



Explicit Formulas and Determinantal Representation for η -Skew-Hermitian Solution to a System of Quaternion Matrix Equations

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Abstract. Some necessary and sufficient conditions for the existence of the η -skew-Hermitian solution quaternion matrix equations the system of matrix equations with η -skew-hermicity,

$$A_1X = C_1, XB_1 = C_2,$$

$$A_2Y = C_3, YB_2 = C_4,$$

$$X = -X^{\eta*}, Y = -Y^{\eta*},$$

$$A_3XA_3^{\eta*} + B_3YB_3^{\eta*} = C_5,$$

are established in this paper by using rank equalities of the coefficient matrices. The general solutions to the system and its special cases are provided when they are consistent. Within the framework of the theory of noncommutative row-column determinants, we also give determinantal representation formulas of finding their exact solutions that are analogs of Cramer's rule. A numerical example is also given to demonstrate the main results.

1. Introduction

In this paper, \mathbb{R} and \mathbb{C} stand for the real number field and the complex field, respectively. The quaternion algebra is denoted by \mathbb{H} and is defined as

$$\mathbb{H} = \{h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, h_0, h_1, h_2, h_3 \in \mathbb{R}\}.$$

The collection of all the matrices of dimension $m \times n$ over \mathbb{H} is denoted by $\mathbb{H}^{m \times n}$. Its subset of matrices with a rank r is specified by $\mathbb{H}_r^{m \times n}$. An identity matrix with conformable shape is denoted by I . For any matrix A over \mathbb{H} , $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the column right space and the row left space of A , respectively. $D[\mathcal{R}(A)]$ denotes the dimension of $\mathcal{R}(A)$. By [29], we have $D[\mathcal{R}(A)] = D[\mathcal{N}(A)]$, which is known as rank of A denoted

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by $r(A)$. The conjugate transpose of A is denoted by A^* . A^\dagger means the Moore-Penrose inverse of $A \in \mathbb{H}^{m \times n}$, i.e. the exclusive matrix $Y \in \mathbb{H}^{m \times m}$ satisfying

$$AYA = A, YAY = Y, (AY)^* = AY, (YA)^* = YA.$$

For more properties on generalized inverses, consult [6, 75] and [30]. Furthermore, $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ are couple of projectors induced by A , respectively. It is evident that $L_A = L_A^* = L_A^\dagger = L_A^2$ and $R_A = R_A^* = R_A^\dagger = R_A^2$.

The idea of quaternions were first time introduced by an Irish mathematician Sir William Rowan Hamilton in his research in [23]. Quaternions have prolific use in diverse areas of mathematics like computation, geometry and algebra; see, e.g. [8, 31, 56, 83]. Presently, quaternion matrices have a central position in control theory, mechanics, altitude control, computer graphics, quantum physics and signal processing; see, e.g. [1, 34, 65]. In skeletal animation systems, quaternions are mostly practiced to interpolate between joint orientations specified with key frames or animation curves [9]. A researcher in [88] gave a comprehensive study on quaternions.

Numerous problems in different areas of sciences and engineering can be converted into matrix equations and hence the investigation of linear matrix equations have crucial function in matrix theory and its applications; see, e.g. [5, 10, 12, 13, 16–18, 20, 22, 26–28, 32, 33, 55, 57, 59, 62–64, 71–73, 76, 78, 80–82, 84, 85, 89, 90]. For example, the most famous Lyapunov equation

$$B_1X + (B_1X)^* = A_1$$

has vital function in optimal control, stability analysis, system theory and model reduction [58, 69]. The general solution of

$$CXC^* + DYD^* = A$$

was analyzed by different authors with different techniques in [11, 52, 86]. In [87], the least squares η -Hermitian solution to $AXB + CXD = E$ was computed. The numerical solution to the two sided Sylvester matrix equation was researched in [14, 15]. We first introduce the definition.

Definition 1.1. [27, 74, 87] A matrix $A \in \mathbb{H}^{n \times n}$ is known to be η -Hermitian and η -skew-Hermitian if $A = A^{\eta*} = -\eta A^* \eta$ and $A = -A^{\eta*} = \eta A^* \eta$, where $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, respectively.

Convergence analysis in statistical signal processing and linear modelling [72–74] are some fields in which the applications of η -Hermitian matrices can be viewed. The singular value decomposition of the η -Hermitian matrix was examined in [27]. Very recently, a researcher in [53] determined the anti- η -Hermitian solution to some significant matrix equations including

$$A_3XA_3^{\eta*} + B_3YB_3^{\eta*} = C_5 \tag{1}$$

and gave general solution to these equations when they are consistent. He and Wang [24] gave the general solution to

$$A_4X + (A_4X)^{\eta*} + B_4YB_4^{\eta*} + C_4ZC_4^{\eta*} = D_4 \tag{2}$$

bearing η -Hermicity over \mathbb{H} . The η -skew-Hermitian solution to the equation (2) was explored in [60]. An iterative algorithm for determining the η -Hermitian and η -skew-Hermitian solutions to $AXB + CYD = E$ were established in [4].

Motivated by the above mentioned work and keeping the latest advancement of η -skew-Hermitian matrices in mind, we in this paper, find some necessary and sufficient conditions for the existence of the η -skew-Hermitian solution to

$$\begin{aligned} A_1X &= C_1, XB_1 = C_2, \\ A_2Y &= C_3, YB_2 = C_4, \\ X &= -X^{\eta*}, Y = -Y^{\eta*}, \\ A_3XA_3^{\eta*} + B_3YB_3^{\eta*} &= C_5, \end{aligned} \tag{3}$$

and provide its general solution when this system is consistent by using the rank equalities of coefficient matrices. Observe that the system (1) is a particular case of our system (3). We will also give more necessary and sufficient conditions for the existence of the solution of (1) than the one presented in [53]. One other reason for the consideration of the system (3) is that the Hermitian solution to a system that is similar to (3) was computed in [19], the η -Hermitian solution was analyzed in [25], and determinantal representations of the least-norm solution are obtained in [61], respectively.

We also get the direct methods of finding exact solutions, namely explicit determinantal representation formulas that are analogs of Cramer’s rule. Our proposed Cramer’s rules are based on the theory of row-column noncommutative determinants introduced in [35, 36], by using determinantal representations of the Moore-Penrose inverse matrix [37]. Within the framework of the theory of noncommutative row-column determinants, determinantal representations of various generalized quaternion inverses and generalized inverse solutions to quaternion matrix equations have been derived by one of the authors (see, e.g. [41–45]) and by other researchers (see, e.g. [66–68]). Moreover, Cramer’s rules for generalized Sylvester matrix equation and for some systems of matrix equations over \mathbb{H} are recently explored in [46–48] and [49–51], respectively.

The remaining part of this paper is composed as follows. In Section 2, we start with some remarkable results which have significant role during the construction of the main results of this paper. Necessary and sufficient conditions for the general solution (X, Y) to (3), where X and Y are η -skew-Hermitian, are presented in Section 3. Some particular cases of (3) are also examined in Section 4. Based on row-column noncommutative determinants, Cramer’s rules of the system (3) and its particular cases are derived in Section 5. A numerical example is presented in Section 6. Finally, in Section 7, the conclusions are drawn.

2. Preliminaries

We begin with some famous results which will be used in the remaining part of this paper.

Lemma 2.1. [54]. Let $A \in \mathbb{H}^{s \times t}$, $B \in \mathbb{H}^{s \times k}$ and $C \in \mathbb{H}^{l \times t}$ be known. Then

$$(1) \quad r(A) + r(R_A B) = r(B) + r(R_B A) = r \begin{bmatrix} A & B \end{bmatrix}.$$

$$(2) \quad r(A) + r(CL_A) = r(C) + r(AL_C) = r \begin{bmatrix} A \\ C \end{bmatrix}.$$

$$(3) \quad r(B) + r(C) + r(R_B A L_C) = r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$$

Lemma 2.2. [24] Let $A \in \mathbb{H}^{m \times n}$ be given. Then

$$(1) \quad (A^\eta)^\dagger = (A^\dagger)^\eta, (A^{\eta*})^\dagger = (A^\dagger)^{\eta*}.$$

$$(2) \quad r(A) = r(A^{\eta*}) = r(A^\eta) = r(A^\eta A^{\eta*}) = r(A^{\eta*} A^\eta).$$

$$(3) \quad (A^\dagger A)^{\eta*} = A^{\eta*} (A^\dagger)^{\eta*} = (A^\dagger A)^\eta = (A^\dagger)^\eta A^\eta.$$

$$(4) \quad (A A^\dagger)^{\eta*} = (A^\dagger)^{\eta*} A^{\eta*} = (A A^\dagger)^\eta = A^\eta (A^\dagger)^\eta.$$

$$(5) \quad (L_A)^{\eta*} = -\eta(L_A)\eta = (L_A)^\eta = L_{A^\eta} = R_{A^{\eta*}}.$$

$$(6) \quad (R_A)^{\eta*} = -\eta(R_A)\eta = (R_A)^\eta = L_{A^{\eta*}} = R_{A^\eta}.$$

Lemma 2.3. [79]. Let A, B and C be given matrices with right sizes over \mathbb{H} . Then

$$(1) \quad A^\dagger = (A^* A)^\dagger A^* = A^* (A A^*)^\dagger.$$

$$(2) \quad L_A = L_A^2 = L_A^*, R_A = R_A^2 = R_A^*.$$

$$(3) L_A(BL_A)^\dagger = (BL_A)^\dagger, (R_AC)^\dagger R_A = (R_AC)^\dagger.$$

Lemma 2.4. [53]. Let A_2, B_2, C_2 and D_2 be given with conformable dimensions over \mathbb{H} . Set

$$E = \begin{bmatrix} A_2 \\ B_2^{\eta^*} \end{bmatrix}, F = \begin{bmatrix} C_2 \\ D_2^{\eta^*} \end{bmatrix}.$$

Then the system $A_2 Y = C_2, Y B_2 = D_2$ has η -Skew-Hermitian solution if and only if $R_E F = 0$ and $EF^{\eta^*} = -FE^{\eta^*}$. Under these terms, its general η -Hermitian solution is

$$Y = E^\dagger F - (E^\dagger F)^{\eta^*} + E^\dagger E (E^\dagger F)^{\eta^*} + L_E V L_E^{\eta^*},$$

where $V = -V^{\eta^*}$ is a free matrix over \mathbb{H} with conformable size.

Remark 2.5. Since for any $\eta_m \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for all $m = 1, 2, 3$, and $q = a_0 + a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3$, the conjugate of q is $q^* = a_0 - a_1 \eta_1 - a_2 \eta_2 - a_3 \eta_3$ and

$$\begin{aligned} q^{\eta_1} &= -\eta_1 q \eta_1 = a_0 + a_1 \eta_1 - a_2 \eta_2 - a_3 \eta_3, \\ q^{-\eta_1} &= \eta_1 q \eta_1 = -a_0 - a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3, \end{aligned}$$

then elements of the main diagonal of an η_1 -Hermitian matrix $A = A^{\eta_1^*}$ must be as

$$a_{ii}^{\eta_1^*} = a_0 + a_2 \eta_2 + a_3 \eta_3,$$

and a pair of elements which are symmetric with respect to the main diagonal can be represented as

$$\begin{aligned} a_{ij}^{\eta_1^*} &= a_0 + a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3, \\ a_{ji}^{\eta_1^*} &= a_0 - a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3. \end{aligned}$$

Similarly, elements of the main diagonal of an η_1 -skew-Hermitian matrix $A = A^{-\eta_1^*}$ must be as

$$a_{ii}^{-\eta_1^*} = a_1 \eta_1,$$

and a pair of elements which are symmetric with respect to the main diagonal can be represented as

$$\begin{aligned} a_{ij}^{-\eta_1^*} &= a_0 + a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3, \\ a_{ji}^{-\eta_1^*} &= -a_0 + a_1 \eta_1 - a_2 \eta_2 - a_3 \eta_3. \end{aligned}$$

where for all $a_m \in \mathbb{R}$ for all $m = 0, \dots, 3$.

Since the Moore-Penrose inverse of a coefficient matrix and its inducted projectors are crucial to expressions of solutions, there is a problem of their construct. The inverse matrix is determined by the adjugate matrix that gives a direct method of its finding by using minors of an initial matrix. Due to minors, this method can be called the determinantal representation of an inverse. The same is desirable for generalized inverses. However, determinantal representations of generalized inverses are not so unambiguous even for complex or real generalized inverses. Through looking for their more applicable explicit expressions, there are various determinantal representations of generalized inverses, in particular for the Moore-Penrose inverse (see, e.g., [3, 21, 39, 40, 70]). By virtue of noncommutativity of quaternions, the problem for determinantal representation of quaternion generalized inverses is even more complicated. All of the previous defined quaternion determinants are derived by transforming a quaternion matrix to an equivalent complex or real matrix (see, e.g., [2, 7, 88]). However, by this way it is impossible to give the determinantal representations of generalized inverses. Only now it became possible due to the theory of column-row noncommutative determinants introduced in [35, 36].

For $A \in \mathbb{H}^{n \times n}$, we define n row determinants and n column determinants. Suppose S_n is the symmetric group on the set $I_n = \{1, \dots, n\}$.

Definition 2.6. [35]. The i -th row determinant of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined for any $i \in I_n$ by setting

$$\text{rdet}_i \mathbf{A} = \sum_{\sigma \in S_n} (-1)^{n-r} (a_{i i_{k_1}} a_{i i_{k_1+1}} \dots a_{i i_{k_1+l_1}}) \dots (a_{i i_{k_r}} a_{i i_{k_r+1}} \dots a_{i i_{k_r+l_r}}),$$

$$\sigma = (i i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}),$$

where σ is the left-ordered permutation. It means that its first cycle from the left starts with i , other cycles start from the left with the minimal of all the integers which are contained in it,

$$i_{k_t} < i_{k_t+s} \text{ for all } t = 2, \dots, r, \quad s = 1, \dots, l_t,$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from left to right of their first elements, $i_{k_2} < i_{k_3} < \dots < i_{k_r}$.

Definition 2.7. [35]. The j -th column determinant of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined for any $j \in I_n$ by setting

$$\text{cdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} (a_{j k_r} a_{j k_r+1} \dots a_{j k_r+l_r}) \dots (a_{j k_1} a_{j k_1+1} \dots a_{j k_1+l_1}),$$

$$\tau = (j_{k_r} j_{k_r+1} \dots j_{k_r+l_r}) \dots (j_{k_2} j_{k_2+1} \dots j_{k_2+l_2}) (j_{k_1} j_{k_1+1} \dots j_{k_1+l_1}),$$

where τ is the right-ordered permutation. It means that its first cycle from the right starts with j , other cycles start from the right with the minimal of all the integers which are contained in it,

$$j_{k_t} < j_{k_t+s} \text{ for all } t = 2, \dots, r, \quad s = 1, \dots, l_t,$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from right to left of their first elements, $j_{k_2} < j_{k_3} < \dots < j_{k_r}$.

Since [36] for Hermitian A we have

$$\text{rdet}_1 A = \dots = \text{rdet}_n A = \text{cdet}_1 A = \dots = \text{cdet}_n A \in \mathbb{R},$$

the determinant of a Hermitian matrix is defined by putting $\det A := \text{rdet}_i A = \text{cdet}_i A$ for all $i = 1, \dots, n$. Its properties are similar to the properties of an usual (commutative) determinant and they have been completely explored by using row and column determinants in [36].

Further, we give determinantal representations of the Moore-Penrose inverse over \mathbb{H} . Let $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. Let A_β^α be a submatrix of $A \in \mathbb{H}^{m \times n}$ whose rows are indexed by α and the columns indexed by β . Similarly, let A_α^α be a principal submatrix of A whose rows and columns indexed by α . If $A \in \mathbb{H}^{n \times n}$ is Hermitian, then $|A|_\alpha^\alpha$ is the corresponding principal minor of $\det A$. For $1 \leq k \leq n$, the collection of strictly increasing sequences of k integers chosen from $\{1, \dots, n\}$ is denoted by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k), 1 \leq \alpha_1 < \dots < \alpha_k \leq n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

Suppose that $a_{.j}$ and $a_{.j}^*$, a_i and a_i^* stand for the j -th columns and the i -th rows of A and A^* , respectively. Let $A_{.j}(b)$ and $A_i(c)$ denote the matrices obtained from A by replacing its j -th column with the column-vector $b \in \mathbb{H}^{m \times 1}$, and its i -th row with the row-vector $c \in \mathbb{H}^{1 \times n}$, respectively.

Theorem 2.8. [37]. If $A \in \mathbb{H}_r^{m \times n}$, then the Moore-Penrose inverse $A^\dagger = (a_{ij}^\dagger) \in \mathbb{H}^{n \times m}$ have the following determinantal representations,

$$a_{ij}^\dagger = \frac{\sum_{\beta \in J_{r,m}\{i\}} \text{cdet}_i \left((A^* A)_{.i} (a_{.j}^*) \right)_\beta^\beta}{\sum_{\beta \in J_{r,n}} |A^* A|_\beta^\beta} = \tag{4}$$

$$= \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j \left((A A^*)_{.j} (a_i^*) \right)_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |A A^*|_\alpha^\alpha}. \tag{5}$$

Remark 2.9. For an arbitrary full-rank matrix $A \in \mathbb{H}_r^{m \times n}$, a column-vector $c \in \mathbb{H}^{1 \times n}$, and a row-vector $b \in \mathbb{H}_r^{m \times 1}$, we put

$$\begin{aligned} \text{cdet}_i((A^*A)_i(c)) &= \sum_{\beta \in J_{n,n}\{i\}} \text{cdet}_i((A^*A)_i(c))_\beta^\beta, \quad \det(A^*A) = \sum_{\beta \in J_{n,n}} |A^*A|_\beta^\beta \quad \text{when } r = n, \\ \text{rdet}_j((AA^*)_j(b)) &= \sum_{\alpha \in I_{m,m}\{j\}} \text{rdet}_j((AA^*)_j(b))_\alpha^\alpha, \quad \det(AA^*) = \sum_{\alpha \in I_{m,m}} |AA^*|_\alpha^\alpha \quad \text{when } r = m. \end{aligned}$$

Corollary 2.10. If $A \in \mathbb{H}_r^{m \times n}$, then the projection matrices $A^+A =: Q_A = (q_{ij})_{n \times n}$ and $AA^+ =: P_A = (p_{ij})_{m \times m}$ have the determinantal representations, respectively,

$$q_{ij} = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((A^*A)_i(\dot{a}_j))_\beta^\beta}{\sum_{\beta \in J_{r,n}} |A^*A|_\beta^\beta}, \tag{6}$$

$$p_{ij} = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j((AA^*)_j(\ddot{a}_i))_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |AA^*|_\alpha^\alpha}, \tag{7}$$

where \dot{a}_j is the j -th column of $A^*A \in \mathbb{H}^{n \times n}$ and \ddot{a}_i is the i -th row of $AA^* \in \mathbb{H}^{m \times m}$.

Corollary 2.11. If $A \in \mathbb{H}_r^{m \times n}$, then the Moore-Penrose inverse $(A^{\eta*})^\dagger = ((a_{ij}^{\eta*})^\dagger) \in \mathbb{H}^{m \times n}$ has the following determinantal representations, respectively,

$$(a_{ij}^{\eta*})^\dagger = -\eta(a_{ji}^{\eta*})^\dagger\eta = -\eta \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j((A^*A)_j(a_i))_\alpha^\alpha}{\sum_{\beta \in I_{r,n}} |A^*A|_\alpha^\alpha} \eta \tag{8}$$

$$= -\eta \frac{\sum_{\beta \in J_{r,m}\{i\}} \text{cdet}_i((AA^*)_i(a_j))_\beta^\beta}{\sum_{\beta \in J_{r,m}} |AA^*|_\beta^\beta} \eta. \tag{9}$$

Lemma 2.12. [77]. Let A , B and C be given matrices matrices of conformable shapes over \mathbb{H} . Then

$$AXB = C \tag{10}$$

is consistent if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$. In this case, the general solution of this equation is

$$X = A^+CB^+ + L_A U_1 + U_2 R_B,$$

where U_1 and U_2 are arbitrary matrices of adequate sizes over \mathbb{H} .

Lemma 2.13. [38] Let $A \in \mathbb{H}_{r_1}^{m \times n}$, $B \in \mathbb{H}_{r_2}^{r \times s}$. Then the partial solution $X = A^+CB^+ = (x_{ij}) \in \mathbb{H}^{n \times r}$ to (10) has determinantal representations,

$$x_{ij} = \frac{\sum_{\beta \in J_{r_1,n}\{i\}} \text{cdet}_i((A^*A)_i(\dot{d}_j^B))_\beta^\beta}{\sum_{\beta \in J_{r_1,n}} |A^*A|_\beta^\beta \sum_{\alpha \in I_{r_2,r}} |BB^*|_\alpha^\alpha} = \frac{\sum_{\alpha \in I_{r_2,r}\{j\}} \text{rdet}_j((BB^*)_j(\dot{d}_i^A))_\alpha^\alpha}{\sum_{\beta \in J_{r_1,n}} |A^*A|_\beta^\beta \sum_{\alpha \in I_{r_2,r}} |BB^*|_\alpha^\alpha},$$

where

$$d_j^B = \left[\sum_{\alpha \in I_{2,r}\{j\}} \text{rdet}_j((BB^*)_j, (\tilde{c}_k)_\alpha)^\alpha \right] \in \mathbb{H}^{n \times 1}, \quad k = 1, \dots, n,$$

$$d_i^A = \left[\sum_{\beta \in J_{1,n}\{i\}} \text{cdet}_i((A^*A)_i, (\tilde{c}_l)_\beta)^\beta \right] \in \mathbb{H}^{1 \times r}, \quad l = 1, \dots, r,$$

are the column vector and the row vector, respectively. \tilde{c}_k and \tilde{c}_l are the k -th row and the l -th column of $\tilde{C} = A^*CB^*$.

Corollary 2.14. Let $A \in \mathbb{H}_k^{m \times n}$, $C \in \mathbb{H}^{n \times s}$ be known and $X \in \mathbb{H}^{n \times s}$ be unknown. Then the matrix equation $AX = C$ is consistent if and only if $AA^\dagger C = C$. In this case, its general solution can be expressed as $X = A^\dagger C + L_A V$, where V is an arbitrary matrix over \mathbb{H} with appropriate dimensions. Its partial solution $X = A^\dagger C$ has the determinantal representation

$$x_{ij} = \frac{\sum_{\beta \in J_{k,n}\{i\}} \text{cdet}_i((A^*A)_i, (\hat{c}_j)_\beta)^\beta}{\sum_{\beta \in J_{k,n}} |A^*A|_\beta^\beta},$$

where \hat{c}_j is the j -th column of $\hat{C} = A^*C$.

Corollary 2.15. Let $B \in \mathbb{H}_k^{r \times s}$, $C \in \mathbb{H}^{n \times s}$ be given, and $X \in \mathbb{H}^{n \times r}$ be unknown. Then the equation $XB = C$ is solvable if and only if $C = CB^\dagger B$ and its general solution is $X = CB^\dagger + WR_B$, where W is a any matrix with conformable dimension. Moreover, its partial solution $X = CB^\dagger$ has the determinantal representation

$$x_{ij} = \frac{\sum_{\alpha \in I_{k,r}\{j\}} \text{rdet}_j((BB^*)_j, (\hat{c}_i)_\alpha)^\alpha}{\sum_{\alpha \in I_{k,r}} |BB^*|_\alpha^\alpha},$$

where \hat{c}_i is the i -th row of $\hat{C} = CB^*$.

3. Some solvability conditions and the general solution to (3)

In this section, we provide some necessary and sufficient conditions for the system (3) to have a solution (X, Y) , where $X = -X^{\eta^*}$ and $Y = -Y^{\eta^*}$. Additionally, its general solution is also given when some solvability conditions are accomplished.

Theorem 3.1. Let $A_1, A_2, A_3, B_1, B_2, B_3, C_1, \dots, C_4$ and $C_5 = -C_5^{\eta^*}$ be known coefficient matrices in (3) over \mathbb{H} with adequate sizes. Denote

$$A_4 = \begin{bmatrix} A_1 \\ B_1^{\eta^*} \end{bmatrix}, \quad C_6 = \begin{bmatrix} C_1 \\ -C_2^{\eta^*} \end{bmatrix}, \quad B_4 = \begin{bmatrix} A_2 \\ B_2^{\eta^*} \end{bmatrix}, \quad C_7 = \begin{bmatrix} C_3 \\ -C_4^{\eta^*} \end{bmatrix},$$

$$A = A_3 L_{A_4}, \quad B = B_3 L_{B_4}, \quad M = R_A B, \quad S = B L_M,$$

$$C = C_5 - A_3 A_4^\dagger C_6 A_3^{\eta^*} + A_3 (A_4^\dagger C_6)^{\eta^*} A_3^{\eta^*} - A_3 A_4^\dagger (A_4 C_6^{\eta^*}) (A_4^\dagger)^{\eta^*} A_3^{\eta^*}$$

$$- B_3 B_4^\dagger C_7 B_3^{\eta^*} + B_3 (B_4^\dagger C_7)^{\eta^*} B_3^{\eta^*} - B_3 B_4^\dagger (B_4 C_7^{\eta^*}) (B_4^\dagger)^{\eta^*} B_3^{\eta^*}$$

Then

- (1) The system (3) has a solution (X, Y) , where $X = -X^{\eta^*}$ and $Y = -Y^{\eta^*}$.

(2)

$$A_4C_6^{\eta^*} = -C_6A_4^{\eta^*}, B_4C_7^{\eta^*} = -C_7B_4^{\eta^*}, R_{A_4}C_6 = 0, R_{B_4}C_7 = 0, \tag{11}$$

$$R_MR_AC = 0, R_ACR_B^{\eta^*} = 0. \tag{12}$$

(3)

$$A_4C_6^{\eta^*} = -C_6A_4^{\eta^*}, B_4C_7^{\eta^*} = -C_7B_4^{\eta^*}, R_{A_4}C_6 = 0, R_{B_4}C_7 = 0,$$

$$MM^{\dagger}R_AC = R_AC = R_AC(B^{\dagger})^{\eta^*}B^{\eta^*}.$$

(4)

$$A_4C_6^{\eta^*} = -C_6A_4^{\eta^*}, \quad B_4C_7^{\eta^*} = -C_7B_4^{\eta^*},$$

$$r \begin{bmatrix} A_1 & C_1 \\ B_1^{\eta^*} & -C_2^{\eta^*} \end{bmatrix} = r \begin{bmatrix} A_1 \\ B_1^{\eta^*} \end{bmatrix}, \quad r \begin{bmatrix} A_2 & C_2 \\ B_2^{\eta^*} & -C_4^{\eta^*} \end{bmatrix} = r \begin{bmatrix} A_2 \\ B_2^{\eta^*} \end{bmatrix},$$

$$r \begin{bmatrix} C_5 & B_3 & A_3 \\ C_1A_3^{\eta^*} & A_2 & 0 \\ -C_2^{\eta^*}A_3^{\eta^*} & B_2^{\eta^*} & 0 \\ C_3B_3^{\eta^*} & 0 & A_1 \\ -C_4^{\eta^*}B_3^{\eta^*} & 0 & B_1^{\eta^*} \end{bmatrix} = r \begin{bmatrix} B_3 & A_3 \\ A_2 & 0 \\ B_2^{\eta^*} & 0 \\ 0 & A_1 \\ 0 & B_1^{\eta^*} \end{bmatrix},$$

$$r \begin{bmatrix} C_5 & A_3 & B_3C_3^{\eta^*} & -B_3C_4 \\ B_3^{\eta^*} & 0 & A_2^{\eta^*} & B_2 \\ C_1A_3^{\eta^*} & A_1 & 0 & 0 \\ -C_2^{\eta^*}A_3^{\eta^*} & B_1^{\eta^*} & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 \\ A_1 \\ B_1^{\eta^*} \end{bmatrix} + r \begin{bmatrix} B_3 \\ A_2 \\ B_2^{\eta^*} \end{bmatrix}.$$

the above mentioned statements are equivalent. Under these conditions, the general solution to the system (3) can be expressed as

$$\begin{aligned} X &= -X^{\eta^*} = A_4^{\dagger}C_6 - (A_4^{\dagger}C_6)^{\eta^*} + A_4^{\dagger}(A_4C_6^{\eta^*})(A_4^{\dagger})^{\eta^*} + L_{A_4}[A^{\dagger}C(A^{\dagger})^{\eta^*} - \frac{1}{2}A^{\dagger}BM^{\dagger}C[I + (B^{\dagger})^{\eta^*}S^{\eta^*}](A^{\dagger})^{\eta^*} \\ &\quad - \frac{1}{2}A^{\dagger}[I + SB^{\dagger}]C(M^{\dagger})^{\eta^*}B^{\eta^*}(A^{\dagger})^{\eta^*} - A^{\dagger}SW_{12}S^{\eta^*}(A^{\dagger})^{\eta^*} - L_AU_{12} + U_{12}^{\eta^*}L_A^{\eta^*}](L_{A_4})^{\eta^*}, \end{aligned} \tag{13}$$

$$\begin{aligned} Y &= -Y^{\eta^*} = B_4^{\dagger}C_7 - (B_4^{\dagger}C_7)^{\eta^*} + B_4^{\dagger}(B_4C_7^{\eta^*})(B_4^{\dagger})^{\eta^*} + L_{B_4}[\frac{1}{2}M^{\dagger}C(B^{\dagger})^{\eta^*}[I + (S^{\dagger}S)^{\eta^*}] \\ &\quad + \frac{1}{2}(I + S^{\dagger}S)B^{\dagger}C(M^{\dagger})^{\eta^*} + L_MW_{12}(L_M)^{\eta^*} - U_{13}L_B^{\eta^*} + L_BU_{13}^{\eta^*} + L_ML_SU_{14} - U_{14}^{\eta^*}(L_ML_S)^{\eta^*}]L_{B_4}^{\eta^*}, \end{aligned} \tag{14}$$

where U_{11}, \dots, U_{14} and $W_{12}^{\eta^*} = -W_{12}$ are arbitrary matrices over \mathbb{H} with allowable dimensions.

Proof. Obviously, (2) \iff (3).

Now we show (2) \iff (4). By means of Lemma 2.1 and Lemma 2.2, we have

$$R_{A_4}C_6 = 0 \iff r(R_{A_4}C_6) = 0 \iff r \begin{bmatrix} A_4 & C_6 \end{bmatrix} = r(A_4)$$

$$\iff r \begin{bmatrix} A_1 & C_1 \\ B_1^{\eta^*} & -C_2^{\eta^*} \end{bmatrix} = r \begin{bmatrix} A_1 \\ B_1^{\eta^*} \end{bmatrix},$$

$$R_{B_4}C_7 = 0 \iff r(R_{B_4}C_7) = 0 \iff r \begin{bmatrix} B_4 & C_7 \end{bmatrix} = r(B_4)$$

$$\iff r \begin{bmatrix} A_2 & C_3 \\ B_2^{\eta^*} & -C_4^{\eta^*} \end{bmatrix} = r \begin{bmatrix} A_2 \\ B_2^{\eta^*} \end{bmatrix}.$$

Observe that

$$A_4(A_4^\dagger C_6 + (A_4^\dagger C_6)^{\eta*} - A_4^\dagger A_4 C_6^{\eta*} (A_4^\dagger)^{\eta*}) = C_6, \tag{15}$$

$$B_4(B_4^\dagger C_7 + (B_4^\dagger C_7)^{\eta*} - B_4^\dagger B_4 C_7^{\eta*} (B_4^\dagger)^{\eta*}) = C_7. \tag{16}$$

It follows that (15)-(16) and Lemma 2.1, we have

$$\begin{aligned} R_M R_A C = 0 &\iff r(R_M R_A C) = 0 \iff r \begin{bmatrix} M & R_A C \end{bmatrix} = r(M) \\ &\iff r \begin{bmatrix} R_A B & R_A C \end{bmatrix} = r(R_A B) \\ &\iff r \begin{bmatrix} A & B & C \end{bmatrix} = r \begin{bmatrix} A & B \end{bmatrix} \\ &\iff r \begin{bmatrix} A_3 L_{A_4} & B_3 L_{B_4} & C \end{bmatrix} = r \begin{bmatrix} A_3 L_{A_4} & B_3 L_{B_4} \end{bmatrix} \\ &\iff r \begin{bmatrix} C & B_3 & A_3 \\ 0 & B_4 & 0 \\ 0 & 0 & A_4 \end{bmatrix} = r \begin{bmatrix} B_3 & A_3 \\ B_4 & 0 \\ 0 & A_4 \end{bmatrix} \\ &\iff r \begin{bmatrix} C_5 & B_3 & A_3 \\ C_7 B_3^{\eta*} & B_4 & 0 \\ C_6 A_3^{\eta*} & 0 & A_4 \end{bmatrix} = r \begin{bmatrix} B_3 & A_3 \\ B_4 & 0 \\ 0 & A_4 \end{bmatrix}, \end{aligned}$$

similarly,

$$\begin{aligned} R_A C (R_B)^\eta = 0 &\iff r(R_A C (R_B)^\eta) = 0 \iff r \begin{bmatrix} C & A \\ B^{\eta*} & 0 \end{bmatrix} = r(A) + r(B) \\ &\iff r \begin{bmatrix} C & A_3 L_{A_4} \\ R_{B_4}^{\eta*} B_3^{\eta*} & 0 \end{bmatrix} = r(A_3 L_{A_4}) + r(R_{B_4}^{\eta*} B_3^{\eta*}) \\ &\iff r \begin{bmatrix} C & A_3 & 0 \\ B_3^{\eta*} & 0 & B_4^{\eta*} \\ 0 & A_4 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} + r \begin{bmatrix} B_3 \\ B_4 \end{bmatrix} \\ &\iff r \begin{bmatrix} C_5 & A_3 & B_3 C_7^{\eta*} \\ B_3^{\eta*} & 0 & B_4^{\eta*} \\ C_6 A_3^{\eta*} & A_4 & 0 \end{bmatrix} = r \begin{bmatrix} A_3 \\ A_4 \end{bmatrix} + r \begin{bmatrix} B_3 \\ B_4 \end{bmatrix}. \end{aligned}$$

Now we show (1) \implies (2): If the system (3) has a solution (Y, Z) , where $Y = -Y^{\eta*}$ and $Z = -Z^{\eta*}$. Then by Lemma 2.4, the general η -skew-Hermitian solution to $A_4 X = C_6$ and $B_4 Y = C_7$ is

$$X = A_4^\dagger C_6 - (A_4^\dagger C_6)^{\eta*} + A_4^\dagger (A_4 C_6^{\eta*}) (A_4^\dagger)^{\eta*} + L_{A_4} U_1 (L_{A_4})^{\eta*} \tag{17}$$

and

$$Y = B_4^\dagger C_7 - (B_4^\dagger C_7)^{\eta*} + B_4^\dagger (B_4 C_7^{\eta*}) (B_4^\dagger)^{\eta*} + L_{B_4} U_2 (L_{B_4})^{\eta*} \tag{18}$$

respectively, where $U_1 = -U_1^{\eta*}$ and $U_2 = -U_2^{\eta*}$ are arbitrary matrices of feasible shapes.

Using (17)-(18) in $A_3 X A_3^{\eta*} - B_3 Y B_3^{\eta*}$ and simplifying, we have

$$A U_1 A^{\eta*} + B U_2 B^{\eta*} = C.$$

From this equation, we have

$$\begin{aligned} \implies R_A [C - A U_1 A^{\eta*} - B U_2 B^{\eta*}] R_B^{\eta*} &= 0 \\ \implies R_A [C] R_B^{\eta*} &= 0 \end{aligned}$$

and

$$R_A[AU_1A^{\eta^*} + BU_2B^{\eta^*}] = R_AC$$

$$\Rightarrow MU_2B^{\eta^*} = R_AC \Rightarrow R_MR_AC = 0.$$

(2) \Rightarrow (1): Now we prove that (X, Y) mentioned in (13)-(14), respectively, is a solution of (3) under (11) and (12). Obviously, X and Y represented in (13) and (14) are η -skew-Hermitian. Put

$$P_0 = \frac{1}{2}M^\dagger C(B^\dagger)^{\eta^*}[I + (S^\dagger S)^{\eta^*}] + \frac{1}{2}(I + S^\dagger S)B^\dagger C(M^\dagger)^{\eta^*}$$

$$+ L_M W_{12}(L_M)^{\eta^*} - U_{13}L_B^{\eta^*} + L_B U_{13}^{\eta^*} + L_M L_S U_{14} - U_{14}^{\eta^*}(L_M L_S)^{\eta^*}, \tag{19}$$

$$Q_0 = A^\dagger(C - BP_0B^{\eta^*})(A^\dagger)^{\eta^*} - L_A U_{12} + U_{12}^{\eta^*}(L_A)^{\eta^*}. \tag{20}$$

Now (14) with the help of (19) can be expressed as

$$Y = -Y^{\eta^*} = B_4^\dagger C_7 - (B_4^\dagger C_7)^{\eta^*} + B_4^\dagger (B_4 C_7^{\eta^*})(B_4^\dagger)^{\eta^*} + L_{B_4} P_0 L_{B_4}^{\eta^*}. \tag{21}$$

By Lemma 2.3, Eq. (12) and $B - S = BM^\dagger M$, we have

$$M^\dagger R_A = M^\dagger, BS^\dagger S - S = (B - S)S^\dagger S = BM^\dagger MS^*(S^\dagger)^* = 0, R_AC = R_AC(B^\dagger)^{\eta^*}B^{\eta^*}. \tag{22}$$

By using (22), Eq. (20) can be written as follows

$$Q_0 = A^\dagger(C - BP_0B^{\eta^*})(A^\dagger)^{\eta^*} - L_A U_{12} + U_{12}^{\eta^*}(L_A)^{\eta^*}$$

$$= A^\dagger C(A^\dagger)^{\eta^*} - A^\dagger BP_0B^{\eta^*}(A^\dagger)^{\eta^*} - L_A U_{12} + U_{12}^{\eta^*}(L_A)^{\eta^*}$$

$$= A^\dagger C(A^\dagger)^{\eta^*} - \frac{1}{2}A^\dagger BM^\dagger C(B^\dagger)^{\eta^*}[I + S^{\eta^*}(S^\dagger)^{\eta^*}]B^{\eta^*}(A^\dagger)^{\eta^*}$$

$$- \frac{1}{2}A^\dagger B[I + S^\dagger S]B^\dagger C(M^\dagger)^{\eta^*}B^{\eta^*}(A^\dagger)^{\eta^*} - A^\dagger S W_{12} S^{\eta^*}(A^\dagger)^{\eta^*} - L_A U_{12} + U_{12}^{\eta^*} L_A^{\eta^*}$$

$$= A^\dagger C(A^\dagger)^{\eta^*} - \frac{1}{2}A^\dagger BM^\dagger C(B^\dagger)^{\eta^*}B^{\eta^*}(A^\dagger)^{\eta^*} - \frac{1}{2}A^\dagger BM^\dagger C(B^\dagger)^{\eta^*}S^{\eta^*}(A^\dagger)^{\eta^*}$$

$$- \frac{1}{2}A^\dagger C(M^\dagger)^{\eta^*}B^{\eta^*}(A^\dagger)^{\eta^*} - \frac{1}{2}A^\dagger S B^\dagger C(M^\dagger)^{\eta^*}B^{\eta^*}(A^\dagger)^{\eta^*}$$

$$- A^\dagger S W_{12} S^{\eta^*}(A^\dagger)^{\eta^*} - L_A U_{12} + U_{12}^{\eta^*} L_A^{\eta^*}$$

$$= A^\dagger C(A^\dagger)^{\eta^*} - \frac{1}{2}A^\dagger BM^\dagger R_A C(B^\dagger)^{\eta^*}B^{\eta^*}(A^\dagger)^{\eta^*} - \frac{1}{2}A^\dagger BM^\dagger R_A C(B^\dagger)^{\eta^*}(S^\dagger S)^{\eta^*}B^{\eta^*}(A^\dagger)^{\eta^*}$$

$$- \frac{1}{2}A^\dagger B B^\dagger C(R_A)^{\eta^*}(M^\dagger)^{\eta^*}B^{\eta^*}(A^\dagger)^{\eta^*} - \frac{1}{2}A^\dagger B S^\dagger S B^\dagger C(M^\dagger)^{\eta^*}B^{\eta^*}(A^\dagger)^{\eta^*}$$

$$- A^\dagger S W_{12} S^{\eta^*}(A^\dagger)^{\eta^*} - L_A U_{12} + U_{12}^{\eta^*} L_A^{\eta^*}, \tag{23}$$

from (23), we have

$$X = -X^{\eta^*} = A_4^\dagger C_6 - (A_4^\dagger C_6)^{\eta^*} + A_4^\dagger (A_4 C_6^{\eta^*})(A_4^\dagger)^{\eta^*} + L_{A_4} Q_0 (L_{A_4})^{\eta^*}. \tag{24}$$

By using $R_{A_4} C_6 = 0$ and $R_{B_4} C_7 = 0$, we have

$$A_4 X = A_4 A_4^\dagger C_6 - A_4 (A_4^\dagger C_6)^{\eta^*} + A_4 A_4^\dagger (A_4 C_6^{\eta^*})(A_4^\dagger)^{\eta^*} + A_4 L_{A_4} Q_0 (L_{A_4})^{\eta^*}$$

$$= C_6 \Rightarrow r \begin{bmatrix} A_1 \\ B_1^{\eta^*} \end{bmatrix} X = r \begin{bmatrix} C_1 \\ -C_2^{\eta^*} \end{bmatrix},$$

$$B_4 Y = B_4 B_4^\dagger C_7 - B_4 (B_4^\dagger C_7)^{\eta^*} + B_4 B_4^\dagger (B_4 C_7^{\eta^*})(B_4^\dagger)^{\eta^*} + B_4 L_{B_4} P_0 (L_{B_4})^{\eta^*}$$

$$= C_7 \Rightarrow r \begin{bmatrix} A_2 \\ B_2^{\eta^*} \end{bmatrix} Y = r \begin{bmatrix} C_3 \\ -C_4^{\eta^*} \end{bmatrix}.$$

By using (21) and (24) in $C_5 - A_3XA_3 - B_3YB_3^{\eta*}$, we have

$$C_5 - A_3XA_3 - B_3YB_3^{\eta*} = C - AQ_0A^{\eta*} - BP_0B^{\eta*}. \tag{25}$$

Notice that

$$ML_B = 0, MS^* = 0, MM^\dagger R_A C = R_A C = R_A C(B^\dagger)^{\eta*} B^{\eta*},$$

$$B^{\eta*} - S^{\eta*} = M^{\eta*} (M^\dagger)^{\eta*} B^{\eta*}, S^{\eta*} = S^{\eta*} (S^\dagger)^{\eta*} B^{\eta*}, P_0 = -P_0^{\eta*},$$

then

$$\begin{aligned} MP_0B^{\eta*} &= \frac{1}{2} [MM^\dagger C(B^\dagger)^{\eta*} B^{\eta*} + MB^\dagger C(M^\dagger)^{\eta*} B^{\eta*} + MM^\dagger C(B^\dagger)^{\eta*} (S^\dagger S)^{\eta*} B^{\eta*}] \\ &= \frac{1}{2} [MM^\dagger R_A C(B^\dagger)^{\eta*} B^{\eta*} \\ &\quad + MB^\dagger C R_A^{\eta*} (M^\dagger)^{\eta*} B^{\eta*} + MM^\dagger R_A C(B^\dagger)^{\eta*} (S^\dagger S)^{\eta*} B^{\eta*}] \\ &= \frac{1}{2} [R_A C + R_A B B^\dagger C R_A^{\eta*} (M^\dagger)^{\eta*} B^{\eta*} + MM^\dagger R_A C(B^\dagger)^{\eta*} S^{\eta*}] \\ &= \frac{1}{2} [R_A C + R_A C L_{A^{\eta*}} (M^\dagger)^{\eta*} B^{\eta*} + R_A C(B^\dagger)^{\eta*} S^{\eta*}] \\ &= \frac{1}{2} [R_A C + R_A C(B^\dagger)^{\eta*} B^{\eta*} L_{A^{\eta*}} (M^\dagger)^{\eta*} B^{\eta*} + R_A C(B^\dagger)^{\eta*} S^{\eta*}] \\ &= \frac{1}{2} [R_A C + R_A C(B^\dagger)^{\eta*} M^{\eta*} (M^\dagger)^{\eta*} B^{\eta*} + R_A C(B^\dagger)^{\eta*} S^{\eta*}] \\ &= \frac{1}{2} [R_A C + R_A C(B^\dagger)^{\eta*} (B^{\eta*} - S^{\eta*}) + R_A C(B^\dagger)^{\eta*} S^{\eta*}] \\ &= \frac{1}{2} [R_A C + R_A C - R_A C(B^\dagger)^{\eta*} S^{\eta*} + R_A C(B^\dagger)^{\eta*} S^{\eta*}] \\ &= R_A C. \end{aligned} \tag{26}$$

By putting Eq. (26) in (25), we have

$$\begin{aligned} C_5 - A_3XA_3 - B_3YB_3^{\eta*} &= C - AQ_0A^{\eta*} - BP_0B^{\eta*} \\ &= C - AA^\dagger C(A^\dagger)^{\eta*} A^{\eta*} + AA^\dagger BP_0B^{\eta*} (AA^\dagger)^{\eta*} - BP_0B^{\eta*} \\ &= C - AA^\dagger C(A^\dagger)^{\eta*} A^{\eta*} + (I - R_A)BP_0B^{\eta*} (AA^\dagger)^{\eta*} - BP_0B^{\eta*} \\ &= C - AA^\dagger C(A^\dagger)^{\eta*} A^{\eta*} + BP_0B^{\eta*} (AA^\dagger)^{\eta*} - MP_0B^{\eta*} (AA^\dagger)^{\eta*} - BP_0B^{\eta*} \\ &= 0. \end{aligned}$$

Next, we want to prove that any solution (X, Y) , where X and Y are η -skew-Hermitian matrices, of the system (3) can be expressed by (14)-(13). Suppose that (X_0, Y_0) , where $X_0 = -X_0^{\eta*}$ and $Y_0 = -Y_0^{\eta*}$, be an arbitrary solution of (3) and we show that its general solution can be expressed by (14)-(13). Observe that

$$L_{A_4} X_0 L_{A_4}^{\eta*} = (I - A_4^\dagger A_4) X_0 (I - A_4^\dagger A_4)^{\eta*} = X_0 - A_4^\dagger C_6 + (A_4^\dagger C_6)^{\eta*} - A_4^\dagger A_4 C_6^{\eta*} (A_4^\dagger)^{\eta*},$$

$$A_3 L_{A_4} X_0 L_{A_4}^{\eta*} A_3^{\eta*} = A_3 (X_0 - A_4^\dagger C_6 + (A_4^\dagger C_6)^{\eta*} - A_4^\dagger A_4 C_6^{\eta*} (A_4^\dagger)^{\eta*}) A_3^{\eta*},$$

that is,

$$AX_0A^{\eta*} = A_3X_0 - A_3A_4^\dagger C_6 A_3^{\eta*} + A_3(A_4^\dagger C_6)^{\eta*} A_3^{\eta*} - A_3A_4^\dagger A_4 C_6^{\eta*} (A_4^\dagger)^{\eta*} A_3^{\eta*} \tag{27}$$

By the same approach, we have

$$BX_0B^{\eta*} = B_3Y_0 - B_3B_4^\dagger C_7 B_3^{\eta*} + B_3(B_4^\dagger C_7)^{\eta*} B_3^{\eta*} - B_3B_4^\dagger B_4 C_7^{\eta*} (B_4^\dagger)^{\eta*} B_3^{\eta*}. \tag{28}$$

From (27) and (28), we can easily get that, $AX_0A^{\eta*} + BY_0B^{\eta*} = C$. Consequently,

$$MY_0B^{\eta*} = R_A C, \quad BY_0M^{\eta*} = C(R_A)^{\eta*}, \quad MY_0M^{\eta*} = R_A C(R_A)^{\eta*} \tag{29}$$

Put

$$U_{14} = \frac{1}{2}Y_0B^{\eta*}(B^\dagger)^{\eta*}M^{\eta*}(M^\dagger)^{\eta*}, \quad U_{12} = \frac{1}{2}[-X_0 - X_0A_1^{\eta*}(A_1^\dagger)^{\eta*}] \tag{30}$$

$$W_{12} = \frac{1}{2}[Y_0B^{\eta*}(B^\dagger)^{\eta*} - B^\dagger BY_0], \quad U_{13} = \frac{1}{2}Y_0. \tag{31}$$

With the help of (30)-(31), $P_0 - Y_0$ can be written as follows

$$\begin{aligned} P_0 - Y_0 &= \frac{1}{2}[M^\dagger C(B^\dagger)^{\eta*} + B^\dagger C(M^\dagger)^{\eta*} + M^\dagger C(B^\dagger)^{\eta*}(S^\dagger S)^{\eta*} + S^\dagger S B^\dagger C(M^\dagger)^{\eta*} \\ &+ L_M L_S Y_0 B^{\eta*}(B^\dagger)^{\eta*} M^{\eta*}(M^\dagger)^{\eta*} + L_M Y_0 B^{\eta*}(B^\dagger)^{\eta*} L_M^{\eta*} + Y_0 L_B^{\eta*} \\ &- L_B Y_0 - M^\dagger M B^\dagger B Y_0 L_S^{\eta*} L_M^{\eta*} - L_M B^\dagger B Y_0 L_M^{\eta*} - 2Y_0]. \end{aligned} \tag{32}$$

To show $P_0 - Y_0 = 0$, we examine the term present in the second line of (32). Since $MS^* = 0$ and $MY_0B^{\eta*} = R_A C$, we have

$$\begin{aligned} &L_M L_S Y_0 B^{\eta*}(B^\dagger)^{\eta*} M^{\eta*}(M^\dagger)^{\eta*} + L_M Y_0 B^{\eta*}(B^\dagger)^{\eta*} L_M^{\eta*} + Y_0 L_B^{\eta*} \\ &= L_M Y_0 B^{\eta*}(B^\dagger)^{\eta*} M^{\eta*}(M^\dagger)^{\eta*} - L_M S^\dagger S Y_0 B^{\eta*}(B^\dagger)^{\eta*} M^{\eta*}(M^\dagger)^{\eta*} + L_M Y_0 B^{\eta*}(B^\dagger)^{\eta*} L_M^{\eta*} + Y_0 L_B^{\eta*} \\ &= L_M Y_0 B^{\eta*}(B^\dagger)^{\eta*} - L_M S^\dagger S Y_0 B^{\eta*}(B^\dagger)^{\eta*} M^{\eta*}(M^\dagger)^{\eta*} + Y_0 L_B^{\eta*} \\ &= Y_0 B^{\eta*}(B^\dagger)^{\eta*} - M^\dagger M Y_0 B^{\eta*}(B^\dagger)^{\eta*} - S^\dagger S Y_0 B^{\eta*}(B^\dagger)^{\eta*} M^{\eta*}(M^\dagger)^{\eta*} + Y_0 L_B^{\eta*} \\ &= Y_0 B^{\eta*}(B^\dagger)^{\eta*} - M^\dagger R_A C(B^\dagger)^{\eta*} - S^\dagger S Y_0 M^{\eta*}(M^\dagger)^{\eta*} + Y_0 L_B^{\eta*} \\ &= Y_0 - M^\dagger C(B^\dagger)^{\eta*} - S^\dagger S Y_0 M^{\eta*}(M^\dagger)^{\eta*}. \end{aligned} \tag{33}$$

The last term of (33) can be computed by using (12) and (29) as follow

$$\begin{aligned} S^\dagger S Y_0 M^{\eta*}(M^\dagger)^{\eta*} &= S^\dagger B L_M Y_0 M^{\eta*}(M^\dagger)^{\eta*} \\ &= S^\dagger B Y_0 M^{\eta*}(M^\dagger)^{\eta*} - S^\dagger B M^\dagger M Y_0 M^{\eta*}(M^\dagger)^{\eta*} \\ &= S^\dagger C L_{A^{\eta*}}(M^\dagger)^{\eta*} - S^\dagger B M^\dagger R_A B B^\dagger C L_{A^{\eta*}}(M^\dagger)^{\eta*} \\ &= S^\dagger C(M^\dagger R_A)^{\eta*} - S^\dagger B M^\dagger M B^\dagger C R_A^{\eta*}(M^\dagger)^{\eta*} \\ &= S^\dagger C(M^\dagger)^{\eta*} - S^\dagger (B - S) B^\dagger C R_A^{\eta*}(M^\dagger)^{\eta*} \\ &= S^\dagger S B^\dagger C(M^\dagger)^{\eta*}. \end{aligned} \tag{34}$$

With the help of (32)-(34), we get that $P_0 = Y_0$. Similarly,

$$\begin{aligned} Q_0 &= A^\dagger(C - B P_0 B^{\eta*})(A^\dagger)^{\eta*} - \frac{1}{2}L_A[-X_0 - X_0A^{\eta*}(A^\dagger)^{\eta*}] + \frac{1}{2}[X_0 + A^\dagger A X_0]L_A^{\eta*} \\ &= A^\dagger A X_0 A^{\eta*}(A^\dagger)^{\eta*} - \frac{1}{2}L_A[-X_0 - X_0A^{\eta*}(A^\dagger)^{\eta*}] + \frac{1}{2}[X_0 + A^\dagger A X_0]L_A^{\eta*} \\ &= 0 \Rightarrow Q_0 = X_0. \end{aligned}$$

Hence

$$X_0 = A_4^\dagger C_6 - (A_4^\dagger C_6)^{\eta*} + A_4^\dagger (A_4 C_6^{\eta*})(A_4^\dagger)^{\eta*} + L_{A_4} X_0 (L_{A_4})^{\eta*},$$

and

$$Y_0 = B_4^\dagger C_7 - (B_4^\dagger C_7)^{\eta*} + B_4^\dagger (B_4 C_7^{\eta*})(B_4^\dagger)^{\eta*} + L_{B_4} Y_0 (L_{B_4})^{\eta*}.$$

This implies that X_0 and Y_0 can be written as (13)-(14), respectively, where

$$U_{14} = \frac{1}{2}Y_0B^{\eta^*}(B^\dagger)^{\eta^*}M^{\eta^*}(M^\dagger)^{\eta^*}, \quad U_{12} = \frac{1}{2}[-X_0 - X_0A_1^{\eta^*}(A_1^\dagger)^{\eta^*}]$$

$$W_{12} = \frac{1}{2}[Y_0B^{\eta^*}(B^\dagger)^{\eta^*} - B^\dagger BY_0], \quad U_{13} = \frac{1}{2}Y_0.$$

Thus the proof is done. \square

Remark 3.2. Due to Lemma 2.3, $L_{A_4}A^\dagger = L_{A_4}(A_3L_{A_4})^\dagger = A^\dagger$, $BM^\dagger = B_3L_{B_4}(R_A B_3L_{B_4})^\dagger = B_3M^\dagger$, and $L_{B_3}B^\dagger = B^\dagger$. Since $L_{B_4}L_M = L_ML_{B_4}$, then $L_{B_4}S^\dagger = S^\dagger$. So, we have the following simplification of the general solution (14)

$$X = -X^{\eta^*} = A_4^\dagger C_6 - (A_4^\dagger C_6)^{\eta^*} + A_4^\dagger (A_4 C_6^{\eta^*})(A_4^\dagger)^{\eta^*} + A^\dagger C(A^\dagger)^{\eta^*} - \frac{1}{2}[A^\dagger B_3 M^\dagger C(A^\dagger)^{\eta^*} + A^\dagger C(A^\dagger B_3 M^\dagger)^{\eta^*}] - \frac{1}{2}[A^\dagger S B^\dagger C(A^\dagger B_3 M^\dagger)^{\eta^*} + A^\dagger B_3 M^\dagger C(A^\dagger S B^\dagger)^{\eta^*}] - A^\dagger S W_{12}(A^\dagger S)^{\eta^*} - L_{A_4} L_A U_{12} L_{A_4}^{\eta^*} + (L_{A_4} L_A U_{12} L_{A_4}^{\eta^*})^{\eta^*}, \tag{35}$$

$$Y = -Y^{\eta^*} = B_4^\dagger C_7 - (B_4^\dagger C_7)^{\eta^*} + B_4^\dagger (B_4 C_7^{\eta^*})(B_4^\dagger)^{\eta^*} + \frac{1}{2}[B^\dagger C(M^\dagger)^{\eta^*} + M^\dagger C(B^\dagger)^{\eta^*}] + \frac{1}{2}[Q_5 B^\dagger C(M^\dagger)^{\eta^*} + M^\dagger C(Q_5 B^\dagger)^{\eta^*}] + L_M W_{12}(L_M)^{\eta^*} - L_{B_4} U_{13} L_B^{\eta^*} L_{B_4}^{\eta^*} + (L_{B_4} U_{13} L_B^{\eta^*} L_{B_4}^{\eta^*})^{\eta^*} + L_{B_4} L_M L_S U_{14} L_{B_4}^{\eta^*} - (L_{B_4} L_M L_S U_{14} L_{B_4}^{\eta^*})^{\eta^*}, \tag{36}$$

where U_{11}, \dots, U_{14} and $W_{12}^{\eta^*} = -W_{12}$ are arbitrary matrices over \mathbb{H} with allowable dimensions.

4. Particular cases of (3)

Now we discuss a particular case of (3). If A_1, A_2, B_1, B_2 and C_1, \dots, C_4 are all zero in (3) then we get the following renowned result.

Corollary 4.1. Suppose that A_3, B_3 and C_3 are given coefficient matrices in (1) over \mathbb{H} of adequate sizes. Denote $M = R_{A_3}B_3, S = B_3L_M$. Then

(1) The system (1) has a solution (X, Y) , where $X = -X^{\eta^*}$ and $Y = -Y^{\eta^*}$.

(2) The coefficient matrices in (1) satisfy:

$$R_M R_{A_3} C_5 = 0, \quad R_{A_3} C_5 R_{B_3}^{\eta^*} = 0. \tag{37}$$

(3) $MM^\dagger R_{A_3} C_5 = R_{A_3} C_5 = R_{A_3} C_5 (B_3)^{\eta^*} (B_3^\dagger)^{\eta^*}$.

$$(4) \quad r \begin{bmatrix} A_3 & C_5 \\ 0 & B_3^{\eta^*} \end{bmatrix} = r(B_3) + r(A_3), \quad r \begin{bmatrix} A_3 & B_3 & C_5 \end{bmatrix} + r \begin{bmatrix} B_3 & A_3 \end{bmatrix}$$

are equivalent statements. Under these conditions, the general solution to the system (3) can be obtained as

$$X = -X^{\eta^*} = A_3^\dagger C_5 (A_3^\dagger)^{\eta^*} - \frac{1}{2} A_3^\dagger B_3 M^\dagger C_5 [I + (B_3^\dagger)^{\eta^*} S^{\eta^*}] (A_3^\dagger)^{\eta^*} - \frac{1}{2} A_3^\dagger [I + S B_3^\dagger] C_5 (M^\dagger)^{\eta^*} B_3^{\eta^*} (A_3^\dagger)^{\eta^*} - A_3^\dagger S W_2 S^{\eta^*} (A_3^\dagger)^{\eta^*} - L_{A_3} U + U^{\eta^*} (L_B A_3)^{\eta^*},$$

$$Y = -Y^{\eta^*} = \frac{1}{2} M^\dagger C_5 (B_3^\dagger)^{\eta^*} [I + (S^\dagger S)^{\eta^*}] + \frac{1}{2} (I + S^\dagger S) B_3^\dagger C_5 (M^\dagger)^{\eta^*} + L_M W_2 (L_M)^{\eta^*} + V L_{B_3}^{\eta^*} - L_{B_3} V^{\eta^*} + L_M L_S W_1 - W_1^{\eta^*} (L_M L_S)^{\eta^*},$$

where U_1, U_2, W_1, U, V and $W_2^{\eta^*} = -W_2$ are arbitrary matrices over \mathbb{H}

In addition, using Theorem 3.1 gives the η -skew-Hermitian solution to the system

$$A_1X = C_1, XB_1 = D_1, A_3XA_3^{\eta*} = C_5. \tag{38}$$

Let A_2, B_2, B_3, C_3 and C_4 are all zero then we get the following consequence.

Corollary 4.2. Let A_1, B_1, C_1, C_2 and A_3 be known coefficient matrices in (38) over \mathbb{H} of suitable shapes. Denote

$$A_4 = \begin{bmatrix} A_1 \\ B_1^{\eta*} \end{bmatrix}, C_6 = \begin{bmatrix} C_1 \\ -C_2^{\eta*} \end{bmatrix}, A = A_3L_{A_4},$$

$$C = C_5 - A_3A_4^\dagger C_6A_3^{\eta*} + A_3(A_4^\dagger C_6)^{\eta*} A_3^{\eta*} - A_3A_4^\dagger (A_4C_6^{\eta*})(A_4^\dagger)^{\eta*} A_3^{\eta*}$$

Then

- (1) The system (38) has a solution X , where $X = -X^{\eta*}$,
- (2) $A_4C_6^{\eta*} = C_6A_4^{\eta*}, R_{A_4}C_6 = 0, R_A C = 0$,
- (3) $r \begin{bmatrix} A_1 & C_1 \\ B_1^{\eta*} & -C_2^{\eta*} \end{bmatrix} = r \begin{bmatrix} A_1 \\ B_1^{\eta*} \end{bmatrix}, r \begin{bmatrix} C_5 & A_3 \\ C_1A_3^{\eta*} & A_1 \\ C_2^{\eta*}A_3^{\eta*} & B_1^{\eta*} \end{bmatrix} = r \begin{bmatrix} A_3 \\ A_1 \\ B_1^{\eta*} \end{bmatrix}$,

the above mentioned statements are equivalent. Under these conditions, the general solution to the system (38) can be demonstrated as

$$X = -X^{\eta*} = A_4^\dagger C_6 - (A_4^\dagger C_6)^{\eta*} + A_4^\dagger (A_4C_6^{\eta*})(A_4^\dagger)^{\eta*} + AC(A^\dagger)^{\eta*} - L_{A_4}[L_A U_{12} - U_{12}^{\eta*} L_A^{\eta*}](L_{A_4})^{\eta*},$$

where U_{12} is any arbitrary matrices over \mathbb{H} .

5. Determinantal representations of solutions to system (3) and its partial cases

Let's put U_{11}, \dots, U_{14} and $W_{12}^{\eta*} = -W_{12}$ as zero matrices in Eqs. (35)-(36). Then we have the following partial solution to the system (3)

$$X = -X^{\eta*} = A_4^\dagger C_6 - (A_4^\dagger C_6)^{\eta*} + A_4^\dagger (A_4C_6^{\eta*})(A_4^\dagger)^{\eta*} + A^\dagger C(A^\dagger)^{\eta*} - \frac{1}{2}[A^\dagger B_3M^\dagger C(A^\dagger)^{\eta*} - (A^\dagger B_3M^\dagger C(A^\dagger)^{\eta*})^{\eta*}] - \frac{1}{2}[A^\dagger SB^\dagger C(A^\dagger B_3M^\dagger)^{\eta*} - (A^\dagger SB^\dagger C(A^\dagger B_3M^\dagger)^{\eta*})^{\eta*}], \tag{39}$$

$$Y = -Y^{\eta*} = B_4^\dagger C_7 - (B_4^\dagger C_7)^{\eta*} + B_4^\dagger (B_4C_7^{\eta*})(B_4^\dagger)^{\eta*} + \frac{1}{2}[B^\dagger C(M^\dagger)^{\eta*} - (B^\dagger C(M^\dagger)^{\eta*})^{\eta*}] + \frac{1}{2}[Q_S B^\dagger C(M^\dagger)^{\eta*} - (Q_S B^\dagger C(M^\dagger)^{\eta*})^{\eta*}]. \tag{40}$$

The following theorem gives determinantal representations of (39)-(40).

Theorem 5.1. Let $A_1 \in \mathbb{H}_{r_1}^{l \times n}, B_1 \in \mathbb{H}_{r_2}^{n \times h}, A_2 \in \mathbb{H}_{r_3}^{p \times k}, B_2 \in \mathbb{H}_{r_4}^{k \times l}, A_3 \in \mathbb{H}_{r_5}^{m \times n}, B_3 \in \mathbb{H}_{r_6}^{m \times k}, r(A_4) = r_7, r(B_4) = r_8, r(A) = r_9, r(B) = r_{10}, r(M) = r_{11}, r(S) = r_{12}$. Then the partial pair solution (39)-(40) to the system (3), $X = (x_{ij}) \in \mathbb{H}^{m \times n}, Y = (y_{pg}) \in \mathbb{H}^{k \times k}$, by the components

$$x_{ij} = x_{ij}^{(1)} + \eta(x_{ji}^{(1)})^* \eta + x_{ij}^{(2)} + x_{ij}^{(3)} - \frac{1}{2}(x_{ij}^{(4)} + \eta(x_{ji}^{(4)})^* \eta) - \frac{1}{2}(x_{ij}^{(5)} + \eta(x_{ji}^{(5)})^* \eta), \tag{41}$$

$$y_{pg} = y_{pg}^{(1)} + \eta(y_{gp}^{(1)})^* \eta + y_{pg}^{(2)} + \frac{1}{2}(y_{pg}^{(3)} + \eta(y_{gp}^{(3)})^* \eta) + \frac{1}{2}(y_{pg}^{(4)} + \eta(y_{gp}^{(4)})^* \eta), \tag{42}$$

possess the following determinantal representations,

(i)

$$x_{ij}^{(1)} = \frac{\sum_{\beta \in J_{7,n}\{i\}} \text{cdet}_i \left((A_4^* A_4)_{.i} (\hat{c}_{.j}^{(6)})_{\beta} \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{7,n}} |A_4^* A_4|_{\beta}^{\beta}}, \tag{43}$$

where $\hat{c}_{.j}^{(6)}$ is the j -th column of $\widehat{C}_6 := A_4^* C_6$.

(ii)

$$x_{ij}^{(2)} = \frac{-\eta \sum_{\alpha \in I_{7,n}\{j\}} \text{rdet}_j \left((A_4^* A_4)_{.j} (u_{.i}^{(1)})_{\alpha}^{\eta} \right)_{\alpha}^{\alpha} \eta}{\left(\sum_{\beta \in J_{7,n}} |A_4^* A_4|_{\beta}^{\beta} \right)^2}, \tag{44}$$

$$= \frac{\sum_{\beta \in J_{7,n}\{i\}} \text{cdet}_i \left((A_4^* A_4)_{.i} (u_{.j})_{\beta} \right)_{\beta}^{\beta}}{\left(\sum_{\beta \in J_{7,n}} |A_4^* A_4|_{\beta}^{\beta} \right)^2}, \tag{45}$$

where

$$u_{.i}^{(1)} = \left[\sum_{\beta \in J_{7,n}\{i\}} \text{cdet}_i \left((A_4^* A_4)_{.i} (\tilde{a}_{.i})_{\beta} \right)_{\beta}^{\beta} \right] \in \mathbb{H}^{1 \times n}, \quad t = 1, \dots, n, \tag{46}$$

$$u_{.j}^{(2)} = \left[-\eta \sum_{\alpha \in I_{7,n}\{j\}} \text{rdet}_j \left((A_4^* A_4)_{.j} (\tilde{a}_{.l}^{\eta})_{\alpha}^{\eta} \right)_{\alpha}^{\alpha} \eta \right] \in \mathbb{H}^{n \times 1}, \quad l = 1, \dots, n, \tag{47}$$

are the row vector and the column vector, $\tilde{a}_{.t}$ is the t -th column of $\widetilde{A}_4 = A_4^* A_4 C_6^{\eta*} A_4^{\eta}$ and $\tilde{a}_{.l}^{\eta}$ is the l -th row of \widetilde{A}_4^{η} .

(iii)

$$x_{ij}^{(3)} = \frac{-\eta \sum_{\alpha \in I_{9,n}\{j\}} \text{rdet}_j \left((A^* A)_{.j} (v_{.i}^{\eta})_{\alpha}^{\eta} \right)_{\alpha}^{\alpha} \eta}{\left(\sum_{\beta \in J_{9,n}} |A^* A|_{\beta}^{\beta} \right)^2} \tag{48}$$

$$= \frac{\sum_{\beta \in J_{9,n}\{i\}} \text{cdet}_i \left((A^* A)_{.i} (v_{.j})_{\beta} \right)_{\beta}^{\beta}}{\left(\sum_{\beta \in J_{9,n}} |A^* A|_{\beta}^{\beta} \right)^2}, \tag{49}$$

where

$$v_{.i}^{\eta} = \left[-\eta \sum_{\beta \in J_{9,n}\{i\}} \text{cdet}_i \left((A^* A)_{.i} (\hat{a}_{.s})_{\beta} \right)_{\beta}^{\beta} \eta \right] \in \mathbb{H}^{1 \times n}, \quad s = 1, \dots, n, \tag{50}$$

$$v_{.j} = \left[-\eta \sum_{\alpha \in I_{9,n}\{j\}} \text{rdet}_j \left((A^* A)_{.j} (\hat{a}_{.l}^{\eta})_{\alpha}^{\eta} \right)_{\alpha}^{\alpha} \eta \right] \in \mathbb{H}^{n \times 1}, \quad l = 1, \dots, n.$$

Here \hat{a}_s is the s -th column of $\widehat{A} = A^*CA^\eta$ and \hat{a}_l^η is the l -th row of \widehat{A}^η .

(iv)

$$x_{ij}^{(4)} = \frac{-\eta \sum_{\alpha \in I_{r_9,n}\{j\}} \text{rdet}_j((A^*A)_j(\widetilde{\phi}_i)_\alpha)^\alpha \eta}{\left(\sum_{\beta \in I_{r_9,n}} |A^*A|_\beta^\beta\right)^2 \sum_{\alpha \in I_{r_{11},m}} |MM^*|_\alpha^\alpha}, \tag{51}$$

where $\widetilde{\phi}_i$ is the i -th row of $\widetilde{\Phi} := \Phi^\eta C^* A$ and $\Phi = (\phi_{iq})$ is such that

$$\phi_{iq} = \sum_{\beta \in I_{r_9,n}\{i\}} \text{cdet}_i((A^*A)_{.i}(\varphi_{.q}^M)_\beta)^\beta = \sum_{\alpha \in I_{r_{11},m}\{q\}} \text{rdet}_q((MM^*)_q(\varphi_i^A)_\alpha)^\alpha, \tag{52}$$

and

$$\varphi_{.q}^M = \left[\sum_{\alpha \in I_{r_{11},m}\{q\}} \text{rdet}_q((MM^*)_q(\hat{b}_f)_\alpha)^\alpha \right] \in \mathbb{H}^{n \times 1}, \quad f = 1, \dots, n,$$

$$\varphi_i^A = \left[\sum_{\beta \in I_{r_9,n}\{i\}} \text{cdet}_i((A^*A)_{.i}(\hat{b}_s)_\beta)^\beta \right] \in \mathbb{H}^{1 \times m}, \quad s = 1, \dots, m.$$

Here \hat{b}_f and \hat{b}_s are the f -th row and the s -th column of $\widehat{B} = A^*BM^*$.

(v)

$$x_{ij}^{(5)} = \frac{\sum_{\beta \in I_{r_9,n}\{j\}} \text{cdet}_i((A^*A)_{.i}(\omega_j^{(1)})_\beta)^\beta}{\left(\sum_{\beta \in I_{r_9,n}} |A^*A|_\beta^\beta\right)^2 \sum_{\alpha \in I_{r_{10},m}} |BB^*|_\alpha^\alpha \sum_{\beta \in I_{r_{11},m}} |MM^*|_\beta^\beta} \tag{53}$$

$$= \frac{-\eta \sum_{\alpha \in I_{r_9,n}\{j\}} \text{rdet}_j((A^*A)_j(\psi_i^{(2)})_\alpha)^\alpha \eta}{\left(\sum_{\beta \in I_{r_9,n}} |A^*A|_\beta^\beta\right)^2 \sum_{\alpha \in I_{r_{10},m}} |BB^*|_\alpha^\alpha \sum_{\beta \in I_{r_{11},m}} |MM^*|_\beta^\beta}, \tag{54}$$

where $\omega_j^{(1)}$ is the j -th column of $\Omega_1 = \Omega\Psi_1$ and $\psi_i^{(2)}$ is the i -th row of $\Psi_2 := \Omega_2^\eta\Psi$. The matrices $\Omega = (\omega_{it}) \in \mathbb{H}^{n \times m}$, $\Psi_1 := (\psi_{tj}^{(1)}) \in \mathbb{H}^{m \times n}$, $\Psi := (\psi_{qj}) \in \mathbb{H}^{m \times n}$, $\Omega_2 = (\omega_{iq}^{(2)})$ are such that

$$\omega_{it} = \sum_{\alpha \in I_{r_{10},m}\{t\}} \text{rdet}_t((BB^*)_t(s_i^{(1)})_\alpha)^\alpha,$$

where $s_i^{(1)}$ is the i -th row of $S_1 = A^*SB^*$;

$$\psi_{tj}^{(1)} = -\eta \sum_{\alpha \in I_{r_9,n}\{j\}} \text{rdet}_j((A^*A)_j(c_t^{(11)})_\alpha)^\alpha \eta,$$

where $c_t^{(11)}$ is the t -th row of $C_{11} := C^\eta\Psi$;

$$\psi_{qj} = \sum_{\beta \in I_{r_{11},m}\{q\}} \text{cdet}_q((MM^*)_q(\hat{b}_f^{(*)})_\beta)^\beta,$$

where $\hat{b}_{.f}^{(*)}$ is f -th column of $\widehat{B}^* = MB^*A$;

$$\omega_{iq}^{(2)} = \sum_{\beta \in I_{r_9, n}(i)} \text{cdet}_i \left((A^*A)_{.i} \left(c_{.q}^{(12)} \right)_{\beta} \right)^{\beta},$$

where $c_{.q}^{(12)}$ is the q -th column of $C_{12} := \Omega C$.

(vi)

$$y_{pg}^{(1)} = \frac{\sum_{\beta \in I_{r_8, k}\{p\}} \text{cdet}_p \left((B_4^*B_4)_{.p} (\tilde{c}_{.g})_{\beta} \right)^{\beta}}{\sum_{\beta \in I_{r_8, k}} |B_4^*B_4|_{\beta}^{\beta}}, \tag{55}$$

where $\tilde{c}_{.g}$ is the g -th column of $\widetilde{C}_7 := B_4^*C_7$.

(vii)

$$y_{pg}^{(2)} = \frac{-\eta \sum_{\alpha \in I_{r_8, k}\{g\}} \text{rdet}_g \left((B_4^*B_4)_{.g} \left((w_p^{(1)})^{\eta} \right)_{\alpha} \right)^{\alpha} \eta}{\left(\sum_{\beta \in I_{r_8, k}} |B_4^*B_4|_{\beta}^{\beta} \right)^2}, \tag{56}$$

$$= \frac{\sum_{\beta \in I_{r_8, k}\{p\}} \text{cdet}_p \left((B_4^*B_4)_{.p} (w_{.g}^{(2)})_{\beta} \right)^{\beta}}{\left(\sum_{\beta \in I_{r_8, k}} |B_4^*B_4|_{\beta}^{\beta} \right)^2}, \tag{57}$$

where

$$w_p^{(1)} = \left[\sum_{\beta \in I_{r_8, k}\{p\}} \text{cdet}_p \left((B_4^*B_4)_{.p} (\tilde{b}_{.s})_{\beta} \right)^{\beta} \right] \in \mathbb{H}^{1 \times n}, s = 1, \dots, k, \tag{58}$$

$$w_{.g}^{(2)} = \left[-\eta \sum_{\alpha \in I_{r_8, k}\{g\}} \text{rdet}_g \left((B_4^*B_4)_{.g} (\tilde{b}_l^{\eta})_{\alpha} \right)^{\alpha} \eta \right] \in \mathbb{H}^{n \times 1}, l = 1, \dots, k. \tag{59}$$

Here $\tilde{b}_{.s}$ is the s -th column of $\widetilde{B}_4 = B_4^*B_4C_7^{\eta*}B_4^{\eta}$ and \tilde{b}_l^{η} is the l -th row of \widetilde{B}_4^{η} .

(viii)

$$y_{pg}^{(3)} = \frac{-\eta \sum_{\alpha \in I_{r_{11}, k}\{g\}} \text{rdet}_g \left((M^*M)_{.g} \left((\omega_p^{(1)})^{\eta} \right)_{\alpha} \right)^{\alpha} \eta}{\sum_{\beta \in I_{r_{10}, k}} |B^*B|_{\beta}^{\beta} \sum_{\alpha \in I_{r_{10}, k}} |M^*M|_{\alpha}^{\alpha}}, \tag{60}$$

$$= \frac{\sum_{\beta \in I_{r_{10}, k}\{p\}} \text{cdet}_p \left((B^*B)_{.p} (\omega_{.g}^{(2)})_{\beta} \right)^{\beta}}{\sum_{\beta \in I_{r_{10}, k}} |B^*B|_{\beta}^{\beta} \sum_{\alpha \in I_{r_{11}, k}} |M^*M|_{\alpha}^{\alpha}}, \tag{61}$$

where

$$\hat{\omega}_p^{(1)} = \left[\sum_{\beta \in I_{r_{10},k}\{p\}} \text{cdet}_p \left((B^*B)_{.p} (\hat{c}_{.l}) \right)_\beta^\beta \right] \in \mathbb{H}^{1 \times k}, \quad l = 1, \dots, k, \tag{62}$$

$$\hat{\omega}_{.g}^{(2)} = \left[-\eta \sum_{\alpha \in I_{r_{11},k}\{g\}} \text{rdet}_g \left((M^*M)_{.g} (\hat{c}_q^\eta) \right)_\alpha^\alpha \eta \right] \in \mathbb{H}^{k \times 1}, \quad q = 1, \dots, k. \tag{63}$$

Here $\hat{c}_{.l}$ is the l -th column of $\widehat{C} := B^*CM^\eta$ and \hat{c}_q^η is the q -th row of \widehat{C}^η .

(ix)

$$y_{pg}^{(4)} = \frac{\sum_{\beta \in I_{r_{12},m}\{p\}} \text{cdet}_p \left((S^*S)_{.p} (\tilde{v}_{.g}) \right)_\beta^\beta}{\sum_{\beta \in I_{r_{12},m}} |S^*S|_\beta^\beta \sum_{\beta \in I_{r_{10},k}} |B^*B|_\beta^\beta \sum_{\alpha \in I_{r_{11},k}} |M^*M|_\alpha^\alpha}, \tag{64}$$

where $\tilde{v}_{.g}$ is the g -th column of $\widetilde{Y} := S^*SY$. Here $Y = (v_{tg})$ is such that

$$v_{tg} = \sum_{\beta \in I_{r_{10},k}\{t\}} \text{cdet}_t \left((B^*B)_{.t} (\hat{\omega}_{.g}^{(2)}) \right)_\beta^\beta = -\eta \sum_{\alpha \in I_{r_{11},k}\{g\}} \text{rdet}_g \left((M^*M)_{.g} (\hat{\omega}_t^{(1)}) \right)_\alpha^\alpha \eta,$$

and $\hat{\omega}_t^{(1)}, \hat{\omega}_{.g}^{(2)}$ are determined by (62) and (63), respectively.

Proof. (i) For the first term, $X_1 = (x_{ij}^{(1)}) := A_4^\dagger C_6$, Eq. (43) follows immediately from Corollary 2.14.

(ii) For the second term, $X_2 = A_4^\dagger A_4 C_6^\eta (A_4^\dagger)^\eta$, due to Corollaries 2.10, 2.11, and 2.14, we get

$$x_{ij}^{(2)} = \frac{\sum_{s=1}^n \sum_{\beta \in I_{r_{7,n}\{i\}}} \text{cdet}_i \left((A_4^*A_4)_{.i} (\hat{a}_{.s}^{(4)}) \right)_\beta^\beta}{\sum_{\beta \in I_{r_{7,n}}} |A_4^*A_4|_\beta^\beta} \left(-\eta \frac{\sum_{\alpha \in I_{r_{7,n}\{j\}}} \text{rdet}_j \left((A_4^*A_4)_{.j} (\hat{c}_s^*) \right)_\alpha^\alpha}{\sum_{\beta \in I_{r_{7,n}}} |A_4^*A_4|_\beta^\beta} \eta \right) =$$

$$\frac{\sum_{s=1}^n \sum_{\beta \in I_{r_{7,n}\{i\}}} \text{cdet}_i \left((A_4^*A_4)_{.i} (\hat{a}_{.s}^{(4)}) \right)_\beta^\beta \left(-\eta \sum_{\alpha \in I_{r_{7,n}\{j\}}} \text{rdet}_j \left((A_4^*A_4)_{.j} (\hat{c}_s^*) \right)_\alpha^\alpha \eta \right)}{\left(\sum_{\beta \in I_{r_{7,n}}} |A_4^*A_4|_\beta^\beta \right)^2},$$

where $\hat{a}_{.s}^{(4)}$ is the s -th column of $A_4^*A_4$ and \hat{c}_s^* is the s -th row of $\widehat{C}_6^* := C_6^*A_4$.

Suppose e_s and $e_{.s}$ are respectively the unit row and column vectors whose components are 0 except the s -th components which are 1. So,

$$x_{ij}^{(3)} = \frac{\sum_q \sum_t \sum_s \sum_{\beta \in I_{r_{7,n}\{i\}}} \text{cdet}_i \left((A_4^*A_4)_{.i} (e_{.l}) \right)_\beta^\beta \hat{a}_{qs} \hat{c}_{st}^{(\eta^*)} \left(-\eta \sum_{\alpha \in I_{r_{7,n}\{j\}}} \text{rdet}_j \left((A_4^*A_4)_{.j} (e_s) \right)_\alpha^\alpha \eta \right)}{\left(\sum_{\beta \in I_{r_{7,n}}} |A_4^*A_4|_\beta^\beta \right)^2},$$

where $\hat{c}_{st}^{(\eta^*)}$ is the st -th entry of $(\widetilde{C}_6^*)^\eta := C_6^{\eta^*} A_4^\eta$. Denote $\widetilde{A}_4 = A_4^* A_4 C_6^{\eta^*} A_4^\eta =: \widetilde{A} = (\tilde{a}_{ij})$. Since $\sum_{s=1}^n \tilde{a}_{qs} \hat{c}_{st}^{(\eta^*)} = \tilde{a}_{st}$, then

$$x_{ij}^{(2)} = \frac{\sum_{s=1}^n \sum_{t=1}^n \sum_{\beta \in I_{r,n}\{i\}} \text{cdet}_i \left((A_4^* A_4)_{.i} (e_{.s}) \right)_\beta^\beta \tilde{a}_{st} \left(-\eta \sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((A_4^* A_4)_{.j} (e_{.t}) \right)_\alpha^\alpha \eta \right)}{\left(\sum_{\beta \in I_{r,n}} |A_4^* A_4|_\beta^\beta \right)^2},$$

If we denote by

$$u_{it}^{(1)} := \sum_{s=1}^n \sum_{\beta \in I_{r,n}\{i\}} \text{cdet}_i \left((A_4^* A_4)_{.i} (e_{.s}) \right)_\beta^\beta \tilde{a}_{st} = \sum_{\beta \in I_{r,n}\{i\}} \text{cdet}_i \left((A_4^* A_4)_{.i} (\tilde{a}_{.t}) \right)_\beta^\beta$$

the t th component of a row vector $u_i^{(1)} = [u_{i1}^{(1)}, \dots, u_{in}^{(1)}]$, then

$$\sum_{t=1}^n u_{it}^{(1)} \left(-\eta \sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((A_4^* A_4)_{.j} (e_{.t}) \right)_\alpha^\alpha \eta \right) = -\eta \left(\sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((A_4^* A_4)_{.j} (u_i^{(1)})^\eta \right)_\alpha^\alpha \right) \eta,$$

where $(u_i^{(1)})^\eta = [(u_{i1}^{(1)})^\eta, \dots, (u_{in}^{(1)})^\eta]$. So, the second term has the determinantal representation (44), where $u_i^{(1)}$ is determined by (46).

If we denote by

$$u_{sj}^{(2)} := \sum_{t=1}^n \tilde{a}_{st} \left(-\eta \sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((A_4^* A_4)_{.j} (e_{.t}) \right)_\alpha^\alpha \eta \right) = -\eta \sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((A_4^* A_4)_{.j} (\tilde{a}_{.s}^\eta) \right)_\alpha^\alpha \eta$$

the s th component of a column vector $u_j^{(2)} = [u_{1j}^{(2)}, \dots, u_{nj}^{(2)}]$, then

$$\sum_{s=1}^n \sum_{\beta \in I_{r,n}\{i\}} \text{cdet}_i \left((A_4^* A_4)_{.i} (e_{.s}) \right)_\beta^\beta u_{sj}^{(2)} = \sum_{\beta \in I_{r,n}\{i\}} \text{cdet}_i \left((A_4^* A_4)_{.i} (u_j^{(2)}) \right)_\beta^\beta.$$

So, another determinantal representation of the second term is (44) with $u_j^{(2)}$ determined by (47).

(iii) For the third term $X_3 = A^\dagger C(A^{\eta^*})^\dagger$, it's evident that Eqs. (48)-(49) follow from Lemma 2.13.

(iv) Consider the fourth term $A^\dagger B_3 M^\dagger C(A^\dagger)^{\eta^*} := X_4 = (x_{ij}^{(4)})$ of (39). Taking into account (8) for the determinantal representation of $(A^{\eta^*})^\dagger = (a_{tj}^{\eta^*,\dagger})$, we have for the multiplier $C(A^{\eta^*})^\dagger$

$$\sum_{t=1}^m c_{qt} a_{tj}^{\eta^*,\dagger} = \sum_{t=1}^m c_{qt} \cdot \left(-\eta \frac{\sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((A^* A)_{.j} (a_{.t}) \right)_\alpha^\alpha}{\sum_{\beta \in I_{r,n}} |A^* A|_\beta^\beta} \eta \right) = \frac{-\eta \sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((A^* A)_{.j} (\tilde{c}_{q.}) \right)_\alpha^\alpha \eta}{\sum_{\beta \in I_{r,n}} |A^* A|_\beta^\beta}$$

where $\tilde{c}_{q.}$ is the q -th row of $\widetilde{C} := C^\eta A$. By applying the determinantal representations (4) and (5) for the Moore-Penrose inverses A^\dagger and M^\dagger , respectively, and due to Lemma 2.13 for the first multiplier $A^\dagger B M^\dagger$, we obtain the matrix $\Phi = (\phi_{iq})$ such that

$$\phi_{iq} = \sum_{\beta \in I_{r,n}\{i\}} \text{cdet}_i \left((A^* A)_{.i} (\varphi_{.q}^M) \right)_\beta^\beta = \sum_{\alpha \in I_{r,11,m}\{q\}} \text{rdet}_q \left((M M^*)_{.q} (\varphi_{.i}^A) \right)_\alpha^\alpha,$$

and

$$\varphi_{.q}^M = \left[\sum_{\alpha \in I_{r_1,m}\{q\}} \text{rdet}_q \left((MM^*)_{.q} (\widehat{b}_{.f})_{\alpha} \right)^{\alpha} \right] \in \mathbb{H}^{n \times 1}, \quad f = 1, \dots, n,$$

$$\varphi_{.i}^A = \left[\sum_{\beta \in I_{r_1,n}\{i\}} \text{cdet}_i \left((A^*A)_{.i} (\widehat{b}_{.s})_{\beta} \right)^{\beta} \right] \in \mathbb{H}^{1 \times m}, \quad s = 1, \dots, m.$$

Here $\widehat{b}_{.f}$ and $\widehat{b}_{.s}$ are the f -th row and the s -th column of $\widehat{B} = A^*BM^*$. So, we have

$$x_{ij}^{(4)} = \frac{\sum_{q=1}^m \phi_{iq} \left(-\eta \sum_{\alpha \in I_{r_9,n}\{j\}} \text{rdet}_j \left((A^*A)_{.j} (\widetilde{c}_{q.})_{\alpha} \right)^{\alpha} \eta \right)}{\left(\sum_{\beta \in I_{r_9,n}} |A^*A|_{\beta}^{\beta} \right)^2 \sum_{\alpha \in I_{r_{11},m}} |MM^*|_{\alpha}^{\alpha}}. \tag{65}$$

Denote $\widetilde{\Phi} := \Phi^{\eta} C^* A$. From this denotation and Eq. (65), it follows (51).

(v) For the fifth term $(A^{\dagger}SB^{\dagger})C((M^{\dagger})^{\eta}B^{\eta*}(A^{\dagger})^{\eta*}) := X_5 = (x_{ij}^{(5)})$ of (39), we have

$$x_{ij}^{(5)} = \frac{\sum_{q=1}^m \sum_{t=1}^m \widetilde{\omega}_{it} c_{tq} \widetilde{\psi}_{qj}}{\left(\sum_{\beta \in I_{r_9,n}} |A^*A|_{\beta}^{\beta} \right)^2 \sum_{\alpha \in I_{r_2,m}} |BB^*|_{\alpha}^{\alpha} \sum_{\beta \in I_{r_{10},m}} |MM^*|_{\beta}^{\beta}}, \tag{66}$$

where

$$\widetilde{\psi}_{qj} = -\eta \phi_{jq}^* \eta = -\eta \sum_{\alpha \in I_{r_9,n}\{j\}} \text{rdet}_j \left((A^*A)_{.j} (\psi_q)_{\alpha} \right)^{\alpha} \eta,$$

and

$$\psi_q = \left[\sum_{\beta \in I_{r_{11},m}\{q\}} \text{cdet}_q \left((MM^*)_{.q} (\widehat{b}_{.f}^{(*)})_{\beta} \right)^{\beta} \right] \in \mathbb{H}^{1 \times n}, \quad f = 1, \dots, n, \tag{67}$$

and $\widehat{b}_{.f}^{(*)}$ is f -th column of $\widehat{B}^* = MB^*A$;

$$\widetilde{\omega}_{it} = \sum_{\beta \in I_{r_1,n}\{i\}} \text{cdet}_i \left((A^*A)_{.i} (\omega_{.t})_{\beta} \right)^{\beta},$$

$$\omega_{.t} = \left[\sum_{\alpha \in I_{r_{10},m}\{t\}} \text{rdet}_t \left((BB^*)_{.t} (s_f^{(1)})_{\alpha} \right)^{\alpha} \right] \in \mathbb{H}^{n \times 1}, \quad f = 1, \dots, n, \tag{68}$$

where $s_f^{(1)}$ is the f -th row of $S_1 = A^*SB^*$. Construct the matrices $\Psi = (\psi_{qf}) \in \mathbb{H}^{m \times n}$ and $\Omega = (\omega_{ft}) \in \mathbb{H}^{n \times m}$ determined by (67) and (68), respectively. Denote $C_{11} := C^{\eta} \Psi$, $\Psi_1 := (\psi_{ij}^{(1)})$, where

$$\psi_{ij}^{(1)} = \sum_{q=1}^m c_{tq} \widetilde{\psi}_{qj} = -\eta \sum_{\alpha \in I_{r_9,n}\{j\}} \text{rdet}_j \left((A^*A)_{.j} (c_t^{(1)})_{\alpha} \right)^{\alpha} \eta,$$

$c_t^{(1)}$ is the t -th row of C_{11} , and $\Omega_1 := \Omega \Psi_1$. From these denotations and Eq. (66), it follows (53).

Another determinantal representation of $x_{ij}^{(5)}$ can be obtained by putting $C_{12} := \Omega C$, $\Omega_2 := (\omega_{ij}^{(2)})$, where

$$\omega_{iq}^{(2)} = \sum_{t=1}^m \tilde{\omega}_{it} c_{tq} = \sum_{\beta \in J_{r_1, n} \setminus \{i\}} \text{cdet}_i \left((A^* A)_{.i} \left(c_{.q}^{(12)} \right) \right)_{\beta}^{\beta}$$

$c_{.q}^{(12)}$ is the q -th column of C_{12} , and $\Psi_2 := \Omega_2^{\eta} \Psi$. From these denotations and Eq. (66), it follows (54).

(vi)-(vii) It's evident that the proofs of the determinantal representations (55) and (56)-(57) for the first term, $Y_1 := B_4^{\dagger} C_7$. and the second term, $Y_2 := B_4^{\dagger} (B_4 C_7^{\eta*}) (B_4^{\dagger})^{\eta*}$, of (40) are similar to the proofs of (43) and (44)-(45), respectively.

(viii) Now consider the third item $B^{\dagger} C (M^{\dagger})^{\eta*} := Y_3 = (y_{pg}^{(3)})$ of (40). Using the determinantal representations (4) for B^{\dagger} , and (8) for $(M^{\dagger})^{\eta*}$, we have

$$y_{pg}^{(3)} = \sum_{t=1}^m \sum_{l=1}^m b_{pt}^{\dagger} c_{tl} \left(m_{lg}^{\eta*} \right)^{\dagger} = \frac{\sum_t \sum_l \sum_{\beta \in J_{r_{10}, k} \setminus \{p\}} \text{cdet}_p \left((B^* B)_{.p} \left(b_{.t}^* \right) \right)_{\beta}^{\beta} c_{tl} \left(-\eta \sum_{\alpha \in I_{r_{11}, k} \setminus \{g\}} \text{rdet}_g \left((M^* M)_{.j} \left(m_{l.} \right) \right)_{\alpha}^{\alpha} \eta \right)}{\sum_{\beta \in J_{r_{10}, k}} |B^* B|_{\beta}^{\beta} \sum_{\beta \in I_{r_{11}, k}} |M^* M|_{\alpha}^{\alpha}},$$

Denote $B^* C M^{\eta} =: \widehat{C} = (\widehat{c}_{ij})$. Then, thinking as in the point (ii), we have

$$y_{pg}^{(3)} = \frac{\sum_t \sum_{\beta \in J_{r_{10}, k} \setminus \{p\}} \sum_{\alpha \in I_{r_{11}, k} \setminus \{g\}} \text{cdet}_p \left((B^* B)_{.p} \left(e_{.t} \right) \right)_{\beta}^{\beta} \widehat{c}_{tl} \left(-\eta \sum_{\alpha \in I_{r_{11}, k} \setminus \{g\}} \text{rdet}_g \left((M^* M)_{.g} \left(e_{l.} \right) \right)_{\alpha}^{\alpha} \eta \right)}{\sum_{\beta \in J_{r_{10}, k}} |B^* B|_{\beta}^{\beta} \sum_{\beta \in I_{r_{11}, k}} |M^* M|_{\alpha}^{\alpha}},$$

where $e_{.t}$ and $e_{.l}$ are respectively the unit row and column vectors.

If we denote by

$$\omega_{pl}^{(1)} := \sum_{t=1}^m \sum_{\beta \in J_{r_{10}, k} \setminus \{p\}} \text{cdet}_p \left((B^* B)_{.p} \left(e_{.t} \right) \right)_{\beta}^{\beta} \widehat{c}_{tl} = \sum_{\beta \in J_{r_{10}, k} \setminus \{i\}} \text{cdet}_p \left((B^* B)_{.p} \left(\widehat{C} \cdot i \right) \right)_{\beta}^{\beta}$$

the l -th component of the row-vector $\omega_p^{(1)} = [\omega_{p1}^{(1)}, \dots, \omega_{pm}^{(1)}]$, then

$$\sum_{l=1}^m \omega_{pl}^{(1)} \left(-\eta \sum_{\alpha \in I_{r_{11}, k} \setminus \{g\}} \text{rdet}_g \left((M^* M)_{.g} \left(e_{l.} \right) \right)_{\alpha}^{\alpha} \eta \right) = -\eta \left(\sum_{\alpha \in I_{r_{11}, k} \setminus \{g\}} \text{rdet}_g \left((M^* M)_{.g} \left((\omega_p^{(1)})^{\eta} \right) \right)_{\alpha}^{\alpha} \right) \eta,$$

where $(\omega_p^{(1)})^{\eta} = [(\omega_{p1}^{(1)})^{\eta}, \dots, (\omega_{pm}^{(1)})^{\eta}]$. So, $y_{pg}^{(3)}$ has the determinantal representation (60), where $\omega_p^{(1)}$ is (62).

If we denote by

$$\omega_{tg}^{(2)} := \sum_{l=1}^m \widehat{c}_{tl} \left(-\eta \sum_{\alpha \in I_{r_{11}, k} \setminus \{g\}} \text{rdet}_g \left((M^* M)_{.g} \left(e_{l.} \right) \right)_{\alpha}^{\alpha} \eta \right) = -\eta \sum_{\alpha \in I_{r_{11}, k} \setminus \{g\}} \text{rdet}_g \left((M^* M)_{.g} \left(\widehat{C} \cdot l \right) \right)_{\alpha}^{\alpha} \eta$$

the t -th component of the column-vector $\omega_{.g}^{(2)} = [\omega_{1g}^{(2)}, \dots, \omega_{mg}^{(2)}]$, then

$$\sum_{t=1}^m \sum_{\beta \in J_{r_{10}, k} \setminus \{p\}} \text{cdet}_p \left((B^* B)_{.p} \left(e_{.t} \right) \right)_{\beta}^{\beta} \omega_{tg}^{(2)} = \sum_{\beta \in J_{r_{10}, k} \setminus \{p\}} \text{cdet}_p \left((B^* B)_{.p} \left(\omega_{.g}^{(2)} \right) \right)_{\beta}^{\beta}.$$

So, another determinantal representation of $y_{pg}^{(3)}$ is (61) with $\omega_{.g}^{(2)}$ determined by (63).

(ix) For the fourth term $Q_S B^\dagger C(M^\dagger)^{\eta^*} = Y_4 = (y_{pg}^{(4)})$ of (40) using (6) for a determinantal representation of Q_S , and similarly as in the point (viii), we have

$$y_{pg}^{(4)} = \frac{\sum_{t=1}^k \sum_{\beta \in J_{r_{12},k}\{p\}} \text{cdet}_p \left((S^*S)_{.p} (\dot{s}_{.t}) \right)_\beta^\beta v_{tg}}{\sum_{\beta \in J_{r_{12},k}} |S^*S|_\beta^\beta \sum_{\beta \in J_{r_{10},k}} |B^*B|_\beta^\beta \sum_{\alpha \in I_{r_{11},k}} |M^*M|_\alpha^\alpha}, \tag{69}$$

where $\dot{s}_{.t}$ is the t -th column of S^*S , and

$$v_{tg} = \sum_{\beta \in J_{r_{10},k}\{t\}} \text{cdet}_t \left((B^*B)_{.t} (\omega_{.g}^{(2)}) \right)_\beta^\beta = -\eta \sum_{\alpha \in I_{r_{11},k}\{g\}} \text{rdet}_g \left((M^*M)_{.g} (\omega_{.t}^{(1)})^\eta \right)_\alpha^\alpha,$$

and $\omega_{.t}^{(1)}, \omega_{.g}^{(2)}$ are determined by (62) and (63), respectively. Construct the matrix $Y = (v_{tg})$ and denote the matrix $\tilde{Y} := S^*S Y$. Using of these denotations in (69) yields to (64).

Since for an arbitrary matrix $D = (d_{ij})$ over \mathbb{H} , $D^{\eta^*} = (d_{ij})^{\eta^*} = (d_{ji}^*)^\eta = (-\eta d_{ji}^* \eta)$, then from this it follows (41)-(42). \square

Now we give determinantal representations for particular cases of (3).

Corollary 5.2. *Suppose that in conditions of Corollary 4.1, we put arbitrary matrices U, V, U_1, U_2, W_1 and W_2 as zeros. Then the generalized Sylvester matrix equation (1) has partial solution*

$$X = -X^{\eta^*} = A_3^\dagger C_5 (A_3^\dagger)^{\eta^*} - \frac{1}{2} [A_3^\dagger B_3 M^\dagger C_5 (A_3^\dagger)^{\eta^*} - (A_3^\dagger B_3 M^\dagger C_5 (A_3^\dagger)^{\eta^*})^{\eta^*}] - \frac{1}{2} [A_3^\dagger S B_3^\dagger C_5 (A_3^\dagger B_3 M^\dagger)^{\eta^*} - (A_3^\dagger S B_3^\dagger C_5 (A_3^\dagger B_3 M^\dagger)^{\eta^*})^{\eta^*}], \tag{70}$$

$$Y = -Y^{\eta^*} = \frac{1}{2} [M^\dagger C_5 (B_3^\dagger)^{\eta^*} - (M^\dagger C_5 (B_3^\dagger)^{\eta^*})^{\eta^*}] + \frac{1}{2} [Q_S B_3^\dagger C_5 (M^\dagger)^{\eta^*} - (Q_S B_3^\dagger C_5 (M^\dagger)^{\eta^*})^{\eta^*}] \tag{71}$$

which can be expressed componentwise by

$$x_{ij} = x_{ij}^{(3)} - \frac{1}{2} (x_{ij}^{(4)} + \eta (x_{ji}^{(4)})^* \eta) - \frac{1}{2} (x_{ij}^{(5)} + \eta (x_{ji}^{(5)})^* \eta),$$

$$y_{pg} = \frac{1}{2} (y_{pg}^{(3)} + \eta (y_{gp}^{(3)})^* \eta) + \frac{1}{2} (y_{pg}^{(4)} + \eta (y_{gp}^{(4)})^* \eta),$$

where $x_{ij}^{(3)}$ is (48) or (49), $x_{ij}^{(4)}$ is (51), $x_{ij}^{(5)}$ is (53) or (54), $y_{pg}^{(3)}$ is (60) or (61), and $y_{pg}^{(4)}$ is (64). Taking into account that $A = A_3, B = B_3$, and $C = C_5$.

Corollary 5.3. *Suppose that in conditions of Corollary 4.2 an arbitrary matrix U_{12} is zero. Then the system (38) has partial solution*

$$X = -X^{\eta^*} = A_4^\dagger C_6 - (A_4^\dagger C_6)^{\eta^*} + A_4^\dagger (A_4 C_6^{\eta^*}) (A_4^\dagger)^{\eta^*} + A C (A^\dagger)^{\eta^*},$$

which by the components can be expressed as

$$x_{ij} = x_{ij}^{(1)} + \eta x_{ji}^{(1)*} \eta + x_{ij}^{(2)} + x_{ij}^{(3)},$$

where $x_{ij}^{(1)}$ is (43), $x_{ij}^{(2)}$ is (44) or (45), and $x_{ij}^{(3)}$ is (48) or (49).

6. An example

Given the matrices:

$$A = \begin{bmatrix} \mathbf{i} & \mathbf{k} \\ \mathbf{j} & -1 \end{bmatrix}, B = \begin{bmatrix} 2\mathbf{i} & -\mathbf{j} \\ \mathbf{j} & \mathbf{i} \end{bmatrix}, C = \begin{bmatrix} \mathbf{i} & -\mathbf{j} + \mathbf{k} \\ \mathbf{j} - \mathbf{k} & \mathbf{i} \end{bmatrix}. \tag{72}$$

Since $C = -C^{\eta*}$ with $\eta = \mathbf{i}$, we shall find the \mathbf{i} -skew-Hermitian solution to Eq. (1) with the given matrices (72). By Theorem 2.8, one can find,

$$A^{\dagger} = \frac{1}{4} \begin{bmatrix} -\mathbf{i} & -\mathbf{j} \\ -\mathbf{k} & -1 \end{bmatrix}, R_A = \frac{1}{2} \begin{bmatrix} 1 & \mathbf{k} \\ -\mathbf{k} & 1 \end{bmatrix}, B^{\dagger} = \begin{bmatrix} -2\mathbf{i} & -\mathbf{j} \\ \mathbf{j} & -\mathbf{i} \end{bmatrix}, R_B = 0,$$

$$M = \frac{1}{2} \begin{bmatrix} \mathbf{i} & 0 \\ -\mathbf{j} & 0 \end{bmatrix}, M^{\dagger} = \begin{bmatrix} -\mathbf{i} & \mathbf{j} \\ 0 & 0 \end{bmatrix}, R_M = \frac{1}{2} \begin{bmatrix} 1 & -\mathbf{k} \\ \mathbf{k} & 1 \end{bmatrix}, L_M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to check that the consistency conditions (37) of Eq. (1) are fulfilled by given matrices. So, Eq. (1) has the η -Hermitian solution. We compute the partial solution (70)-(71) by Cramer’s rule from Corollary 5.2. So,

$$A^* = \begin{bmatrix} -\mathbf{i} & -\mathbf{j} \\ -\mathbf{k} & -1 \end{bmatrix}, A^{\eta} = \begin{bmatrix} \mathbf{i} & -\mathbf{k} \\ -\mathbf{j} & -1 \end{bmatrix}, A^*A = \begin{bmatrix} 2 & 2\mathbf{j} \\ -2\mathbf{j} & 2 \end{bmatrix}, \widehat{A} = A^*CA^{\eta} = 4 \begin{bmatrix} \mathbf{i} & -\mathbf{k} \\ \mathbf{k} & \mathbf{i} \end{bmatrix}.$$

Since $r(A^*A) = 1$, then by (50),

$$v_{1.}^{\eta} = [4\mathbf{i} \quad 4\mathbf{k}], v_{2.}^{\eta} = [-4\mathbf{k} \quad 4\mathbf{i}].$$

Further, by (48), we obtain

$$x_{11}^{(1)} = \frac{-\mathbf{i}(4\mathbf{i})\mathbf{i}}{16} = 0.25\mathbf{i}, x_{12}^{(1)} = \frac{-\mathbf{i}(4\mathbf{k})\mathbf{i}}{16} = -0.25\mathbf{k}, x_{21}^{(1)} = \frac{-\mathbf{i}(-4\mathbf{k})\mathbf{i}}{16} = 0.25\mathbf{k}, x_{22}^{(1)} = \frac{-\mathbf{i}(4\mathbf{i})\mathbf{i}}{16} = 0.25\mathbf{i}.$$

Now, we find $x_{ij}^{(2)}$ by (51). Since

$$\widehat{B} = A^*BM^* = 1.5 \begin{bmatrix} -\mathbf{i} & \mathbf{j} \\ -\mathbf{k} & 1 \end{bmatrix}, \tag{73}$$

then

$$\varphi_{1.}^A = [-1.5\mathbf{i}, 1.5\mathbf{j}], \varphi_{2.}^A = [-1.5\mathbf{k}, 1.5].$$

Taking into account (52), we get $\Phi = \begin{bmatrix} -1.5\mathbf{i} & 1.5\mathbf{j} \\ -1.5\mathbf{k} & 1.5 \end{bmatrix}$ and $\widetilde{\Phi} = \Phi^{\eta}C^*A = \begin{bmatrix} -3 & -3\mathbf{j} \\ -3\mathbf{j} & 3 \end{bmatrix}$. Finally, we have

$$x_{11}^{(2)} = \frac{-\mathbf{i}(-3)\mathbf{i}}{32} = -\frac{3}{32}, x_{12}^{(2)} = \frac{-\mathbf{i}(-3\mathbf{j})\mathbf{i}}{32} = \frac{3}{32}\mathbf{j}, x_{21}^{(2)} = \frac{-\mathbf{i}(-3\mathbf{j})\mathbf{i}}{32} = \frac{3}{32}\mathbf{j}, x_{22}^{(2)} = \frac{-\mathbf{i}(3)\mathbf{i}}{32} = \frac{3}{32}.$$

Now, we find X_3 by (53). The following algorithm of finding X_3 is given by Theorem 5.1.

1. Find the matrix $S_1 = A^*SB^* = 2 \begin{bmatrix} -\mathbf{i} & -\mathbf{j} \\ -\mathbf{k} & -1 \end{bmatrix}$.
2. Find the matrix $BB^* = \begin{bmatrix} 5 & -3\mathbf{k} \\ 3\mathbf{k} & 2 \end{bmatrix}$.

3. By (68), construct the matrix the matrix Ω . Since $r(BB^*) = 2$, then

$$\omega_{11} = \text{rdet}_1 \left((B_1 B_1^*)_{1.} (s_1^{(1)}) \right) = \text{rdet}_1 \begin{bmatrix} -2i & -2j \\ 3k & 2 \end{bmatrix} = 2i.$$

By similar computing, we get

$$\Omega = \begin{bmatrix} 2i & -4j \\ 2k & -4 \end{bmatrix}.$$

4. Obtain the matrix $C_{12} = \Omega C$.

$$C_{12} = \begin{bmatrix} 2 + 4k & -2j + 2k \\ -2j + 4k & -2 - 2i \end{bmatrix}.$$

5. Construct the matrix Ω_2 . Since $r(A^*A) = 1$, then $\Omega_2 = C_{12}$.

6. Construct the matrix the matrix Ψ . Since $r(MM^*) = 1$, then $\Psi = \widehat{B}^*$, where \widehat{B} is obtained in (73).

7. Get the matrix $\Psi_2 = \Omega_2^\eta \Psi$.

$$\Psi_2 = \begin{bmatrix} 3 - 6j & 6 + 3j \\ -3j & 3 \end{bmatrix}.$$

8. Finally, by (54) and due to $r(A^*A) = 1$, we have $X_3 = \frac{1}{8} \Psi_2^\eta$. So,

$$X_3 = \frac{1}{8} \begin{bmatrix} 3 + 6j & 6 - 3j \\ 3j & 3 \end{bmatrix}.$$

Hence,

$$X = X_1 - \frac{1}{2} (X_2 - X_2^{\eta*}) - \frac{1}{2} (X_3 - X_3^{\eta*}) = \frac{1}{4} \begin{bmatrix} i & -k \\ k & i \end{bmatrix} - \frac{1}{8} \begin{bmatrix} 0 & 3 - 3j \\ 3j - 3 & 0 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 2i & -3 + 3j - 2k \\ 3 - 3j + 2k & 2i \end{bmatrix}.$$

Further, we find y_{pg} by (36) for all $p, g = 1, 2$. Since

$$\hat{C} = B^* C M^\eta = \begin{bmatrix} -1.5 & 0 \\ k & 0 \end{bmatrix}, \quad M^* M = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad B^* B = \begin{bmatrix} 5 & 3k \\ -3k & 2 \end{bmatrix}$$

and $\omega_1^{(1)} = [0, 0]$, $\omega_2^{(1)} = [0.5k, 0]$, then by (60) we have

$$y_{11}^{(1)} = y_{12}^{(1)} = y_{22}^{(1)} = 0, \quad y_{21}^{(1)} = \frac{0.5k}{0.5} = k.$$

Further, we find Y_2 by (64). Since, as it is obtained before, $\Upsilon = \frac{1}{2} Y = \begin{bmatrix} 0 & 0 \\ 0.5k & 0 \end{bmatrix}$, and

$$S = B L_M = \begin{bmatrix} 0 & -j \\ 0 & i \end{bmatrix}, \quad \tilde{\Upsilon} = S^* S \Upsilon = \begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix},$$

then by (64), $Y_2 = \begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix}$. Hence,

$$Y = \frac{1}{2} (Y_1 - Y_1^{\eta*}) + \frac{1}{2} (Y_2 - Y_2^{\eta*}) = \frac{1}{2} \left(\begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix} - \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} \right) + \frac{1}{2} \left(\begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix} - \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}$$

So,

$$X = \frac{1}{8} \begin{bmatrix} 2i & -3 + 3j - 2k \\ 3 - 3j + 2k & 2i \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}$$

is the partial i -skew-Hermitian solution to Eq.(1) with the given matrices (72).

7. Conclusion

Some necessary and sufficient conditions for the existence of the η -skew-Hermitian solution to the quaternion system (3) are constructed in this paper. The expression of the general η -skew-Hermitian solution to this system is presented when the system is consistent. Some particular cases of (3) are also discussed in this paper. Within the framework of the theory of noncommutative row-column determinants, we give determinantal representation formulas of finding its exact solution that are analogs of Cramer's rule. A numerical example is also given to demonstrate the main results.

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