x_s \mid x_1^{p^{r_1}} = \cdots = x_s^{p^{r_s}} = 1, x_i x_j = x_j x_i$ for $i, j \in \{1, \cdots, s\}\rangle$. Clearly, $G_w = \langle x_1, x_2, x_3, \cdots, x_s\rangle$ and $G_w = G$ as a set. Since powers of each element in G and G_w are same, so $x_1^{p^{r_1}} = x_2^{p^{r_2}} = \cdots = x_s^{p^{r_s}} = 1$ in G_w. Also, $x_i \circ x_j = x_j \circ x_i$ for all i, j. Thus, the generators of G_w satisfy the relations of K, so by Von Dyck's Theorem [19, Page 51], there is a surjective homomorphism $\phi : K \longrightarrow G_w$ with $x_i \to x_i$ for all $i \in \{1, \cdots, s\}$. Moreover, $|G_w| = |G|$. So, $|G_w| = |K|$. Thus, $G_w \cong K$. This completes the proof. □

4.1. Power Graph of a Generalized Extraspecial p-Group, p Odd

In this subsection, we find the structure of power graph of a generalized extraspecial p-group G (p odd) and as a consequence, we also find the cardinality of the set $\mathcal{M}_G$.

Let G be a generalized extraspecial p-group of order p^{2n+m}, $m \geq 1$ and p odd (for $m = 1$, G will be extraspecial p-group). Then G has generators $a_1, a_2, \cdots, a_{2n}, b$ which satisfy the following conditions:

$$Z(G) = \langle b\rangle, b^{p^m} = 1, a_i^p = 1 \text{ for } i \in \{2, \cdots, 2n\}$$
$$[a_{2i-1}, a_{2i}] = b^{p^{m-1}}, i \in \{1, 2, \cdots, n\}$$
$$[a_{2i-1}, a_j] = 1, j \neq 2i$$
$$[a_{2i}, a_k] = 1, k \neq 2i - 1,$$

and either $a_1^p = 1$ (in this case, G is called generalized extraspecial p-group of exponent p^m) or $a_1^p = b$ (in this case, G is called generalized extraspecial p-group of exponent p^{m+1}). For more details see [19].

Proposition 4.7.
1. *If G is a generalized extraspecial p-group of order p^{2n+m} with exponent p^m and p odd, then $\mathcal{P}(G) \cong \mathcal{P}(A)$, where $A \cong \mathbb{Z}_{p^m} \times \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{2n \text{ factors}}$.*
2. *If G is a generalized extraspecial p-group of order p^{2n+m} with exponent p^{m+1} and p odd, then $\mathcal{P}(G) \cong \mathcal{P}(A)$, where $A \cong \mathbb{Z}_{p^{m+1}} \times \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{(2n-1) \text{ factors}}$.*

Proof. This follows from Theorem 4.5 and Proposition 4.6. □

Now the problem reduces to the problem of determining the power graph of the abelian group $E \cong \mathbb{Z}_{p^m} \times \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{\text{n-1 factors}} \cong \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_n \rangle$, where $o(x_1) = p^m$ and $o(x_i) = p$, for $2 \leq i \leq n$ and $n > 1$.

Theorem 4.8. *The power graph $\mathcal{P}(E)$ is isomorphic to the graph*

$$K_1 + \Big[\Gamma_1 \cup \Big(K_{\phi(p)} + \Big[\Gamma_2 \cup \Big(K_{\phi(p^2)} + \Big[\Gamma_3 \cup \Big(K_{\phi(p^3)} + \Big[\cdots + \Big[\Gamma_{m-1} \cup \Big(K_{\phi(p^{m-1})} + \Big[\Gamma_m \cup K_{\phi(p^m)}\Big]\Big)\Big]\cdots\Big]\Big)\Big]\Big)\Big]\Big)\Big],$$

where $\Gamma_j = \cup_{i=1}^{p^{n-1}-1} K_{\phi(p^j)}$, *for* $j \in \{2, 3, \cdots, m\}$ *and* $\Gamma_1 = \cup_{i=1}^{\frac{p^n-1}{p-1}-1} K_{\phi(p)}$.

Proof. Let us identify E with $\langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_n \rangle$, where $o(x_1) = p^m$ and $o(x_i) = p$, for $2 \leq i \leq n$. Then by Corollary 3.3, E has $\frac{p^n-1}{p-1}$ cyclic subgroups of order p and these cyclic subgroups are given as:

$$\langle x_1^{p^{m-1}} \rangle, \langle x_1^{\alpha_1 p^{m-1}} x_2 \rangle, \langle x_1^{\alpha_1 p^{m-1}} x_2^{\alpha_2} x_3 \rangle, \cdots, \langle x_1^{\alpha_1 p^{m-1}} x_2^{\alpha_2} \ldots x_{n-1}^{\alpha_{n-1}} x_n \rangle,$$

where $\alpha_i \in \{1, 2, \ldots, p\}$ for $1 \leq i \leq n-1$. For $m = 1$, these are the only non-trivial cyclic subgroups of E. Assume $m \geq 2$.

By Lemma 3.1, except the cyclic subgroup $\langle x_1^{p^{m-1}} \rangle$, none of the other cyclic subgroups of order p are contained in cyclic subgroups of a higher order. Moreover, cyclic subgroup $\langle x_1^{p^{m-t+1}} \rangle$ of order p^{t-1}, $t > 1$ is contained in p^{n-1} cyclic subgroups of order p^t. Since, the number of all cyclic subgroups of order $p^t, t > 1$ in the group E is p^{n-1} (Corollary 3.3), the cyclic subgroup $\langle x^{p^{m-t+1}} \rangle$ of order p^{t-1} is contained in all cyclic subgroups of order p^t.

Recall that $C'(E) = \{[x] \mid \langle x \rangle \in C(G)\}$, where $[x] = \{y \in G \mid \langle y \rangle = \langle x \rangle\}$. Thus, the set $C'(E)$ has p^{n-1} equivalence classes of cardinality $\phi(p^t)$ for $1 < t \leq m$, $\frac{p^n-1}{p-1}$ equivalence classes of cardinality $\phi(p)$ and one equivalence class of cardinality one.

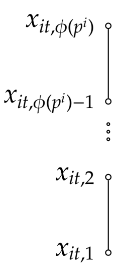
Following (1), we write

$C'(E) = \Big\{[V_{00}], [V_{it}] \mid i \in \{1, \cdots, m\}$ and $1 \leq t \leq \frac{p^n-1}{p-1}$, for i=1 and $1 \leq t \leq p^{n-1}$, for $i > 1\Big\}$, where $[V_{it}]$ denotes the equivalence class of cardinality $\phi(p^i)$. Moreover, $[V_{00}] = \{1\}$ and $[V_{it}] = \{x_{it,1}, \cdots, x_{it,\phi(p^i)}\}$. By Remark 2.7, each element $[V_{it}]$ gives a chain

$$x_{it,1} \leq \cdots \leq x_{it,\phi(p^i)}$$

of length $\phi(p^i)$ in the poset $\mathcal{L}_E$. Clearly, the identity element of the group E is comparable with every element of E in $\mathcal{L}_E$. Now, collecting all arguments, we draw the Hasse diagram of the poset $\mathcal{L}_E$ in Figure 1.

In Figure 1, V_{it} denotes the chain



corresponding to the elements of $[V_{it}]$. Here $x_{it,1}$ is called the minimal element and $x_{it,\phi(p^i)}$ is called the maximal element of the chain. In Figure 1, edge between V_{it} and $V_{i't'} (i < i')$ means there is an edge between the maximal element of V_{it} and minimal element of $V_{i't'}$.

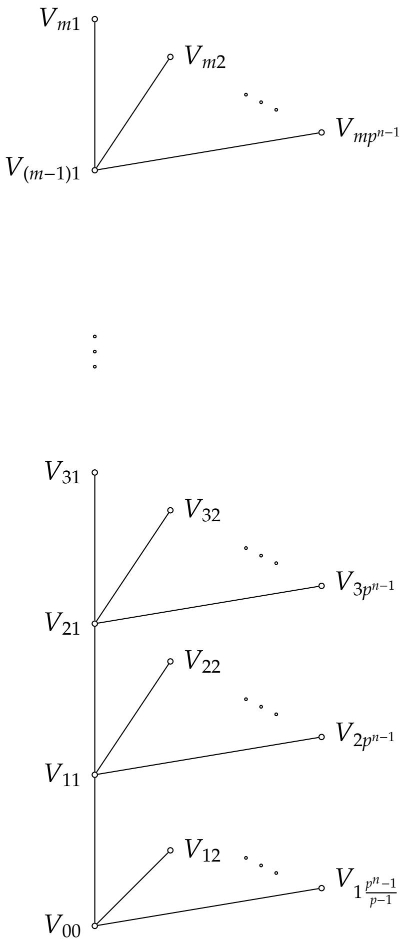


Figure 1: Hasse Diagram of $\mathcal{L}_E$

We know that the comparability graph $\mathcal{T}_{\mathcal{L}_E}$ of the poset $\mathcal{L}_E$ is equal to the power graph of E. Now we deduce the $\mathcal{P}(E)$ with the help of Figure 1. Each $[V_{ij}]$ is a chain of length $\phi(p^i)$, so the vertices corresponding to the elements of $[V_{ij}]$ give a complete graph $K_{\phi(p^i)}$ in $\mathcal{P}(E)$. Moreover, each $[V_{ij}]$ is a homogeneous chain. Therefore if $x_{ij,m} \preceq x_{i'j',m'}$ or $x_{i'j',m'} \preceq x_{ij,m}$ for some $x_{ij,m} \in [V_{ij}]$ and $x_{i'j',m'} \in [V_{i'j'}]$, then we get $K_{\phi(p^i)} + K_{\phi(p^{i'})}$ in $\mathcal{P}(E)$ corresponding the vertex subset $[V_{ij}] \cup [V_{i'j'}]$, otherwise vertices corresponding to subset $[V_{ij}] \cup [V_{i'j'}]$ give union of graphs $K_{\phi(p^i)}$ and $K_{\phi(p^{i'})}$ in $\mathcal{P}(E)$. Now by Figure 1, we can conclude the result. □

By Propositions 4.7 and Theorem 4.8, we deduce the following corollaries.

Corollary 4.9. *Let G be a generalized extraspecial p-group of order p^{2n+m} with exponent p^m, p odd. Then $\mathcal{P}(G)$ is isomorphic to the graph*

$$K_1 + \Big[\Gamma_1 \cup \Big(K_{\phi(p)} + \Big[\Gamma_2 \cup \Big(K_{\phi(p^2)} + \Big[\Gamma_3 \cup \Big(K_{\phi(p^3)} + \Big[\cdots + \Big[\Gamma_{m-1} \cup \Big(K_{\phi(p^{m-1})} + \Big[\Gamma_m \cup K_{\phi(p^m)}\Big]\Big)\Big]\cdots\Big]\Big)\Big]\Big)\Big]\Big)\Big],$$

where $\Gamma_j = \cup_{i=1}^{p^{2n}-1} K_{\phi(p^j)}$, for $j \in \{2, 3, \cdots, m\}$ and $\Gamma_1 = \cup_{i=1}^{\frac{p^{2n+1}-1}{p-1}-1} K_{\phi(p)}$.

Corollary 4.10. *Let G be a generalized extraspecial p-group of order p^{2n+m} with exponent p^{m+1}, p odd. Then $\mathcal{P}(G)$ is isomorphic to the graph*

$$K_1 + \Big[\Gamma_1 \cup \Big(K_{\phi(p)} + \Big[\Gamma_2 \cup \Big(K_{\phi(p^2)} + \Big[\Gamma_3 \cup \Big(K_{\phi(p^3)} + \Big[\cdots + \Big[\Gamma_m \cup \Big(K_{\phi(p^m)} + \Big[\Gamma_{m+1} \cup K_{\phi(p^{m+1})}\Big]\Big)\Big]\cdots\Big]\Big)\Big]\Big)\Big]\Big)\Big],$$

where $\Gamma_j = \cup_{i=1}^{p^{2n-1}-1} K_{\phi(p^j)}$, *for* $j \in \{2, 3, \cdots, m+1\}$ *and* $\Gamma_1 = \cup_{i=1}^{\frac{p^{2n}-1}{p-1}-1} K_{\phi(p)}$.

Theorem 4.11. *Let G be a generalized extraspecial p-group of order* p^{2n+m}, *(p odd). Then*

$$|\mathcal{M}_G| = \begin{cases} p^{a-1} + (b-2)(p^{a-1}-1) + \left(\frac{p^a-1}{p-1} - 1\right), & b \geq 2 \\ \frac{p^a-1}{p-1}, & b = 1, \end{cases}$$

where $a = 2n+1$, $b = m$, *when exponent of G is* p^m *and* $a = 2n, b = m+1$, *when exponent of G is* p^{m+1}.

Proof. Firstly, we find the number of maximal cyclic subgroup of $E \cong \mathbb{Z}_{p^m} \times \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{\text{n-1 factors}}$. By Figure 1, it is clear that for $x \in [V_{ij}]$ $(j > 1)$, $\langle x\rangle$ is a maximal cyclic subgroup of E. Also for $x \in [V_{m1}]$, $\langle x\rangle$ is a maximal cyclic subgroup of E. Since for $x, y \in [V_{ij}]$, $\langle x\rangle = \langle y\rangle$. So we need to count V_{ij} for $j > 1$ and V_{m1}. Thus, by Figure 5.1, we have

$$|\mathcal{M}_E| = \begin{cases} p^{n-1} + (m-2)(p^{n-1}-1) + \left(\frac{p^n-1}{p-1} - 1\right), & m \geq 2 \\ \frac{p^n-1}{p-1}, & m = 1. \end{cases} \tag{2}$$

By Theorem 3.5 and Theorem 4.7, for generalized extraspecial p-group G of exponent p^m $|\mathcal{M}_G| = |\mathcal{M}_{A_1}|$, where $A_1 \cong \mathbb{Z}_{p^m} \times \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{2n \text{ factors}}$ and for generalized extraspecial p-group G of exponent p^{m+1}, $|\mathcal{M}_G| = |\mathcal{M}_{A_2}|$, where $A_2 \cong \mathbb{Z}_{p^{m+1}} \times \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{(2n-1) \text{ factors}}$. Thus, by (2), we can complete the proof. □

4.2. Power Graph of a Group of Order p^4, p Odd

In this subsection, we find the structure of power graph of a group of order p^4 (p odd). Following [2], there are 15 groups of order p^4 up to isomorphism. We number them P_1 to P_{15}. The groups P_1 to P_5 are abelian, P_6 to P_{10} and P_{14} are of class 2, and P_{11} to P_{13} and P_{15} are of class 3. Here we list the all non-isomorphic groups of order p^4.

1. $P_1 = \mathbb{Z}_{p^4}$.
2. $P_2 = \mathbb{Z}_{p^3} \times \mathbb{Z}_p$.
3. $P_3 = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$.
4. $P_4 = \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$.
5. $P_5 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.
6. $P_6 = \langle u, v \mid u^{p^3} = v^p = 1, v^{-1}uv = u^{1+p^2}\rangle$.
7. $P_7 = \langle u, v, w \mid u^{p^2} = v^p = w^p = 1, uv = vu, wu = uw, w^{-1}vw = vu^p\rangle$.
8. $P_8 = \langle u, v \mid u^{p^2} = v^{p^2} = 1, v^{-1}uv = u^{1+p}\rangle$.
9. $P_9 = \langle u, v, w \mid u^{p^2} = v^p = w^p = 1, w^{-1}uw = u^{1+p}, vu = uv, wv = vw\rangle$.
10. $P_{10} = \langle u, v, w \mid u^{p^2} = v^p = w^p = 1, uv = vu, w^{-1}uw = uv, vw = wv\rangle$.
11. $P_{11} = \langle u, v, w \mid u^{p^2} = v^p = w^p = 1, v^{-1}uv = u^{1+p}, w^{-1}uw = uv, vw = wv\rangle$.
12. (a) $P_{12} = \langle u, v, w \mid u^{p^2} = v^p = 1, w^p = 1, v^{-1}uv = u^{1+p}, w^{-1}uw = uv, w^{-1}vw = u^pv\rangle$, $p > 3$.
    (b) $P_{12} = \langle u, v, w \mid u^{p^2} = v^p = 1, w^p = u^p, v^{-1}uv = u^{1+p}, w^{-1}uw = uv^{-1}, vw = wv\rangle$, $p = 3$.

13. (a) $P_{13} = \langle u, v, w \mid u^{p^2} = v^p = w^p = 1, v^{-1}uv = u^{1+p}, w^{-1}uw = uv, w^{-1}vw = u^{dp}v$, $p > 3$ and d is any non residue mod p.
    (b) $P_{13} = \langle u, v, w \mid u^{p^2} = v^p = 1, w^p = u^{-p}, v^{-1}uv = u^{1+p}, w^{-1}uw = uv^{-1}, vw = wv\rangle$, $p = 3$.
14. $P_{14} = \langle u, v, w, x \mid u^p = v^p = w^p = x^p = 1, x^{-1}wx = wu, vx = xv, ux = xu, vw = wv, uw = wu, uv = vu\rangle$.
15. (a) $P_{15} = \langle u, v, w, x \mid u^p = v^p = w^p = x^p = 1, x^{-1}wx = wv, x^{-1}vx = vu, xu = ux, vw = wv, uw = wu, uv = vu\rangle, p > 3$.
    (b) $P_{15} = \langle u, v, w \mid u^{p^2} = v^p = w^p = 1, uv = vu, w^{-1}uw = uv, w^{-1}vw = u^{-p}v\rangle$, $p = 3$.

Lemma 4.12. *The following hold in groups of order p^4, $p > 3$.*

1. $\mathcal{P}(P_6) \cong \mathcal{P}(P_2)$.
2. $\mathcal{P}(P_8) \cong \mathcal{P}(P_3)$.
3. $\mathcal{P}(P_7) \cong \mathcal{P}(P_9) \cong \mathcal{P}(P_{10}) \cong \mathcal{P}(P_{11}) \cong \mathcal{P}(P_{12})) \cong \mathcal{P}(P_{13}) \cong \mathcal{P}(P_4)$.
4. $\mathcal{P}(P_{14}) \cong \mathcal{P}(P_{15}) \cong \mathcal{P}(P_5)$.

Proof. This follows from Theorem 4.5 and Proposition 4.6. □

Lemma 4.13. *The following hold in groups of order p^4, $p = 3$.*

1. $\mathcal{P}(P_6) \cong \mathcal{P}(P_2)$.
2. $\mathcal{P}(P_8) \cong \mathcal{P}(P_3)$.
3. $\mathcal{P}(P_7) \cong \mathcal{P}(P_9) \cong \mathcal{P}(P_{10}) \cong \mathcal{P}(P_4)$.
4. $\mathcal{P}(P_{14}) \cong \mathcal{P}(P_5)$.

Proof. $P_6, P_7, P_8, P_9, P_{10}, P_{14}$ are p-groups of class 2 and P_2, P_3, P_4, P_5 are abelian. Thus, by Theorem 4.1 and Proposition 4.6, we can conclude the result. □

Lemma 4.14. *The following hold in groups of order p^4, where p is any prime.*

1. $\mathcal{P}(P_1) = K_{p^4}$.
2. $\mathcal{P}(P_2) = K_1 + \left[\cup_{i=1}^{p} K_{\phi(p)} \cup \left(K_{\phi(p)} + \left[\cup_{i=1}^{p-1} K_{\phi(p^2)} \cup \left(K_{\phi(p^2)} + \cup_{i=1}^{p} K_{\phi(p^3)}\right)\right]\right)\right]$.
3. $\mathcal{P}(P_3) = K_1 + \cup_{i=1}^{p+1}\left[K_{\phi(p)} + \cup_{i=1}^{p} K_{\phi(p^2)}\right]$.
4. $\mathcal{P}(P_4) = K_1 + \left[\cup_{i=1}^{p+p^2} K_{\phi(p)} \cup \left(K_{\phi(p)} + \cup_{i=1}^{p^2} K_{\phi(p^2)}\right)\right]$.
5. $\mathcal{P}(P_5) = K_1 + \left[\cup_{i=1}^{\frac{p^4-1}{p-1}} K_{\phi(p)}\right]$.

Proof. Since P_1 is a cyclic group of order p^4, $\mathcal{P}(P_1) = K_{p^4}$. Now, 2, 4, and 5 are determined by using Theorem 4.8 and 3 from Theorem 3.4. □

Now, we find the structure of power graphs of groups $P_{11}, P_{12}, P_{13}, P_{15}$, for $p = 3$.

Lemma 4.15. *For $p = 3$, the following hold:*

1. $\mathcal{P}(P_{12}) = K_1 + \left[\cup_{i=1}^{3} K_2 \cup \left(K_2 + \cup_{i=1}^{12} K_6\right)\right]$.
2. $\mathcal{P}(P_{13}) = K_1 + \left[\cup_{i=1}^{12} K_2 \cup \left(K_2 + \cup_{i=1}^{9} K_6\right)\right]$.
3. $\mathcal{P}(P_{11}) = K_1 + \left[\cup_{i=1}^{21} K_2 \cup \left(K_2 + \cup_{i=1}^{6} K_6\right)\right]$.
4. $\mathcal{P}(P_{15}) = K_1 + \left[\cup_{i=1}^{30} K_2 \cup \left(K_2 + \cup_{i=1}^{3} K_6\right)\right]$.

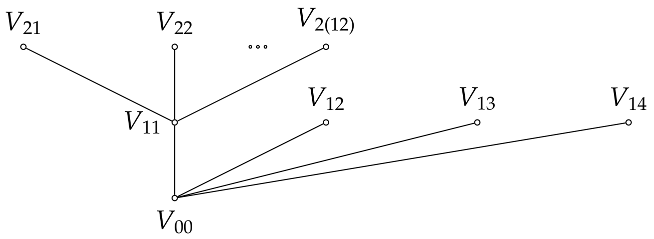


Figure 2: Hasse Diagram of $\mathcal{L}_{P_{12}}$

Proof. Let P_{11}, P_{12}, P_{13}, and P_{15} be the groups of order 81. Further, let $T = \langle u, v, w \mid u^9 = v^3 = 1, w^3 = u^{3\beta}, uv = vu^4, w^{-1}uw = uv^{-1}, vw = wv\rangle$, $\beta \in \{1, -1\}$. Clearly $[u, w] = 1$ and $[u^3, v] = u^9 = 1$. Thus, $u^3 \in Z(G)$. By using relations $wv = vw$, $wu = uvw$, and $vu = u^7v$, we can show that $v^ju^i = u^{i(1+6j)}v^j$ and $w^ku^i = u^{i+3ki(i-1)}v^{ik}w^k$, where $1 \le i \le 9, 1 \le j \le 3$, and $1 \le k \le 3$. Thus, each element of the group T can be written in the form $u^iv^jw^k$ for some $i, j, k \ge 1$. By using above relations, we can deduce that $(u^iv^jw^k)^3 = u^{3(i+2i^2k)}w^{3k}$. Now, for P_{12}, $\beta = 1$. Thus, $(u^iv^jw^k)^3 = u^{3(i+2i^2k+k)}$. So, $(u^iv^jw^k)^3 = 1$ for $k = 3$, $1 \le j \le 3$, and $i \in \{3, 6, 9\}$. Therefore, P_{12} has 8 elements of order 3 and $81 - 9 = 72$ elements of order 9 (exponent of P_{12} is 9). Hence, P_{12} has 4 cyclic subgroups of order 3 and 12 cyclic subgroups of order 9.

For $\beta = -1$, $T = P_{13}$. Thus, $(u^iv^jw^k)^3 = u^{3(i+2i^2k-k)}$. In similar manner as above, we can obtain that P_{13} has 13 cyclic subgroups of order 3 and 9 cyclic subgroups of order 9.

Now for P_{11}, $[u^3, v] = u^9 = 1$. By using relations $wu = uv^2w$, $vu = u^7b$, and $wv = vw$, we have $v^ju^i = u^{i(1+6i)}v^j$ and $w^ku^i = u^{i+6ik(i-1)}v^{2ki}c^k$, where $1 \le i \le 9, 1 \le j \le 3, 1 \le k \le 3$. Thus, each element of the group P_{11} can be written in the form $u^iv^jw^k$ for some $i, j, k \ge 1$. By using above relations, we can obtain that $(u^iv^jw^k)^3 = u^{3(i+ki^2)}$. Using this relation, similarly as above, we can obtain that P_{11} has 22 cyclic subgroups of order 3 and 6 cyclic subgroups of order 9.

Again for P_{15}, $u^3 \in Z(G)$. Using relations $uv = vu, wu = u^7v^2w$, and $wv = u^3vw$, we have $w^kv^j = u^{3jk}v^jc^k$, $w^ku^i = u^{(1+6k)i+3ik(k-1)}v^{2ik}w^k$, and $(u^iv^jc^k)^3 = u^{3i(1+2k^2)}$. Thus, using last relation, we can deduce that P_{15} has 31 cyclic subgroups of order 3 and 3 cyclic subgroups of order 9.

In all four groups, observe that the cyclic subgroup $\langle u^3\rangle$ is contained in all cyclic subgroups of order 9. Therefore, we obtain the structure of power graph of the group P_{12} and for remaining groups, power graphs can be obtained by doing similar process. Now, we find $\mathcal{P}(P_{12})$. Since P_{12} has 12 cyclic subgroups of order 9 and 4 cyclic subgroups of order 3, the set $C'(P_{12})$ has 12 equivalence classes of cardinality 6 and 4 equivalence classes of cardinality 2.

Following (1), we write

$C'(P_{12}) = \Big\{[V_{00}], [V_{it}] \mid i \in \{1, 2\} \text{ and } 1 \le t \le 4, \text{ for i=1 and } 1 \le t \le 12, \text{for } i = 2\Big\}$, where $[V_{it}]$ denotes the equivalence class of cardinality $\phi(3^i)$. Moreover, $[V_{00}] = \{1\}$ and $[V_{it}] = \{x_{it,1}, \cdots, x_{it,\phi(3^i)}\}$.

The Hasse diagram of the poset $\mathcal{L}_{P_{12}}$ is given in Figure 2 . Since only one cyclic group of order 3 is contained in all cyclic subgroups of order 9, so only one V_{1t} say V_{11} is connected to V_{2t} for all t in Hasse diagram of the poset $\mathcal{L}_{P_{12}}$.

In Figure 2, recall that V_{it} denote the a chain of length $\phi(3^i)$ corresponding to element $[V_{it}]$ (see proof of Theorem 4.8). Thus, we get $K_{\phi(3^i)}$ in $\mathcal{P}(P_{12})$ corresponding vertex subset $[V_{it}]$.

We know that the comparability graph $\mathcal{T}_{\mathcal{L}_{P_{12}}}$ of the poset $\mathcal{L}_{P_{12}}$ is equal to the power graph of P_{12}. Thus, by Figure 2, we can determine that $\mathcal{P}(P_{12}) = K_1 + \Big[\cup_{i=1}^{3} K_2 \cup \Big(K_2 + \cup_{i=1}^{12} K_6\Big)\Big]$. This complete the proof. □

Theorem 4.16. *For $p = 3$, there are 8 non-isomorphic power graphs for groups of order 81 and there are 5 non-isomorphic power graphs for groups of order p^4, $p > 3$.*

Proof. This follows from Lemmas 4.12, 4.13, 4.14, and 4.15. □

Remark 4.17. *For $p = 3$, P_4 and P_{13} are conformal and their power graphs are also same and P_3, P_{12} are conformal but have different power graphs.*

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References

[1] J. Abawajy, A. Kelarev, M. Chowdhary, Power graphs: A survey, Electronic Journal of Graph Theory and Application 1 (2013) 125–147.
[2] W. Burnside, Theory of Finite Groups, (2nd edition), Dover Publication Inc. (1955), reprint of 2nd edition, Cambridge (1911).
[3] P. J. Cameron, The power graph of a finite group II, Journal of Group Theory 13 (2010) 779-783.
[4] P. J. Cameron, S. Ghosh, The power graph of a finite group, Discrete Mathematics 311 (2011) 1220-1222.
[5] P. J. Cameron, H. Guerra, Š. Jurina, The power graph of a torsion-free group, Journal of Algebraic Combinatorics 49 (2019) 83–98.
[6] C. M. Campbell, E. F. Robertson, G. C. Smith, Groups St Andrews 2001 in Oxford Vol II, London Mathematical Society Lecture Notes Series, 305, 2003.
[7] T. T. Chelevam, M. Sattanathan, Power graph of finite abelian groups, Algebra and Discrete Mathematics 16 (2013) 33–41.
[8] B. Curtin, G. R. Pourgholi, Edge-maximality of power graphs of finite cyclic groups, Journal of Algebraic Combinatorics 40 (2014) 313–330.
[9] M. Feng, X. Ma, K. Wang, The structure and metric dimension of the power graph of a finite group, European Journal of Combinatorics 43 (2015) 82–97.
[10] M. Ghorbani, F. A. Barfaraz, On the characteristic polynomial of power graphs, Filomat 32 (12) (2018) 4375–4387.
[11] I. M. Isaacs, Finite Group Theory, Graduate Studies in Mathematics Vol–92. AMS, 2008.
[12] A.V. Kelarev, S. J. Quinn, A combinatorial property and power graphs of groups, Contributions to general algebra, 12 (Vienna, 1999), Heyn, Klagenfurt, 2000 229–235.
[13] X. Ma, R. Fu, X. Lu, On the independence number of the power graph of a finite group, Indagationes Mathematicae 29 (2) (2018) 794–806.
[14] E. Mckemmie, Power graphs of finite group, BA project Supervised by Peter M. Neumann at Oxford University, 2014.
[15] Z. Mehranian, A. Gholami, A. R. Ashrafi, A note on the power graph of a finite group, International Journal of Group Theory 5 (2016) 1–10.
[16] G. A. Miller, H. F. Blichfeldt, L. E. Dickson, Theory and Applications of Finite Groups, Dover Publications, Inc., New York, 1916.
[17] M. Mirzargar, A. R. Ashrafi, M. J. Nadjafi-Arani, On the power graph of a finite group, Filomat 26 (6) (2012) 1201–1208.
[18] A. R. Moghaddamfar, S. Rahbariyan, S. N. Salehy, S. N. Salehy, The number of spanning trees of power graphs associated with specific groups and some applications, Ars Combinatoria 133 (2017) 269–296.
[19] D. J. S. Robinson, A Course in The Theory of Groups, Springer-Verlag, New York, 1996.