Filomat 34:8 (2020), 2643–2653
https://doi.org/10.2298/FIL2008643W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Cubic ρ-Functional Equation in Matrix Non-Archimedean Random Normed Spaces

Zhihua Wang[a], Chaozhu Hu[a]

[a] *School of Science, Hubei University of Technology, Wuhan, Hubei 430068, P.R. China*

Abstract. Using the direct method and fixed point method, we investigate the Hyers-Ulam stability of the following cubic ρ-functional equation

$$\begin{aligned} &f(x+2y)+f(x-2y)-2f(x+y)-2f(x-y)-12f(x)\\ &\quad=\rho\Big(4f(x+\frac{y}{2})+4f(x-\frac{y}{2})-f(x+y)-f(x-y)-6f(x)\Big) \end{aligned}$$

in matrix non-Archimedean random normed spaces, where ρ is a fixed real number with $\rho \neq 2$.

1. Introduction

The concept of stability for a functional equation arising when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1940, Ulam [25] posed the first stability problem concerning group homomorphisms. In 1941, Hyers [7] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Subsequently, Hyers' result was generalized by Aoki [1] for additive mappings and by Rassias [20] for linear mappings by considering an unbounded Cauchy difference. Găvruţă [6] obtained generalized Rassias' result which allows the Cauchy difference to be controlled by a general unbounded function in the spirit of Rassias' approach. The stability problems of several functional equations have been extensively investigated by several mathematicians (see [8, 9, 11, 21, 22] and references therein); as well as various stability results of functional equations and inequalities were investigated [12, 13, 17–19] in matrix normed spaces, matrix paranormed spaces and matrix fuzzy normed spaces.

In 2016, Park [15] considered the functional equation

$$\begin{aligned} &f(x+2y)+f(x-2y)-2f(x+y)-2f(x-y)-12f(x)\\ &\quad=\rho\Big(4f(x+\frac{y}{2})+4f(x-\frac{y}{2})-f(x+y)-f(x-y)-6f(x)\Big) \end{aligned} \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 39B82; Secondary 39B72, 47H10, 46L07

Keywords. Cubic ρ-functional equation; fixed point method; Hyers-Ulam stability; matrix non-Archimedean random normed spaces; non-Archimedean random normed spaces.

Received: 19 August 2019; Accepted: 19 April 2020

Communicated by Dragan S. Djordjević

Corresponding author: Zhihua Wang

This research work is supported by the National Natural Science Foundation of China (Grant Nos. 11401190 and 11701161).

Email addresses: matwzh2000@126.com (Zhihua Wang), huchaozhu@126.com (Chaozhu Hu)

where ρ is a fixed real number with $\rho \neq 2$. And he established the general solution and proved the generalized Hyers-Ulam stability of the functional equation (1) in fuzzy Banach spaces by using the fixed point method. The main purpose of this paper is to apply the direct method and fixed point method to investigate the Hyers-Ulam stability of functional equation (1) in matrix non-Archimedean random normed spaces.

2. Preliminaries

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 2.1. (cf. [23]). *A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:*
(1) *T is commutative and associative;*
(2) *T is continuous;*
(3) *$T(a,1) = a$ for all $a \in [0,1]$;*
(4) *$T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.*

Typical examples of continuous t-norms are the Lukasiewicz t-norm T_L, where $T_L(a,b) = \max(a+b-1,0), \forall a,b \in [0,1]$ and the t-norms T_P, T_M, T_D, where $T_P(a,b) := ab$, $T_M(a,b) := \min(a,b)$,

$$T_D(a,b) := \begin{cases} \min(a,b), & \text{if } \max(a,b) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r+s| \leq \max\{|r|,|s|\}$ for $r,s \in \mathbb{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a non-Archimedean norm if it satisfies the following conditions:
(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) For any $r \in \mathbb{K}$ and $x \in X$, $\|rx\| = |r|\|x\|$;
(iii) For all $x,y \in X$, $\|x+y\| \leq \max\{\|x\|,\|y\|\}$ (the strong triangle inequality).
Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space.

We adopt the usual terminology, notions and conventions of the theory of non-Archimedean random normed space as in [3, 14, 16, 23, 24]. Throughout this paper, Δ^+ is the space of all probability distribution functions, i.e., the space of all mappings $F : \mathbb{R} \cup \{-\infty,\infty\} \to [0,1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-f(x)$ denotes the left limit of the function f at the point x, that is, $l^-f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordered of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$.

Definition 2.2. (cf. [10, 24]). *A non-Archimedean random normed space (briefly, NA-RN-space) is a triple (X,μ,T), where X is a linear space over a non-Archimedean field $\mathbb{K}$, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:*
(NA-RN1) *$\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;*
(NA-RN2) *$\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $t > 0$, and $\alpha \neq 0$;*
(NA-RN3) *$\mu_{x+y}(\max(t,s)) \geq T(\mu_x(t),\mu_y(s))$ for all $x,y \in X$ and $t,s \geq 0$;*
It is easy to see that if (NA-RN3) *holds, then*
(RN3) *$\mu_{x+y}(t+s) \geq T(\mu_x(t),\mu_y(s))$.*

Definition 2.3. (cf. [10]) *Let (X,μ,T) be an NA-RN-space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that*

$$\lim_{n\to\infty} \mu_{x_n-x}(t) = 1$$

for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$.

A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and $t > 0$, there exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $\mu_{x_{n+p}-x_n}(t) > 1 - \varepsilon$. If each Cauchy sequence is convergent, then the random norm is said to be complete, and the NA-RN-space is called a non-Archimedean random Banach space.

We will also use the following notations. The set of all $m \times n$-matrices in X will be denoted by $M_{m,n}(X)$. When $m = n$, the matrix $M_{m,n}(X)$ will be written as $M_n(X)$. The symbols $e_j \in M_{1,n}(\mathbb{C})$ will denote the row vector whose jth component is 1 and the other components are 0. Similarly, $E_{ij} \in M_n(\mathbb{C})$ will denote the $n \times n$ matrix whose (i, j)-component is 1 and the other components are 0. The $n \times n$ matrix whose (i, j)-component is x and the other components are 0 will be denoted by $E_{ij} \otimes x \in M_n(X)$.

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

We introduce the concept of a matrix non-Archimedean random normed space.

Definition 2.4. (cf. [5]) *Let (X, μ, T) be an NA-RN-space. Then:*
(1) *$(X, \{\mu^{(n)}\}, T)$ is called a matrix non-Archimedean random normed space if for each positive integer n, $(M_n(X), \mu^{(n)}, T)$ is a non-Archimedean random normed space and $\mu^{(k)}_{AxB}(t) \geq \mu^{(n)}_x(\frac{t}{\|A\|\cdot\|B\|})$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \neq 0$.*
(2) *$(X, \{\mu^{(n)}\}, T)$ is called a matrix non-Archimedean random Banach space if (X, μ, T) is a non-Archimedean random Banach space and $(X, \{\mu^{(n)}\}, T)$ is a matrix non-Archimedean random normed space.*

Example 2.5. *Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space and $\alpha, \beta > 0$. Define*

$$\mu^{(n)}_x(t) = \frac{\alpha t}{\alpha t + \beta\|x\|_n}, \qquad t > 0, x = [x_{ij}] \in M_n(X).$$

Since

$$\mu^{(k)}_{AxB}(t) = \frac{\alpha t}{\alpha t + \beta\|AxB\|_k} \geq \frac{\alpha t}{\alpha t + \beta(\|A\|\|x\|_n\|B\|)} = \frac{\alpha \frac{t}{\|A\|\|B\|}}{\alpha \frac{t}{\|A\|\|B\|} + \beta\|x\|_n} = \mu^{(n)}_x(\frac{t}{\|A\|\|B\|})$$

for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \neq 0$, $(X, \{\mu^{(n)}\}, T_M)$ is a matrix non-Archimedean random normed space.

3. Stability of the cubic ρ-functional equation (1): Direct method

From now on, let $(X, \{\mu^{(n)}\}, T)$ be a matrix non-Archimedean random normed space, $(Y, \{\mu^{(n)}\}, T)$ be a matrix non-Archimedean random Banach space. In this section, we prove the Hyers-Ulam stability of the cubic ρ-functional equation (1) in matrix non-Archimedean random normed spaces by using the direct method. We need the following lemmas:

Lemma 3.1. (cf. [15]). *Let V and W be real vector spaces. If a mapping $f : V \to W$ satisfies (1), then f is cubic.*

Lemma 3.2. *Let $(X, \{\mu^{(n)}\}, T)$ be a matrix non-Archimedean random normed space. Then*
(1) *$\mu^{(n)}_{E_{kl}\otimes x}(t) = \mu_x(t)$ for all $t > 0$ and $x \in X$;*
(2) *For all $[x_{ij}] \in M_n(X)$ and $t = \sum_{i,j=1}^{n} t_{ij} > 0$,*

$$\mu_{x_{kl}}(t) \geq \mu^{(n)}_{[x_{ij}]}(t) \geq T(\mu_{x_{ij}}(t_{ij}) : i, j = 1, 2, \ldots, n),$$

$$\mu_{x_{kl}}(t) \geq \mu^{(n)}_{[x_{ij}]}(t) \geq T(\mu_{x_{ij}}(\frac{t}{n^2}) : i, j = 1, 2, \ldots, n);$$

(3) $\lim_{m\to\infty} x_m = x$ *if and only if* $\lim_{m\to\infty} x_{ijm} = x_{ij}$ *for* $x_m = [x_{ijm}], x = [x_{ij}] \in M_k(X)$.

Proof. (1) It is easy to see that $E_{kl} \otimes x = e_k^* x e_l$. By the definition of $M_{n,m}(\mathbb{R})$, it follow that $\|E_{kk}\| = \|E_{kk}E_{kk}^*\| = \|E_{kk}\|^2$, so we have

$$\|E_{kk}\| = 1 \quad \text{and} \quad \|e_k^* e_k\| = \|E_{kk}\| = \|e_k\|^2,$$

thus

$$\|e_k\| = \|e_k^*\| = 1.$$

By the definition of matrix non-Archimedean random normed space, for all $t > 0$, we

$$\mu^{(n)}_{E_{kl}\otimes x}(t) = \mu^{(n)}_{e_k^* x e_l}(t) \geq \mu_x(\frac{t}{\|e_k^*\|\|e_l\|}) = \mu_x(t).$$

Since $e_k(E_{kl} \otimes x)e_l^* = x$,

$$\mu_x(t) = \mu_{e_k(E_{kl}\otimes x)e_l^*}(t) \geq \mu^{(n)}_{E_{kl}\otimes x}(\frac{t}{\|e_k^*\|\|e_l\|}) = \mu^{(n)}_{E_{kl}\otimes x}(t).$$

Thus, we have $\mu^{(n)}_{E_{kl}\otimes x}(t) = \mu_x(t)$.

(2) Since $\mu_{x_{kl}}(t) = \mu_{e_k[x_{ij}]e_l^*}(t) \geq \mu^{(n)}_{[x_{ij}]}(\frac{t}{\|e_k\|\|e_l\|}) = \mu^{(n)}_{[x_{ij}]}(t)$,

$$\mu^{(n)}_{[x_{ij}]}(t) = \mu^{(n)}_{\sum_{i,j=1}^{n} E_{ij}\otimes x_{ij}}(t) \geq T(\mu^{(n)}_{E_{ij}\otimes x_{ij}}(t_{ij}) : i, j = 1, 2, \ldots, n) = T(\mu_{x_{ij}}(t_{ij}) : i, j = 1, 2, \ldots, n),$$

where $t = \sum_{i,j=1}^{n} t_{ij}$. So $\mu^{(n)}_{[x_{ij}]}(t) \geq T(\mu_{x_{ij}}(\frac{t}{n^2}) : i, j = 1, 2, \ldots, n)$.

(3) For $x_m = [x_{ijm}], x = [x_{ij}] \in M_k(X), t > 0$. If $\lim_{m\to\infty} x_m = x$, then $\lim_{m\to\infty} \mu_{x_m - x}(t) = 1$. It follows from (2) that

$$\mu_{x_{kl}}(t) \geq \mu^{(n)}_{[x_{ij}]}(t),$$

which is to see that

$$\lim_{m\to\infty} \mu_{x_{ijm}-x_{ij}}(t) \geq \lim_{m\to\infty} \mu^{(n)}_{[x_{ijm}-x_{ij}]}(t) = \lim_{m\to\infty} \mu^{(n)}_{x_m - x}(t) = 1,$$

that is,

$$\lim_{m\to\infty} x_{ijm} = x_{ij}.$$

On the contrary, by $\mu_{x_{kl}}(t) \geq \mu^{(n)}_{[x_{ij}]}(t) \geq T(\mu_{x_{ij}}(\frac{t}{n^2}) : i, j = 1, 2, \ldots, n)$, we obtain the result. □

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X)^2 \to M_n(Y)$ by

$$\begin{aligned} Df(a, b) &:= f(2a + b) + f(2a - b) - 2f(a + b) - 2f(a - b) - 12f(a) \\ &\quad - \rho\Big(4f(a + \frac{b}{2}) + 4f(a - \frac{b}{2}) - f(a + b) - f(a - b) - 6f(a)\Big), \\ Df_n([x_{ij}], [y_{ij}]) &:= f_n(2[x_{ij}] + [y_{ij}]) + f_n(2[x_{ij}] - [y_{ij}]) - 2f_n([x_{ij}] + [y_{ij}]) - 2f_n([x_{ij}] - [y_{ij}]) - 12f_n([x_{ij}]) \\ &\quad - \rho\Big(4f_n([x_{ij}] + \frac{[y_{ij}]}{2}) + 4f_n([x_{ij}] - \frac{[y_{ij}]}{2}) - f_n([x_{ij}] + [y_{ij}]) - f_n([x_{ij}] - [y_{ij}]) - 6f_n([x_{ij}])\Big) \end{aligned}$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 3.3. *Let $\varphi : X^2 \to \mathcal{D}^+$ be a function such that there exists $\alpha \in \mathbb{R}$ with $0 < |8| < |\alpha|$ such that*

$$\varphi(\frac{a}{2}, \frac{b}{2})(t) \geq \varphi(a, b)(|\alpha|t) \tag{2}$$

for all $a, b \in X$ and all $t > 0$ and $\lim_{m\to\infty} \mathcal{T}_{\ell=m}^{\infty} \varphi(a, 0)(\frac{|2||\alpha|^{\ell}}{|8|^{\ell}} t) = 1$ for all $a \in X$ and all $t > 0$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\mu^{(n)}_{Df_n([x_{ij}],[y_{ij}])}(t) \geq \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij})(t) \tag{3}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu^{(n)}_{f_n([x_{ij}])-C_n([x_{ij}])}(t) \geq \mathcal{T}\Big(\mathcal{T}_{\ell=1}^{\infty} \varphi(x_{ij}, 0)(\frac{|2||\alpha|^{\ell+1}}{|8|^{\ell} n^2} t) : i, j = 1, \ldots, n\Big) \tag{4}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. When $n = 1$, (3) is equivalent to

$$\mu_{Df(a,b)}(t) \geq \varphi(a, b)(t) \tag{5}$$

for all $a, b \in X$ and all $t > 0$. Letting $b = 0$ in (5), we get

$$\mu_{f(a)-8f(\frac{a}{2})}(t) \geq \varphi(a, 0)(|2||\alpha|t) \tag{6}$$

for all $a \in X$ and all $t > 0$. Replacing a by $\frac{a}{2^m}$ in (6) and using the inequality (2), we get

$$\mu_{f(\frac{a}{2^m})-8f(\frac{a}{2^{m+1}})}(t) \geq \varphi(a, 0)(|2||\alpha|^{m+1} t) \tag{7}$$

for all $a \in X$ and all $t > 0$. It follows from (2) and (7) that

$$\mu_{8^m f(\frac{a}{2^m})-8^{m+1} f(\frac{a}{2^{m+1}})}(t) \geq \varphi(a, 0)(\frac{|2||\alpha|^{m+1}}{|8|^m} t) \tag{8}$$

for all $a \in X$ and all $t > 0$. Hence

$$\mu_{8^m f(\frac{a}{2^m})-8^{m+p} f(\frac{a}{2^{m+p}})}(t) \geq \mathcal{T}_{\ell=m}^{m+p}(\mu_{8^{\ell} f(\frac{a}{2^{\ell}})-8^{\ell+p} f(\frac{a}{2^{\ell+p}})}(t)) \geq \mathcal{T}_{\ell=m}^{m+p} \varphi(a, 0)(\frac{|2||\alpha|^{\ell+1}}{|8|^{\ell}} t) \tag{9}$$

for all $a \in X$ and all $t > 0$. Since $\lim_{m\to\infty} \mathcal{T}_{\ell=m}^{\infty} \varphi(a, 0)(\frac{|2||\alpha|^{\ell+1}}{|8|^{\ell}} t) = 1$ for all $a \in X$ and all $t > 0$, it follows that the sequence $\{8^m f(\frac{a}{2^m})\}$ is a Cauchy sequence in the matrix non-Archimedean random Banach space $(Y, \mu^{(n)}, \mathcal{T})$. Hence, we can define the mapping $C : X \to Y$ by

$$\lim_{m\to\infty} \mu_{8^m f(\frac{a}{2^m})-C(a)}(t) = 1 \tag{10}$$

for all $a \in X$ and all $t > 0$. Next, we have

$$\mu_{f(a)-8^m f(\frac{a}{2^m})}(t) = \mu_{\Sigma_{\ell=0}^{m-1}(8^{\ell} f(\frac{a}{2^{\ell}})-8^{\ell+1} f(\frac{a}{2^{\ell+1}}))}(t) \geq \mathcal{T}_{\ell=0}^{m-1}(\mu_{8^{\ell} f(\frac{a}{2^{\ell}})-8^{\ell+1} f(\frac{a}{2^{\ell+1}})}(t)) \geq \mathcal{T}_{\ell=0}^{m-1} \varphi(a, 0)(\frac{|2||\alpha|^{\ell+1}}{|8|^{\ell}} t) \tag{11}$$

for all $a \in X$ and all $t > 0$. Therefore,

$$\mu_{f(a)-C(a)}(t) \geq \mathcal{T}(\mu_{f(a)-8^m f(\frac{a}{2^m})}(t), \mu_{8^m f(\frac{a}{2^m})-C(a)}(t)) \geq \mathcal{T}(\mathcal{T}_{\ell=0}^{m-1} \varphi(a, 0)(\frac{|2||\alpha|^{\ell+1}}{|8|^{\ell}} t), \mu_{8^m f(\frac{a}{2^m})-C(a)}(t)). \tag{12}$$

Moreover, letting $m \to \infty$ in (12), we obtain

$$\mu_{f(a)-C(a)}(t) \geq \mathcal{T}_{\ell=1}^{\infty}\varphi(a,0)(\frac{|2||\alpha|^{\ell+1}}{|8|^{\ell}}t). \tag{13}$$

As $\mathcal{T}$ is continuous, from a well known result in the probabilistic metric space (see [23, Chapter 12]), it follows that

$$\begin{aligned}&\lim_{m\to\infty}\mu_{8^m f(2^{-m}(2a+b))+8^m f(2^{-m}(2a-b))-2\cdot 8^m f(2^{-m}(a+b))-2(8^m)f(2^{-m}(a-b))-12(8^m)f(2^{-m}a)}\\ &\qquad {}_{-\rho(4(8^m)f(2^{-m}(a+\frac{b}{2}))+4(8^m)f(2^{-m}(a-\frac{b}{2}))-8^m f(2^{-m}(a+b))-8^m f(2^{-m}(a-b))-6(8^m)f(2^{-m}a))}(t)\\ &\qquad = \mu_{C(2a+b)+C(2a-b)-2C(a+b)-2C(a-b)-12C(a)-\rho(4C(a+\frac{b}{2})+4C(a-\frac{b}{2})-C(a+b)-C(a-b)-6C(a))}(t)\end{aligned}$$

for all $a, b \in X$ and all $t > 0$. On the other hand, replacing a, b by $2^{-m}a, 2^{-m}b$ in (5), and using (2), we get

$$\begin{aligned}&\lim_{m\to\infty}\mu_{8^m f(2^{-m}(2a+b))+8^m f(2^{-m}(2a-b))-2\cdot 8^m f(2^{-m}(a+b))-2(8^m)f(2^{-m}(a-b))-12(8^m)f(2^{-m}a)}\\ &\qquad {}_{-\rho(4(8^m)f(2^{-m}(a+\frac{b}{2}))+4(8^m)f(2^{-m}(a-\frac{b}{2}))-8^m f(2^{-m}(a+b))-8^m f(2^{-m}(a-b))-6(8^m)f(2^{-m}a))}(t)\\ &\qquad \geq \varphi(a,b)(\frac{|\alpha|^m}{|8|^m}t)\end{aligned}$$

for all $a, b \in X$ and all $t > 0$. Since $\lim_{m\to\infty}\varphi(a,b)(\frac{|\alpha|^m}{|8|^m}t) = 1$, we infer that C is a cubic mapping.

To prove the uniqueness of C, let $C' : X \to Y$ is another cubic mapping such that

$$\mu_{C'(a)-f(a)}(t) \geq \mathcal{T}_{\ell=1}^{\infty}\varphi(a,0)(\frac{|2||\alpha|^{\ell+1}}{|8|^{\ell}}t)$$

for all $a \in X$ and all $t > 0$. Let $n = 1$. Then we have

$$\mu_{C'(a)-C(a)}(t) \geq \mathcal{T}(\mu_{C(a)-8^m f(\frac{a}{2^m})}(t), \mu_{8^m f(\frac{a}{2^m})-C'(a)}(t)).$$

Thanks to (10), we can conclude that $C(a) = C'(a)$ for all $a \in X$, which gives the conclusion. Thus the mapping $C : X \to Y$ is a unique cubic mapping.

By Lemma 3.2 and (13), we get

$$\mu^{(n)}_{f_n([x_{ij}])-C_n([x_{ij}])}(t) \geq \mathcal{T}\Big(\mu_{f([x_{ij}])-C([x_{ij}])}(\frac{t}{n^2}) : i,j = 1,\ldots,n\Big) \geq \mathcal{T}\Big(\mathcal{T}_{\ell=1}^{\infty}\varphi(x_{ij},0)(\frac{|2||\alpha|^{\ell+1}}{|8|^{\ell}n^2}t) : i,j = 1,\ldots,n\Big)$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$. Thus $C : X \to Y$ is a unique cubic mapping satisfying (4), as desired. This completes the proof of the theorem. □

Corollary 3.4. *Let r, θ be positive real numbers with $r < 3$. Suppose that $f : X \to Y$ is a mapping satisfying*

$$\mu^{(n)}_{Df_n([x_{ij}],[y_{ij}])}(t) \geq \frac{t}{t + \sum_{i,j=1}^{n}\theta(\|x_{ij}\|^r + \|y_{ij}\|^r)} \tag{14}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. If

$$\lim_{m\to\infty}\mathcal{T}_{\ell=m}^{\infty}(\frac{t}{t + \frac{|8|^{\ell}}{|2||2|^{\ell r}}\theta\|a\|^r}) = 1$$

for all $a \in X$ and all $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu^{(n)}_{f_n([x_{ij}])-C_n([x_{ij}])}(t) \geq \mathcal{T}\Big(\mathcal{T}_{\ell=1}^{\infty}(\frac{t}{t + \frac{|8|^{\ell}n^2}{|2||2|^{(\ell+1)r}}\theta\|x_{ij}\|^r}) : i,j = 1,\ldots,n\Big) \tag{15}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. The proof follows immediately by taking $\varphi(a,b)(t) = \frac{t}{t+\theta(\|a\|^r+\|b\|^r)}$ for all $a,b \in X$ and all $t > 0$ in Theorem 3.3. □

Theorem 3.5. *Let $\varphi : X^2 \to \mathcal{D}^+$ be a function such that there exists $\alpha \in \mathbb{R}$ with $0 < |\alpha| < |8|$ such that*

$$\varphi(2a,2b)(t) \geq \varphi(a,b)(\frac{t}{|\alpha|}) \tag{16}$$

for all $a,b \in X$ and all $t > 0$ and $\lim_{m\to\infty} \mathcal{T}_{\ell=m}^{\infty}\varphi(a,0)(\frac{|2||8|^{\ell}}{|\alpha|^{\ell}}t) = 1$ for all $a \in X$ and all $t > 0$. Suppose that $f : X \to Y$ is a mapping satisfying (3) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu^{(n)}_{f_n([x_{ij}])-C_n([x_{ij}])}(t) \geq \mathcal{T}\Big(\mathcal{T}_{\ell=1}^{\infty}\varphi(x_{ij},0)(\frac{|2||8|^{\ell+1}}{|\alpha|^{\ell}n^2}t) : i,j = 1,\ldots,n\Big) \tag{17}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. Letting $b = 0$ in (5), we get

$$\mu_{f(a)-f(\frac{2a}{8})}(t) \geq \varphi(a,0)(|2||8|t) \tag{18}$$

for all $a \in X$ and all $t > 0$. Replacing a by $2^m a$ in (18) and using the inequality (16), we get

$$\mu_{f(2^m a)-\frac{f(2^{m+1}a)}{8}}(t) \geq \varphi(a,0)(\frac{|2||8|}{|\alpha|^m}t) \tag{19}$$

for all $a \in X$ and all $t > 0$. It follows from (16) and (19) that

$$\mu_{\frac{f(2^m a)}{8^m}-\frac{f(2^{m+1}a)}{8^{m+1}}}(t) \geq \varphi(a,0)(\frac{|2||8|^{m+1}}{|\alpha|^m}t) \tag{20}$$

for all $a \in X$ and all $t > 0$. Hence

$$\mu_{\frac{f(2^m a)}{8^m}-\frac{f(2^{m+p}a)}{8^{m+p}}}(t) \geq \mathcal{T}_{\ell=m}^{m+p}(\mu_{\frac{f(2^\ell a)}{8^\ell}-\frac{f(2^{\ell+p}a)}{8^{\ell+p}}}(t)) \geq \mathcal{T}_{\ell=m}^{m+p}\varphi(a,0)(\frac{|2||8|^{\ell+1}}{|\alpha|^{\ell}}t) \tag{21}$$

for all $a \in X$ and all $t > 0$. Since $\lim_{m\to\infty} \mathcal{T}_{\ell=m}^{\infty}\varphi(a,0)(\frac{|2||8|^{\ell+1}}{|\alpha|^{\ell}}t) = 1$ for all $a \in X$ and all $t > 0$, it follows that the sequence $\{\frac{f(2^m a)}{8^m}\}$ is a Cauchy sequence in the matrix non-Archimedean random Banach space $(Y, \mu^{(n)}, \mathcal{T})$. Hence, we can define the mapping $C : X \to Y$ by

$$\lim_{m\to\infty} \mu_{\frac{f(2^m a)}{8^m}-C(a)}(t) = 1 \tag{22}$$

for all $a \in X$ and all $t > 0$. Next, we have

$$\mu_{f(a)-\frac{f(2^m a)}{8^m}}(t) = \mu_{\Sigma_{\ell=0}^{m-1}(\frac{f(2^\ell a)}{8^\ell}-\frac{f(2^{\ell+1}a)}{8^{\ell+1}})}(t) \geq \mathcal{T}_{\ell=0}^{m-1}(\mu_{\frac{f(2^\ell a)}{8^\ell}-\frac{f(2^{\ell+1}a)}{8^{\ell+1}}})(t)) \geq \mathcal{T}_{\ell=0}^{m-1}\varphi(a,0)(\frac{|2||8|^{\ell+1}}{|\alpha|^{\ell}}t) \tag{23}$$

for all $a \in X$ and all $t > 0$. Therefore,

$$\mu_{f(a)-C(a)}(t) \geq \mathcal{T}(\mu_{f(a)-\frac{f(2^m a)}{8^m}}(t), \mu_{\frac{f(2^m a)}{8^m}-C(a)}(t)) \geq \mathcal{T}(\mathcal{T}_{\ell=0}^{m-1}\varphi(a,0)(\frac{|2||8|^{\ell+1}}{|\alpha|^{\ell}}t), \mu_{\frac{f(2^m a)}{8^m}-C(a)}(t)). \tag{24}$$

Moreover, letting $m \to \infty$ in (24), we obtain

$$\mu_{f(a)-C(a)}(t) \geq \mathcal{T}_{\ell=1}^{\infty}\varphi(a,0)(\frac{|2||8|^{\ell+1}}{|\alpha|^{\ell}}t). \tag{25}$$

The rest of the proof is similar to the proof of Theorem 3.3 and thus it is omitted. This completes the proof of the theorem. □

Corollary 3.6. *Let r, θ be positive real numbers with $r > 3$. Suppose that $f : X \to Y$ is a mapping satisfying (14) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. If*

$$\lim_{m\to\infty} \mathcal{T}_{\ell=m}^{\infty}\Big(\frac{t}{t + \frac{|2|^{\ell r}}{|2||8|^{\ell}}\theta\|a\|^r}\Big) = 1$$

for all $a \in X$ and all $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu^{(n)}_{f_n([x_{ij}])-C_n([x_{ij}])}(t) \geq \mathcal{T}\Big(\mathcal{T}_{\ell=1}^{\infty}\Big(\frac{t}{t + \frac{|2|^{\ell r} n^2}{|2||8|^{\ell+1}}\theta\|x_{ij}\|^r}\Big) : i, j = 1, \ldots, n\Big) \tag{26}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. Letting $\varphi(a, b)(t) = \frac{t}{t+\theta(\|a\|^r+\|b\|^r)}$ for all $a, b \in X$ and all $t > 0$ in Theorem 3.5, we obtain the result.□

4. Stability of the cubic ρ-functional equation (1): Fixed point method

In this section, we prove the Hyers-Ulam stability of the cubic ρ-functional equation (1) in matrix non-Archimedean random normed spaces by using fixed point method. We begin with the definition of a generalized metric on a set.

Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a generalized metric on E if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$, $\forall x, y \in E$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in E$.

Before proceeding to the proof of the main results, we begin with a result due to Diaz and Margolis [4].

Lemma 4.1. (cf. [4]). *Let (E, d) be a complete generalized metric space and $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each fixed element $x \in E$, either*

$$d(J^n x, J^{n+1} x) = \infty \quad \forall n \geq 0,$$

or

$$d(J^n x, J^{n+1} x) < \infty \quad \forall n \geq n_0,$$

for some natural number n_0. Moreover, if the second alternative holds then:
(i) The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;
(ii) y^* is the unique fixed point of J in the set $E^* := \{y \in E \mid d(J^{n_0} x, y) < +\infty\}$ and $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$, $\forall x, y \in E^*$.

Theorem 4.2. *Let $s \in \{1, -1\}$ be fixed and let $\varphi : X^2 \to \mathcal{D}^+$ be a function such that there exists there exists an L with $0 < L < 1$ and*

$$\varphi(a, b)(t) \geq \varphi\Big(\frac{a}{2^s}, \frac{b}{2^s}\Big)\Big(\frac{t}{|8|^s L}\Big) \tag{27}$$

for all $a, b \in X$ and all $t > 0$. Suppose that $f : X \to Y$ is a mapping satisfying (3) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu^{(n)}_{f_n([x_{ij}])-C_n([x_{ij}])}(t) \geq \begin{cases} \mathcal{T}\Big(\varphi(x_{ij}, 0)\big(\frac{|2||8|(1-L)}{n^2 L} t\big) : i, j = 1, \ldots, n\Big), & \text{if } s = -1, \\ \mathcal{T}\Big(\varphi(x_{ij}, 0)\big(\frac{|2||8|(1-L)}{n^2} t\big) : i, j = 1, \ldots, n\Big), & \text{if } s = 1 \end{cases} \tag{28}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$.

Proof. When $n = 1$, similar to the proof of Theorem 3.3, we have

$$\mu_{2f(2a)-16f(a)}(t) \geq \varphi(a, 0)(t) \tag{29}$$

for all $a \in X$ and all $t > 0$.

Let $S := \{g : X \to Y\}$, and introduce a generalized metric d on S as follows:

$$d(g, h) := \inf\left\{\lambda \in \mathbb{R}_+ \middle| \mu_{g(a)-h(a)}(\lambda t) \geq \varphi(a, 0)(t), \forall a \in X, \forall t > 0\right\}.$$

It is easy to prove that (S, d) is a complete generalized metric space (cf. [2]). Now, we define the mapping $\mathcal{J} : S \to S$ by

$$\mathcal{J}g(a) := \frac{1}{8^s} g(2^s a), \quad \text{for all } g \in S \text{ and } a \in X, s \in \{1, -1\}. \tag{30}$$

Let $g, h \in S$ and let $\lambda \in \mathbb{R}_+$ be an arbitrary constant with $d(g, h) \leq \lambda$. From the definition of d, we get

$$\mu_{g(a)-h(a)}(\lambda t) \geq \varphi(a, 0)(t)$$

for all $a \in X$ and $t > 0$. Therefore, using (27), we get

$$\mu_{\mathcal{J}g(a)-\mathcal{J}h(a)}(\lambda L t) = \mu_{(\frac{1}{8^s} g(2^s a) - \frac{1}{8^s} h(2^s a))}(\lambda L t) \geq \varphi(2^s a, 0)(|8|^s L t) \geq \varphi(a, 0)(t) \tag{31}$$

for some $L < 1$, all $a \in X$ and all $t > 0$. Hence, it holds that $d(\mathcal{J}g, \mathcal{J}h) \leq \lambda L$, that is, $d(\mathcal{J}g, \mathcal{J}h) \leq L d(g, h)$ for all $g, h \in S$.

Furthermore, by (27) and (29), we obtain the inequality

$$d(f, \mathcal{J}f) \leq \begin{cases} \frac{L}{|2||8|}, & \text{if } s = -1, \\ \frac{1}{|2||8|}, & \text{if } s = 1. \end{cases}$$

It follows from Lemma 4.1 that the sequence $\mathcal{J}^m f$ converges to a fixed point C of $\mathcal{J}$, that is, for all $a \in X$ and all $t > 0$,

$$C : X \to Y, \quad \lim_{m\to\infty} \mu_{\frac{1}{8^{sm}} f(2^{sm} a) - C(a)}(t) = 1 \tag{32}$$

and

$$C(2^s a) = 8^s C(a). \tag{33}$$

Meanwhile, C is the unique fixed point of $\mathcal{J}$ in the set $S^* = \{g \in S : d(f, g) < \infty\}$. Thus there exists a $\lambda \in \mathbb{R}_+$ such that

$$\mu_{f(a)-C(a)}(\lambda t) \geq \varphi(a, 0)(t)$$

for all $a \in X$ and all $t > 0$. Also,

$$d(f, C) \leq \frac{1}{1 - L} d(f, \mathcal{J}f) \leq \begin{cases} \frac{L}{|2||8|(1-L)}, & \text{if } s = -1, \\ \frac{1}{|2||8|(1-L)}, & \text{if } s = 1. \end{cases}$$

This means that the following inequality

$$\mu_{f(a)-C(a)}(t) \geq \begin{cases} \varphi(a, 0)(\frac{|2||8|(1-L)}{L} t), & \text{if } s = -1, \\ \varphi(a, 0)(|2||8|(1 - L)t), & \text{if } s = 1 \end{cases} \tag{34}$$

holds for all $a \in X$ and all $t > 0$. As $\mathcal{T}$ is continuous, from a well known result in the probabilistic metric space (see [23, Chapter 12]), it follows that

$$\begin{aligned}&\lim_{m\to\infty}\mu_{8^{-sm}f(2^{sm}(2a+b))+8^{-sm}f(2^{sm}(2a-b))-2\cdot 8^{-sm}f(2^{sm}(a+b))-2(8^{-sm})f(2^{sm}(a-b))-12(8^{-sm})f(2^{sm}a)}\\&\quad{}_{-\rho(4(8^{-sm})f(2^{sm}(a+\frac{b}{2}))+4(8^{-sm})f(2^{sm}(a-\frac{b}{2}))-8^{-sm}f(2^{sm}(a+b))-8^{-sm}f(2^{sm}(a-b))-6(8^{-sm})f(2^{sm}a))}(t)\\&\quad=\mu_{C(2a+b)+C(2a-b)-2C(a+b)-2C(a-b)-12C(a)-\rho(4C(a+\frac{b}{2})+4C(a-\frac{b}{2})-C(a+b)-C(a-b)-6C(a))}(t)\end{aligned}$$

for all $a, b \in X$ and all $t > 0$. On the other hand, it follows from (3), (27) and (32) that

$$\begin{aligned}&\lim_{m\to\infty}\mu_{8^{-sm}f(2^{sm}(2a+b))+8^{-sm}f(2^{sm}(2a-b))-2\cdot 8^{-sm}f(2^{sm}(a+b))-2(8^{-sm})f(2^{sm}(a-b))-12(8^{-sm})f(2^{sm}a)}\\&\quad{}_{-\rho(4(8^{-sm})f(2^{sm}(a+\frac{b}{2}))+4(8^{-sm})f(2^{sm}(a-\frac{b}{2}))-8^{-sm}f(2^{sm}(a+b))-8^{-sm}f(2^{sm}(a-b))-6(8^{-sm})f(2^{sm}a))}(t)\\&\quad\geq\varphi(a,b)\Big(\frac{|8|^{sm}}{|8|^{sm}L^m}t\Big)=\varphi(a,b)\Big(\frac{1}{L^m}t\Big)\end{aligned}$$

for all $a, b \in X$ and all $t > 0$. Since $\lim_{m\to\infty}\varphi(a,b)(\frac{1}{L^m}t) = 1$, we infer that C is a cubic mapping.

By Lemma 3.2 and (34), we get

$$\begin{aligned}\mu^{(n)}_{f_n([x_{ij}])-C_n([x_{ij}])}(t)&\geq\mathcal{T}\Big(\mu_{f([x_{ij}])-C([x_{ij}])}\Big(\frac{t}{n^2}\Big):i,j=1,\ldots,n\Big)\\&\geq\begin{cases}\mathcal{T}\Big(\varphi(x_{ij},0)\big(\frac{|2||8|(1-L)}{n^2L}t\big):i,j=1,\ldots,n\Big), & \text{if } s=-1,\\ \mathcal{T}\Big(\varphi(x_{ij},0)\big(\frac{|2||8|(1-L)}{n^2}t\big):i,j=1,\ldots,n\Big), & \text{if } s=1\end{cases}\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$ and all $t > 0$. Thus $C : X \to Y$ is a unique cubic mapping satisfying (28), as desired. This completes the proof of the theorem. □

Corollary 4.3. *Let r, θ be positive real numbers with $r \neq 3$. Suppose that $f : X \to Y$ is a mapping satisfying (14) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\mu^{(n)}_{f_n([x_{ij}])-C_n([x_{ij}])}(t)\geq\begin{cases}\mathcal{T}\Big(\frac{|8|(|2|^r-|2|^3)t}{|8|(|2|^r-|2|^3)t+\theta n^2|2|^2\|x_{ij}\|^r}:i,j=1,\ldots,n\Big), & \text{if } 0<r<3,\\ \mathcal{T}\Big(\frac{|8|(|2|^3-|2|^r)t}{|8|(|2|^3-|2|^r)t+\theta n^2|2|^2\|x_{ij}\|^r}:i,j=1,\ldots,n\Big), & \text{if } r>3\end{cases} \quad (35)$$

for all $x \in X$.

Proof. The proof follows immediately by taking $\varphi(a,b)(t) = \frac{t}{t+\theta(\|a\|^r+\|b\|^r)}$ for all $a, b \in X$ and all $t > 0$, and choosing

$$L=\begin{cases}|2|^{3-r}, & \text{if } 0<r<3,\\ |2|^{r-3}, & \text{if } r>3\end{cases}$$

in Theorem 4.2. □

Acknowledgement:

The authors thank the referee for careful reading and helpful comments.

References

[1] T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan* **2**(1950), 64-66.
[2] L. Cădariu and V. Radu, On the stability of the Cauchy functional equation: A fixed point approach, *Grazer Math. Ber.* **346**(2004), 43-52.
[3] Y. J. Cho, C. Park and R. Saadati, Functional inequalities in non-Archimedean in Banach spaces, *Appl. Math. Lett.* **60**(2010), 1994-2002.
[4] J. B. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.* **74**(1968), 305-309.
[5] A. Ebadian, S. Zolfaghan, S. Ostadbash and C. Park, Approximation on the reciprocal functional equation in several variables in matrix non-Archimedean random normed spaces, *Adv. Differ. Equ.* **2015**(2015), Article ID 314.
[6] P. Găvruţă, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184**(1994), 431-436.
[7] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* **27**(1941), 222-224.
[8] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkh*ä*user, Basel, 1998.
[9] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Science, New York, 2011.
[10] J. I. Kang and R. Saadati, Approximation of homomorphisms and derivations on non-Archimedean random Lie C^*-algebras via fixed point method, *J. Ineq. Appl.* **2012**(2012), Article ID 251.
[11] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Science, New York, 2009.
[12] J. Lee, C. Park and D. Shin, An AQCQ-functional equation in Matrix normed spaces, *Result. Math.* **64**(2013), 305-318.
[13] J. Lee, D. Shin and C. Park, Hyers-Ulam stability of functional equations in matrix normed spaces, *J. Ineq. Appl.* **2013**(2013), Article ID 22.
[14] D. Miheţ and V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces. *J. Math. Anal. Appl.* **343**(2008), 567-572.
[15] C. Park, Cubic and quartic ρ-functional inqualities in fuzzy Banach spaces, *J. Math. Ineq.* **10**(2016), 1123-1136.
[16] C. Park, Y. J. Cho and H. A. Kenary, Orthogonal stability of a generalized quadratic functional equation in non-Archimedean spaces, *J. Comput. Anal. Appl.* **14**(2012), 526-535.
[17] C. Park, J. Lee and D. Shin, An AQCQ-functional equation in matrix Banach spaces, *Adv. Diff. Equ.* **2013**(2013), Article ID 146.
[18] C. Park, J. Lee and D. Shin, Functional equations and inequalities in matrix paranormed spaces, *J. Ineq. Appl.* **2013**(2013), Article ID 547.
[19] C. Park, D. Shin and J. Lee, Fuzzy stability of functional inequalities in matrix fuzzy normed spaces, *J. Ineq. Appl.* **2013**(2013), Article ID 224.
[20] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72**(1978), 297-300.
[21] Th. M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic, Dordrecht, 2003.
[22] P. K. Sahoo and Pl. Kannappan, *Introduction to Functional Equations*, CRC Press, Boca Raton, 2011.
[23] B. Schweizer and A. Sklar, *Probabilistic metric spaces*, North-Holland, New York, 1983.
[24] A. N. Šerstnev, On the notion of a random normed space (in Russian), *Dokl. Akad. Nauk. SSSR* **149**(1963), 280-283.
[25] S. M. Ulam, *Problems in Modern Mathematics, Chapter VI*, Science Editions, Wiley, New York, 1964.