Filomat 34:8 (2020), 2667–2681
https://doi.org/10.2298/FIL2008667E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Pointfree Version of Grills

Ali Akbar Estaji[a], Toktam Haghdadi[b]

[a]*Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran*
[b]*Department of Basic Sciences, Birjand University of Technology, Birjand, Iran*

Abstract. In this paper, the pointfree version of grills is introduced. We consider a Boolean algebra B and a subframe L instead of a topological space (X, τ), and present the concept of approximation θ over B. Moreover, some properties of them are given. Also, we introduce and study the new concepts grill and θ-grill on Boolean algebras.

1. Introduction

The idea of grill on a topological space was first introduced by Choquet in [4]. It is proved that, grills, nets and filters, are useful and important for studying some topological concepts such as proximity spaces, closure spaces, the theory of compactifications and other similar extension problems. Chattopadhyay and Thorn [3] proved that grills are always unions of ultra filters. Further, Roy and Mukherjee [10] defined and studied the typical topology associated with a grill defined on a given topological space. Recently, Hatir and Jafari [7] and Al-Omari and Noiri [1, 2] have developed the study of grill topological spaces with continuities and generalized continuities. The notions of soft grill, soft operators ϕ_G, ψ_G and soft topology τ_G were defined and discussed in [8]. Latif in [5] introduced more properties of soft grills.

Traditionally, a topological space consists of a set of points together with a topology, a system of subsets called open sets that with the operations of intersection and union forms a lattice with certain properties. Point-free topology focuses on the open sets rather than the points of the space. The main purpose of this article is to introduce a point-free version of grill and θ-grills on a bounded lattice D.

This article is organized as follows. In the next section some basic notions and properties of lattices and grills are reviewed. In Section 3, we introduce the concept of approximation on a bounded lattice. Also, by an approximation θ, we introduce new concept θ-grill on D and we show that the set of all of grills and the set of all of θ-grills on a bounded lattice, ordered by inclusion, are complete lattices. In Section 4, we consider a Boolean algebra B, a subframe L of B, a θ-grill $\mathcal{G}$ on B and define two operators $\varphi^L_{\mathcal{G}}$ and $\varphi^L_{(\theta,\mathcal{G})}$. Then, we study some basic properties of them. In Section 5, we consider new operators $\psi^L_{\mathcal{G}}$ and $\psi^L_{(\theta,\mathcal{G})}$ induced by the operators $\varphi^L_{\mathcal{G}}$ and $\varphi^L_{(\theta,\mathcal{G})}$, respectively. Moreover, we introduce frames $L_{\mathcal{G}}$ and $L_{(\theta,\mathcal{G})}$ induced by $\psi^L_{\mathcal{G}}$ and $\psi^L_{(\theta,\mathcal{G})}$.

2010 *Mathematics Subject Classification*. Primary 06D22; Secondary 03G10, 06D10, 54A05
Keywords. Frame, Grill, Boolean algebra, θ-grill
Received: 20 August 2019; Revised: 25 April 2020; Accepted: 30 April 2020
Communicated by Ljubiša D.R. Kočinac
Email addresses: aaestaji@hsu.ac.ir (Ali Akbar Estaji), haghdadi@birjandut.ac.ir (Toktam Haghdadi*)

2. Preliminaries

In what follows, by a space X we shall mean a topological space(X,τ).

A nonempty collection $\mathcal{G}$ of subsets of a nonempty set X is said to be a grill on X, if it satisfies the following conditions:

1. $\emptyset \notin \mathcal{G}$
2. if $A \in \mathcal{G}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{G}$, and
3. if $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ then $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

In [10], Roy and Mukherjee introduced a new topology from a topological space (X,τ), constructed by use of a grill on X, and it is described as follows: Let $\mathcal{G}$ be a grill on a space X. Consider the operator $\phi_{\mathcal{G}} \colon P(X) \to P(X)$ is given by

$$\phi_{\mathcal{G}}(A) = \{x \in X \colon x \in O \text{ implies } O \cap A \in \mathcal{G} \text{ for all } O \in \tau\}$$

for all $A \in P(X)$. Then the map $\psi_{\mathcal{G}} \colon P(X) \to P(X)$ given by $\psi_{\mathcal{G}}(A) = A \cup \phi_{\mathcal{G}}(A)$ is a Kuratowski's closure operator and hence induces a topology

$$\tau_{\mathcal{G}} = \{G \subseteq X \colon \psi_{\mathcal{G}}(X - G) = X - G\}$$

on X, strictly finer than τ in general. A simple base for the open sets of $\tau_{\mathcal{G}}$ is described as follows:

$$\beta(\mathcal{G},\tau) = \{V - A \colon V \in \tau, A \notin \mathcal{G}\}.$$

We denote the top element and the bottom element of a bounded lattice by $\top$ and $\perp$, respectively. An element a of a bounded lattice D is called an atom if $a \neq \perp$, and $\perp < b \leq a$ implies $b = a$ for every $b \in D$. In what follows, the set of all atom elements of a bounded lattice D is denoted by $At(D)$.

A lattice L is said to be complemented if every $x \in L$ has a complement; that is, for every $x \in L$, there exists an element y of L such that $x \wedge y = \perp$ and $x \vee y = \top$. A distributive complemented lattice is called a Boolean algebra. Notice that every element x of a Boolean algebra has a unique complement, which is denoted by x' (see [6]).

A frame is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \colon s \in S\}$$

holds for all $x \in L$ and $S \subset L$. It is well known that every frame is isomorphic to a subframe of a complete Boolean algebra (see [9, Corollary 2.6, page 53]). In what follows, B will denote a complete Boolean algebra. Also, L will denote a subframe of the complete Boolean algebra B.

3. Approximation

In this section, we introduce the concept of approximation on a bounded lattice. Also, by an approximation θ, we introduce new concept θ-grill on D.

Definition 3.1. Let D be a bounded lattice. The function $\theta \colon D \to D$ is called an approximation if

1. $\theta(\perp) = \perp$,
2. $\theta\big(\theta(a)\big) = \theta(a)$,
3. $a \leq \theta(a)$,
4. $\theta(a \vee b) = \theta(a) \vee \theta(b)$, and
5. $\theta(a \wedge b) \leq \theta(a) \wedge \theta(b)$

for all $a, b \in D$.

Proposition 3.2. *Let $\theta\colon D \to D$ be an approximation on a bounded lattice D. Then $\theta(D)$ is a lattice such that for every $\alpha, \beta \in \theta(D)$,*

$$\alpha \vee^{\theta(D)} \beta = \alpha \vee^{D} \beta \text{ and } \alpha \wedge^{\theta(D)} \beta = \theta(\alpha \wedge^{D} \beta).$$

Proof. It is clear that $\theta(D)$ is a poset. Since $\theta(\perp) = \perp$ and $\theta(\top) = \top$, then $\theta(D)$ is a bounded poset. Let α, β be any elements of $\theta(D)$. Then, there exist $a, b \in D$ such that $\alpha = \theta(a)$ and $\beta = \theta(b)$. Since $\alpha \vee^D \beta = \theta(a) \vee^D \theta(b) = \theta(a \vee^D b) \in \theta(D)$, we conclude that $\theta(a) \vee^D \theta(b)$ is an upper bound of $\{\alpha, \beta\}$ in $\theta(D)$. Also $\alpha \vee^D \beta \leq \alpha \vee^{\theta(D)} \beta$, then $\alpha \vee^D \beta = \alpha \vee^{\theta(D)} \beta$.
On the other hand,

$$\theta\Big(\theta(a) \wedge \theta(b)\Big) \geq \theta\Big(\theta(a \wedge b)\Big) = \theta(a \wedge b),$$

and

$$\theta\Big(\theta(a) \wedge^D \theta(b)\Big) \leq \theta\Big(\theta(a)\Big) \wedge \theta\Big(\theta(b)\Big) = \theta(a) \wedge \theta(b).$$

Then $\theta\Big(\theta(a) \wedge^D \theta(b)\Big)$ is a lower bound of $\{\theta(a), \theta(b)\}$. Now, let $d \in D$ be a lower bound of $\{\theta(a), \theta(b)\}$. Then, $\theta(d) \leq \theta(a)$ and $\theta(d) \leq \theta(b)$ and so $\theta(d) \leq \theta(a) \wedge^D \theta(b)$ which implies that $\theta(d) = \theta\Big(\theta(d)\Big) \leq \theta\Big(\theta(a) \wedge^D \theta(b)\Big)$. Therefore, $\theta(a) \wedge^{\theta(D)} \theta(b) = \theta\Big(\theta(a) \wedge^D \theta(b)\Big)$, which means that $\theta(D)$ is a lattice. □

Let D be a bounded lattice. In what follows, $\overline{Apr}(D)$ will denote the set of all approximations on D. It is clear that the set $\overline{Apr}(D)$ with the following relation is a poset.

$$\theta \leq \psi \Longleftrightarrow \forall x \in L;\ \theta(x) \leq \psi(x).$$

Definition 3.3. Let D be a bounded lattice, and let $\theta \in \overline{Apr}(D)$ be given. A non-empty subset $\mathcal{G} \subseteq D$ is called

1. a grill on D, if the following conditions hold:
   (a) $\perp \notin \mathcal{G}$.
   (b) For every $a, b \in D$, if $a \in \mathcal{G}$ and $a \leq b$, then $b \in \mathcal{G}$.
   (c) For every $a, b \in D$, if $a \vee b \in \mathcal{G}$, then $a \in \mathcal{G}$ or $b \in \mathcal{G}$.
2. a θ-grill on D, if the following conditions hold:
   (a) $\perp \notin \mathcal{G}$.
   (b) For every $a, b \in D$, if $a \in \mathcal{G}$ and $\theta(a) \leq \theta(b)$, then $b \in \mathcal{G}$.
   (c) For every $a, b \in D$, if $\theta(a) \vee \theta(b) \in \mathcal{G}$, then $\theta(a) \in \mathcal{G}$ or $\theta(b) \in \mathcal{G}$.

Let D be a bounded lattice. In what follows, by $\theta - RG(D)$ we denote the set all of θ- grills on D. It is easy to see that $(\theta - RG(D), \subseteq)$ is a poset.

Remark 3.4. Notice that there is no generally connection between filters and grills. For example, let $X = \{a, b, c, d\}$, $\mathcal{G} = \Big\{\{a\}, \{b\}, \{a, c\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{c, b, d\}, \{a, b, d\}, X\Big\}$ and $\mathcal{F} = \Big\{X, \{a, b, c\}\Big\}$. It is easy to see that $\mathcal{G}$ is a grill and $\mathcal{F}$ is a filter. Moreover, if $A = \{a\}$ and $B = \{b, c\}$, then $A \cup B \in \mathcal{F}$ but $A, B \notin \mathcal{F}$ which means that $\mathcal{F}$ is not a grill. Also, $\{a, c\}, \{b, c\} \in \mathcal{G}$ but $\{a, c\} \cap \{b, c\} \notin \mathcal{G}$. Hence $\mathcal{G}$ is not a filter.

Proposition 3.5. *Let D be a bounded lattice. The set $\theta - RG(D)$, ordered by inclusion, is a complete lattice such that for every $\{\mathcal{G}_\lambda\}_{\lambda \in \Lambda} \subseteq \theta - RG(D)$,*

$$\bigvee_{\lambda \in \Lambda} \mathcal{G}_\lambda = \bigcup_{\lambda \in \Lambda} \mathcal{G}_\lambda \text{ and } \bigwedge_{\lambda \in \Lambda} \mathcal{G}_\lambda = \bigcup\{\mathcal{G} \in \theta - RG(D) \colon \mathcal{G} \subseteq \bigcap_{\lambda \in \Lambda} \mathcal{G}_\lambda\}.$$

Also, the set all of grills on D, ordered by inclusion, is a complete lattice.

Proof. It is straightforward. □

Proposition 3.6. *Let D be a bounded lattice and $\mathcal{G} \in \theta - RG(D)$. Then the following statements hold.*

(1) $\theta(D) = \{d \in D \colon \theta(d) = d\}$.
(2) *For every $d \in D$, $\theta(d) \in \mathcal{G}$ if and only if $d \in \mathcal{G}$.*
(3) *The set $\mathcal{G}$ is a grill on D.*

Proof. It is straightforward. □

4. Frame Induced by a Grill

In this section, we introduce two operators $\varphi^L_{\mathcal{G}}$ and $\varphi^L_{(\theta,\mathcal{G})}$ and give some properties of them. Recall that L is a subframe of B and for every subset X of B and $x \in B$, we write $\downarrow X = \{y \in B : y \leq x \text{ for some } x \in X\}$, $\uparrow X = \{y \in B : y \geq x \text{ for some } x \in X\}$, $\downarrow x = \downarrow\{x\}$ and $\uparrow x = \uparrow\{x\}$.

Definition 4.1. Let $\mathcal{G} \in \theta - RG(B)$ be given. We define $\varphi^L_{\mathcal{G}}\colon B \to B$ is given by $b \mapsto \bigvee\{x \in At(B)\colon b \wedge y \in \mathcal{G} \text{ for all } y \in L \cap \uparrow x\}$, and $\varphi^L_{(\theta,\mathcal{G})}\colon B \to B$ is given by $b \mapsto \bigvee\{x \in At(B)\colon b \wedge \theta(y) \in \mathcal{G} \text{ for all } y \in L \cap \uparrow x\}$.

For every $b \in B$, we put $H^b_{\mathcal{G}} := \{x \in At(B)\colon b \wedge y \in \mathcal{G} \text{ for all } y \in L \cap \uparrow x\}$, and $H^b_{(\theta,\mathcal{G})} := \{x \in At(B)\colon b \wedge \theta(y) \in \mathcal{G} \text{ for all } y \in L \cap \uparrow x\}$. Hence $\varphi^L_{\mathcal{G}}(b) = \bigvee H^b_{\mathcal{G}}$ and $\varphi^L_{(\theta,\mathcal{G})}(b) = \bigvee H^b_{(\theta,\mathcal{G})}$.

Definition 4.2. If $p \in B$, the closure of p on L is the element

$$\mathrm{cl}_L(p) := \bigwedge\{q \in B\colon q' \in L,\ p \leq q\}.$$

For every $p \in B$, we put $C_L(p) := \{q \in B\colon q' \in L,\ p \leq q\}$. For every $x \in At(B)$, it is clear that $x \leq \mathrm{cl}_L(p)$ if and only if for any $a \in L$, $x \leq a$ implies $a \wedge p \neq \perp$.

Proposition 4.3. *Let $\mathcal{G}$ be a θ-grill on B. Then the following statements hold.*

(1) $\varphi^L_{\mathcal{G}}(\perp) = \perp = \varphi^L_{(\theta,\mathcal{G})}(\perp)$.

(2) *For every $b \in B$, $\varphi^L_{\mathcal{G}}(b) \leq \varphi^L_{(\theta,\mathcal{G})}(b)$ and, in particular, equality holds if and only if $H^b_{\mathcal{G}} = H^b_{(\theta,\mathcal{G})}$.*

(3) *For every $a, b \in B$, if $a \leq b$, then $\varphi^L_{\mathcal{G}}(a) \leq \varphi^L_{\mathcal{G}}(b)$ and $\varphi^L_{(\theta,\mathcal{G})}(a) \leq \varphi^L_{(\theta,\mathcal{G})}(b)$.*

(4) *For every $b \in \mathcal{G} \cap At(B)$, $b \leq \varphi^L_{\mathcal{G}}(b) \leq \varphi^L_{(\theta,\mathcal{G})}(b)$ and in particular, $\varphi^L_{\mathcal{G}}(b) \leq \varphi^L_{\mathcal{G}}\big(\varphi^L_{\mathcal{G}}(b)\big) \leq \varphi^L_{(\theta,\mathcal{G})}\big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)$.*

Proof. (1) Obvious.

(2) Since $H^b_{\mathcal{G}} \subseteq H^b_{(\theta,\mathcal{G})}$, we conclude that for every $b \in B$, $\varphi^L_{\mathcal{G}}(b) \leq \varphi^L_{(\theta,\mathcal{G})}(b)$. The rest is evident.

(3) Let $x \in At(B)$ and $y \in L \cap \uparrow x$ such that $a \wedge \theta(y) \in \mathcal{G}$. Since $a \leq b$ and $\mathcal{G}$ is a θ- grill on B, we have $\theta\big(a \wedge \theta(y)\big) \leq \theta\big(b \wedge \theta(y)\big)$ and so $b \wedge \theta(y) \in \mathcal{G}$, which implies that $x \in H^b_{(\theta,\mathcal{G})}$. Then, $H^a_{(\theta,\mathcal{G})} \subseteq H^b_{(\theta,\mathcal{G})}$ and so $\varphi^L_{(\theta,\mathcal{G})}(a) \leq \varphi^L_{(\theta,\mathcal{G})}(b)$.

(4) For every $y \in L \cap \uparrow b$, we have $\theta(y) \geq y \geq b$ and so $b \wedge \theta(y) = b \wedge y = b \in \mathcal{G}$, which implies that $b \in H^b_{\mathcal{G}} \cap H^b_{(\theta,\mathcal{G})}$ and so $b \leq \varphi^L_{\mathcal{G}}(b) \leq \varphi^L_{(\theta,\mathcal{G})}(b)$. The rest is evident. □

Example 4.4. Let B be the set of all positive integers which are integral divisors of 70. Thus

$$B = \{1, 2, 5, 7, 10, 14, 35, 70\}.$$

For any $x, y \in B$, let $x \wedge y$ be the greatest common divisor of x and y, let $x \vee y$ be the least common multiple of x and y, and let $x' = \frac{70}{x}$. Then $(B, \vee, \wedge, 1, 70)$ is a Boolean algebra. The function $\theta\colon B \to B$ given by

$$\theta = \begin{pmatrix} 1 & 2 & 5 & 7 & 10 & 14 & 35 & 70 \\ 1 & 2 & 10 & 14 & 10 & 14 & 70 & 70 \end{pmatrix}$$

is an approximation. Consider the subframe $L = \{1, 7, 10, 70\}$ of B and let $\mathcal{G} = \{2, 5, 7, 10, 14, 35, 70\}$. It is easy to see that $\mathcal{G}$ is a θ-grill on B. Let $b = 2$. Then, $H^2_{\mathcal{G}} = \{2, 5\}$, and $H^2_{(\theta,\mathcal{G})} = \{2, 5, 7\}$. Therefore, $\varphi^L_{\mathcal{G}}(2) = 10$ and $\varphi^L_{(\theta,\mathcal{G})}(2) = 70$ and so $\varphi^L_{\mathcal{G}}(2) \lneqq \varphi^L_{(\theta,\mathcal{G})}(2)$.

Proposition 4.5. *Let $\mathcal{G} \in \theta - RG(B)$ and $b \in B$ be given. The following statements hold.*

(1) $\varphi^L_{\mathcal{G}}(b) \leq \mathrm{cl}_L(b)$.

(2) *If there exists a subset A of $At(B)$ such that* $\mathrm{cl}_L\Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big) = \bigvee A$, *then*

$$\mathrm{cl}_L\Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big) = \varphi^L_{(\theta,\mathcal{G})}(b) \text{ and } \Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big)' \in L.$$

(3) *If there exists a subset A of $At(B)$ such that* $\mathrm{cl}_L\Big(\varphi^L_{\mathcal{G}}(b)\Big) = \bigvee A$, *then*

$$\varphi^L_{\mathcal{G}}\Big(\varphi^L_{\mathcal{G}}(b)\Big) \leq \mathrm{cl}_L\Big(\varphi^L_{\mathcal{G}}(b)\Big) = \varphi^L_{\mathcal{G}}(b) \text{ and } \Big(\varphi^L_{\mathcal{G}}(b)\Big)' \in L.$$

Proof. (1) Let $x \in H^b_{\mathcal{G}}$ and $x \nleq \mathrm{cl}_L(b)$. Then there exists an element t of L such that $x \nleq t'$ and $b \leq t'$. Since x is an atom element of B, we conclude that $t \in L \cap \uparrow x$ and $\perp = b \wedge t \in \mathcal{G}$, which is a contradiction. Therefore, $\varphi^L_{\mathcal{G}}(b) \leq \mathrm{cl}_L(b)$.

(2) Let $x \in A$ and $a \in L \cap \uparrow x$ be given. If $a \wedge \varphi^L_{(\theta,\mathcal{G})}(b) = \perp$, then

$$\begin{aligned} a \wedge \varphi^L_{(\theta,\mathcal{G})}(b) = \perp &\Rightarrow \varphi^L_{(\theta,\mathcal{G})}(b) \leq a' \leq x' \\ &\Rightarrow \mathrm{cl}_L\Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big) \leq x' \\ &\Rightarrow x \leq \Big(\mathrm{cl}_L\Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big)\Big)' \\ &\Rightarrow x = \perp, \end{aligned}$$

which is a contradiction. Therefore,

$$\begin{aligned} \perp &\neq a \wedge \varphi^L_{(\theta,\mathcal{G})}(b) \\ &= a \wedge \bigvee\{z \in At(B) \colon \theta(c) \wedge b \in \mathcal{G} \text{ for all } c \in L \cap \uparrow z\} \\ &= \bigvee\{a \wedge z \colon z \in At(B) \text{ and } \theta(c) \wedge b \in \mathcal{G} \text{ for all } c \in L \cap \uparrow z\}. \end{aligned}$$

So that there exists an atom element z of B such that $a \wedge z \neq \perp$ and $\theta(c) \wedge b \in \mathcal{G}$ for all $c \in L \cap \uparrow z$. Hence, $z \leq a$, which implies that $b \wedge \theta(a) \in \mathcal{G}$, that is $x \in H^b_{(\theta,\mathcal{G})}$. Thus,

$$\mathrm{cl}_L\Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big) = \bigvee A \leq \bigvee H^b_{(\theta,\mathcal{G})} = \varphi^L_{(\theta,\mathcal{G})}(b),$$

and since $\varphi^L_{(\theta,\mathcal{G})}(b) \leq \mathrm{cl}_L\Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big)$, we conclude that $\mathrm{cl}_L\Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big) = \varphi^L_{\mathcal{G}}(b)$.

(3) The proof of (3) is similar to the proof of part (2). □

The following example shows that Proposition 4.5 (1) is not true for operation $\varphi^L_{(\theta,\mathcal{G})}$.

Example 4.6. Consider the Boolean algebra, subframe L and θ-grill $\mathcal{G}$ as in Example 4.4. Then, $\varphi^L_{(\theta,\mathcal{G})}(2) = 70$ and $cl_L(2) = 10$. Therefore, $\varphi^L_{(\theta,\mathcal{G})}(2) \nleq cl_L(2)$.

Proposition 4.7. *For every $\mathcal{G} \in \theta - RG(B)$, $\varphi^L_{\mathcal{G}}(b) = \perp$ if and only if $H^b_{\mathcal{G}} = \emptyset$ and also $\varphi^L_{(\theta,\mathcal{G})}(b) = \perp$ if and only if $H^b_{(\theta,\mathcal{G})} = \emptyset$.*

Proof. It is straightforward. □

Proposition 4.8. *For every $\mathcal{G} \in \theta - RG(B)$, the following statements hold.*

1. *If $\varphi^L_{\mathcal{G}}(b) = \perp$ or $\varphi^L_{(\theta,\mathcal{G})}(b) = \perp$, then $b \notin \mathcal{G} \cap At(B)$.*
2. *If $b \notin \mathcal{G}$, then $\varphi^L_{(\theta,\mathcal{G})}(b) = \perp = \varphi^L_{\mathcal{G}}(b)$.*

Proof. It is straightforward. □

Proposition 4.9. *If $\mathcal{G}_1, \mathcal{G}_2 \in \theta - RG(B)$ with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then for all $b \in B$,*

$$\varphi^L_{(\theta,\mathcal{G}_1)}(b) \leq \varphi^L_{(\theta,\mathcal{G}_2)}(b) \text{ and } \varphi^L_{\mathcal{G}_1}(b) \leq \varphi^L_{\mathcal{G}_2}(b).$$

Proof. It is clear. □

Proposition 4.10. *For every $\mathcal{G} \in \theta - RG(B)$ and every $a, b \in B$,*

$$\varphi^L_{(\theta,\mathcal{G})}(a \vee b) = \varphi^L_{(\theta,\mathcal{G})}(a) \vee \varphi^L_{(\theta,\mathcal{G})}(b) \text{ and } \varphi^L_{\mathcal{G}}(a \vee b) = \varphi^L_{\mathcal{G}}(a) \vee \varphi^L_{\mathcal{G}}(b).$$

Proof. By Proposition 4.3(3),

$$\varphi^L_{(\theta,\mathcal{G})}(a) \vee \varphi^L_{(\theta,\mathcal{G})}(b) \leq \varphi^L_{(\theta,\mathcal{G})}(a \vee b).$$

Let $x \in At(B) \setminus \left(H^a_{(\theta,\mathcal{G})} \cup H^b_{(\theta,\mathcal{G})}\right)$ be given. Then, there exist $y_1, y_2 \in L \cap \uparrow x$ such that $a \wedge \theta(y_1) \notin \mathcal{G}$ and $b \wedge \theta(y_2) \notin \mathcal{G}$, which implies that

$$\left(a \wedge \theta(y_1)\right) \vee \left(b \wedge \theta(y_2)\right) \notin \mathcal{G},$$

and since

$$\begin{aligned}(a \vee b) \wedge \theta(y_1 \wedge y_2) &\leq (a \vee b) \wedge \left(\theta(y_1) \wedge \theta(y_2)\right)\\ &= \left(a \wedge \theta(y_1) \wedge \theta(y_2)\right) \vee \left(b \wedge \theta(y_1) \wedge \theta(y_2)\right)\\ &\leq \left(a \wedge \theta(y_1)\right) \vee \left(b \wedge \theta(y_2)\right),\end{aligned}$$

we conclude that $(a \vee b) \wedge \theta(y_1 \wedge y_2) \notin \mathcal{G}$, that is $x \notin H^{a\vee b}_{(\theta,\mathcal{G})}$. Hence, $H^{a\vee b}_{(\theta,\mathcal{G})} \subseteq H^a_{(\theta,\mathcal{G})} \cup H^b_{(\theta,\mathcal{G})}$. Therefore, $\varphi^L_{(\theta,\mathcal{G})}(a \vee b) \leq \varphi^L_{(\theta,\mathcal{G})}(a) \vee \varphi^L_{(\theta,\mathcal{G})}(b)$. □

Proposition 4.11. *For every $\mathcal{G} \in \theta - RG(B)$ and every $(a, b) \in L \times B$, the following statements hold.*

1. $\varphi^L_{(\theta,\mathcal{G})}(b) = \varphi^L_{(\theta,\mathcal{G})}(\theta(a) \wedge b)$ *and* $\varphi^L_{\mathcal{G}}(b) = \varphi^L_{\mathcal{G}}(a \wedge b)$.
2. $\varphi^L_{\mathcal{G}}(a') = \perp$ *and* $\varphi^L_{(\theta,\mathcal{G})}\left(\left(\theta(a)\right)'\right) = \perp$.

Proof. (1) By Proposition 4.3(3), $\varphi^L_{(\theta,\mathcal{G})}(\theta(a) \wedge b) \leq \varphi^L_{(\theta,\mathcal{G})}(b)$. Let $x \in H^b_{(\theta,\mathcal{G})}$ with $a \wedge x \neq \perp$. Since x is an atom element of B, we conclude that $x \leq a \leq \theta(a)$. Now, suppose that $y \in L \cap \uparrow x$, then $y \wedge a \in L \cap \uparrow x$ and $b \wedge \theta(a \wedge y) \in \mathcal{G}$, and so $b \wedge \theta(a) \wedge \theta(y) \in \mathcal{G}$. Then, $x \in H^{\theta(a)\wedge b}_{(\theta,\mathcal{G})}$ and therefore, $\varphi^L_{(\theta,\mathcal{G})}(\theta(a) \wedge b) = \varphi^L_{(\theta,\mathcal{G})}(b)$.

(2) By statement (1) and Proposition 4.3(1), we have

$$\varphi^L_{\mathcal{G}}(a') = \varphi^L_{\mathcal{G}}(a \wedge a') = \varphi^L_{\mathcal{G}}(\perp) = \perp,$$

and

$$\varphi^L_{(\theta,\mathcal{G})}\left(\left(\theta(a)\right)'\right) = \varphi^L_{(\theta,\mathcal{G})}\left(\theta(a) \wedge \left(\theta(a)\right)'\right) = \varphi^L_{(\theta,\mathcal{G})}(\perp) = \perp. \quad \square$$

□

Proposition 4.12. *Let $\mathcal{G} \in \theta - RG(B)$ and $L \setminus \{\perp\} \subseteq \mathcal{G}$. Then the following statements hold.*

1. *The Boolean algebra B is atomic if and only if $\varphi^L_{\mathcal{G}}(\top) = \top$.*
2. *The Boolean algebra B is atomic if and only if $\varphi^L_{(\theta,\mathcal{G})}(\top) = \top$.*
3. *If the Boolean algebra B is atomic, then $b \leq \varphi^L_{(\theta,\mathcal{G})}\left(\theta(b)\right)$ and $b \leq \varphi^L_{\mathcal{G}}(b)$ for every $b \in L$.*

Proof. (1) *Necessity.* To prove that $\varphi^L_{\mathcal{G}}(\top) = \top$, it suffices to show that if $x \in At(B)$, then $x \leq \varphi^L_{\mathcal{G}}(\top)$. Let $x \in At(B)$ be given. If $x \nleq \varphi^L_{\mathcal{G}}(\top)$, then $x \notin H^{\top}_{\mathcal{G}}$, which implies that there exists an element c of $L \cap \uparrow x$ such that $c = c \wedge \top \notin \mathcal{G}$, which is a contradiction.

Sufficiency. By Lemma 3 in [6], it is clear.

(2) It is clear that $\varphi^L_{\mathcal{G}}(\top) = \top$ if and only if $\varphi^L_{(\theta,\mathcal{G})}(\top) = \top$.

(3) For every $b \in L$, by statement (1) and Proposition 4.11, we have

$$b = b \wedge \top = b \wedge \varphi^L_{\mathcal{G}}(\top) = b \wedge \varphi^L_{\mathcal{G}}(b \wedge \top) = b \wedge \varphi^L_{\mathcal{G}}(b),$$

which implies that $b \leq \varphi^L_{\mathcal{G}}(b)$. □

Proposition 4.13. *For every $\mathcal{G} \in \theta - RG(B)$ and every $a, b \in B$, the following statements hold.*

1. $\varphi^L_{\mathcal{G}}(a) \wedge \left(\varphi^L_{\mathcal{G}}(b)\right)' = \varphi^L_{\mathcal{G}}(a \wedge b') \wedge \left(\varphi^L_{\mathcal{G}}(b)\right)'$.
2. $\varphi^L_{(\theta,\mathcal{G})}(a) \wedge \left(\varphi^L_{(\theta,\mathcal{G})}(b)\right)' = \varphi^L_{(\theta,\mathcal{G})}(a \wedge b') \wedge \left(\varphi^L_{(\theta,\mathcal{G})}(b)\right)'$.
3. *If $b \notin \mathcal{G}$, then $\varphi^L_{\mathcal{G}}(a \vee b) = \varphi^L_{\mathcal{G}}(a) = \varphi^L_{\mathcal{G}}(a \wedge b')$.*
4. *If $b \notin \mathcal{G}$, then $\varphi^L_{(\theta,\mathcal{G})}(a \vee b) = \varphi^L_{(\theta,\mathcal{G})}(a) = \varphi^L_{(\theta,\mathcal{G})}(a \wedge b')$.*

Proof. (1) By Propositions 4.3 and 4.10, we have

$$\varphi^L_{\mathcal{G}}(a) = \varphi^L_{\mathcal{G}}(a \wedge b') \vee \varphi^L_{\mathcal{G}}(a \wedge b) \leq \varphi^L_{\mathcal{G}}(a \wedge b') \vee \varphi^L_{\mathcal{G}}(b),$$

which implies that

$$\begin{aligned}\varphi^L_{\mathcal{G}}(a) \wedge \left(\varphi^L_{\mathcal{G}}(b)\right)' &\leq \left(\varphi^L_{\mathcal{G}}(a \wedge b') \vee \varphi^L_{\mathcal{G}}(b)\right) \wedge \left(\varphi^L_{\mathcal{G}}(b)\right)' \\ &= \varphi^L_{\mathcal{G}}(a \wedge b') \wedge \left(\varphi^L_{\mathcal{G}}(b)\right)'.\end{aligned}$$

Also, we have

$$\varphi^L_{\mathcal{G}}(a \wedge b') \wedge \left(\varphi^L_{\mathcal{G}}(b)\right)' \leq \varphi^L_{\mathcal{G}}(a) \wedge \left(\varphi^L_{\mathcal{G}}(b)\right)'.$$

The proof is now complete.

(2) Similar to the proof of statement (1).

(3) By Proposition 4.8 and 4.10, we have

$$\varphi^L_{\mathcal{G}}(a \vee b) = \varphi^L_{\mathcal{G}}(a) \vee \varphi^L_{\mathcal{G}}(b) = \varphi^L_{\mathcal{G}}(a).$$

Again, by Proposition 4.8 and statement (1), we have

$$\varphi^L_{\mathcal{G}}(a) = \varphi^L_{\mathcal{G}}(a) \wedge \left(\varphi^L_{\mathcal{G}}(b)\right)' = \varphi^L_{\mathcal{G}}(a \wedge b') \wedge \left(\varphi^L_{\mathcal{G}}(b)\right)' = \varphi^L_{\mathcal{G}}(a \wedge b').$$

(4) Similar to the proof of statement (3). □

5. ψ-Operator

In this section, we consider new operators $\psi^L_{\mathcal{G}}$ and $\psi^L_{(\theta,\mathcal{G})}$ induced by the operators $\varphi^L_{\mathcal{G}}$ and $\varphi^L_{(\theta,\mathcal{G})}$, respectively. Moreover, we introduce frames $L_{\mathcal{G}}$ and $L_{(\theta,\mathcal{G})}$ induced by $\psi^L_{\mathcal{G}}$ and $\psi^L_{(\theta,\mathcal{G})}$.

Definition 5.1. Let $\mathcal{G} \in \theta - RG(B)$ be given. Operators $\psi^L_{\mathcal{G}} \colon B \to B$ and $\psi^L_{(\theta,\mathcal{G})} \colon B \to B$ are defined as follow for every $b \in B$,

$$\psi^L_{\mathcal{G}}(b) := b \vee \varphi^L_{\mathcal{G}}(b) \text{ and } \psi^L_{(\theta,\mathcal{G})}(b) := b \vee \varphi^L_{(\theta,\mathcal{G})}(b).$$

Several basic facts concerning the behavior of the operators $\psi^L_{\mathcal{G}}$ and $\psi^L_{(\theta,\mathcal{G})}$ are included in the following proposition.

Proposition 5.2. *For every $\mathcal{G} \in \theta - RG(B)$ and every $a, b \in B$, the following statements hold.*

1. $\psi^L_{\mathcal{G}}(\perp) = \perp$ *and* $\psi^L_{(\theta,\mathcal{G})}(\perp) = \perp$.
2. $b \leq \psi^L_{\mathcal{G}}(b) \leq \psi^L_{(\theta,\mathcal{G})}(b)$.
3. *If $a \leq b$, then $\psi^L_{\mathcal{G}}(a) \leq \psi^L_{\mathcal{G}}(b)$ and $\psi^L_{(\theta,\mathcal{G})}(a) \leq \psi^L_{(\theta,\mathcal{G})}(b)$.*

Proof. By Proposition 4.3, it is straightforward. □

Proposition 5.3. *For every $\mathcal{G} \in \theta - RG(B)$ and every $a, b \in B$,*

$$\psi^L_{\mathcal{G}}(a \vee b) = \psi^L_{\mathcal{G}}(a) \vee \psi^L_{\mathcal{G}}(b) \text{ and } \psi^L_{(\theta,\mathcal{G})}(a \vee b) = \psi^L_{(\theta,\mathcal{G})}(a) \vee \psi^L_{(\theta,\mathcal{G})}(b).$$

Proof. This follows from Proposition 4.10. □

Definition 5.4. Corresponding to a θ- grill $\mathcal{G}$ on B, we define

$$L_{\mathcal{G}} := \{b \in B \colon \psi^L_{\mathcal{G}}(b') = b'\}.$$

and

$$L_{(\theta,\mathcal{G})} := \{b \in B \colon \psi^L_{(\theta,\mathcal{G})}(b') = b'\}.$$

Proposition 5.5. *For every $\mathcal{G} \in \theta - RG(B)$, $L_{\mathcal{G}}$ and $L_{(\theta,\mathcal{G})}$ are frames, and $L_{(\theta,\mathcal{G})}$ is a subframe of $L_{\mathcal{G}}$.*

Proof. It is clear that $\top, \perp \in L_{\mathcal{G}}$. For every $a, b \in L_{\mathcal{G}}$,

$$\psi^L_{\mathcal{G}}\Big((a \wedge b)'\Big) = \psi^L_{\mathcal{G}}(a' \vee b') = \psi^L_{\mathcal{G}}(a') \vee \psi^L_{\mathcal{G}}(b') = a' \vee b' = (a \wedge b)'.$$

Hence, $L_{\mathcal{G}}$ is closed under finite meets. Now, suppose that $\{b_\lambda\}_{\lambda\in\Lambda} \subseteq L_{\mathcal{G}}$. For every $\lambda \in \Lambda$, $\psi^L_{\mathcal{G}}(\bigwedge_{\lambda\in\Lambda} b'_\lambda) \leq \psi^L_{\mathcal{G}}(b'_\lambda) = b'_\lambda$. Let $u \in B$ be a lower bound of the set $\{b'_\lambda\}_{\lambda\in\Lambda}$. By Proposition 5.2, $u \leq \psi^L_{\mathcal{G}}(u) \leq \psi^L_{\mathcal{G}}(\bigwedge_{\lambda\in\Lambda} b'_\lambda)$. Therefore,

$$\psi^L_{\mathcal{G}}((\bigvee_{\lambda\in\Lambda} b_\lambda)') = \psi^L_{\mathcal{G}}(\bigwedge_{\lambda\in\Lambda} b'_\lambda) = \bigwedge_{\lambda\in\Lambda} b'_\lambda = (\bigvee_{\lambda\in\Lambda} b_\lambda)'.$$

Hence, $L_{\mathcal{G}}$ is closed under arbitrary join. Since B is complete Boolean algebra, we infer that B is a frame, which implies that $L_{\mathcal{G}}$ is a frame. Similarly, $L_{(\theta,\mathcal{G})}$ is a frame.

If $b \in L_{(\theta,\mathcal{G})}$, then $\varphi^L_{\mathcal{G}}(b') \leq \varphi^L_{(\theta,\mathcal{G})}(b') \leq b'$, which implies that $b \in L_{\mathcal{G}}$. Therefore, $L_{(\theta,\mathcal{G})}$ is a subframe $L_{\mathcal{G}}$. □

Proposition 5.6. *For every $\mathcal{G} \in \theta - RG(B)$ and every $b \in B$, if $b \notin \mathcal{G}$, then $b', \Big(\varphi^L_{\mathcal{G}}(b)\Big)' \in L_{\mathcal{G}}$ and $b', \Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big)' \in L_{(\theta,\mathcal{G})}$.*

Proof. This follows from Propositions 4.3 and 4.8. □

Proposition 5.7. *If $\mathcal{G}_1, \mathcal{G}_2 \in \theta - RG(B)$ with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $L_{\mathcal{G}_2} \subseteq L_{\mathcal{G}_1}$ and $L_{(\theta,\mathcal{G}_2)} \subseteq L_{(\theta,\mathcal{G}_1)}$.*

Proof. By Proposition 4.9,

$$b' \leq \psi^L_{\mathcal{G}_1}(b') \leq \psi^L_{\mathcal{G}_2}(b') = b',$$

for every $b \in L_{\mathcal{G}_2}$. Hence, $L_{\mathcal{G}_2} \subseteq L_{\mathcal{G}_1}$. Similarly, $L_{(\theta,\mathcal{G}_2)} \subseteq L_{(\theta,\mathcal{G}_1)}$. □

Throughout this article, for every $\mathcal{G} \in \theta - RG(B)$, we put

$$\mathfrak{B}(\mathcal{G}, L) := \{x \wedge y' : x \in L,\ y \notin \mathcal{G}\},$$

and

$$\mathfrak{B}(\theta, \mathcal{G}, L) := \{\theta(x) \wedge y' : \theta(x) \in L,\ y \notin \mathcal{G}\}.$$

A base $\mathfrak{B}$ of a frame L is a subset of L such that every element of L is a join of elements of $\mathfrak{B}$.

Proposition 5.8. *For every $\mathcal{G} \in \theta - RG(B)$, the following statements hold.*

1. *$L \subseteq \mathfrak{B}(\mathcal{G}, L) \subseteq L_{\mathcal{G}}$ and $L \subseteq \mathfrak{B}(\theta, \mathcal{G}, L) \subseteq L_{(\theta,\mathcal{G})}$.*
2. *If B is an atomic complete Boolean algebra, then $\mathfrak{B}(\theta, \mathcal{G}, L)$ is a base of $L_{(\theta,\mathcal{G})}$.*
3. *If B is an atomic complete Boolean algebra, then $\mathfrak{B}(\mathcal{G}, L)$ is a base of $L_{\mathcal{G}}$.*
4. *The set $\mathfrak{B}(\mathcal{G}, L)$ is closed under finite meets.*
5. *If $\theta|_L : L \to B$ is a lattice homomorphism, then $\mathfrak{B}(\theta, \mathcal{G}, L)$ is closed under finite meets.*

Proof. (1) For every $v \in L$, since $\perp \notin \mathcal{G}$, we infer that $\theta(v) = \theta(v) \wedge \perp' \in \mathfrak{B}(\theta, \mathcal{G}, L)$. Therefore, $L \subseteq \mathfrak{B}(\theta, \mathcal{G}, L)$. Let $\theta(x) \in L$ and $y \notin \mathcal{G}$. Then, by Propositions 4.8(2) and 4.11(2), $\varphi^L_{(\theta,\mathcal{G})}(y) = \perp$ and $\varphi^L_{(\theta,\mathcal{G})}\Big(\big(\theta(x)\big)'\Big) = \perp$. Therefore,

$$\begin{aligned}
\psi^L_{(\theta,\mathcal{G})}\Big(\big(\theta(x) \wedge y'\big)'\Big) &= \psi^L_{(\theta,\mathcal{G})}\Big(\big(\theta(x)\big)' \vee y\Big) \\
&= \big(\theta(x)\big)' \vee y \vee \varphi^L_{(\theta,\mathcal{G})}\Big(\big(\theta(x)\big)' \vee y\Big) \\
&= \big(\theta(x)\big)' \vee y \vee \varphi^L_{(\theta,\mathcal{G})}\Big(\big(\theta(x)\big)'\Big) \vee \varphi^L_{(\theta,\mathcal{G})}(y) \\
&= \big(\theta(x)\big)' \vee y \\
&= \big(\theta(x) \wedge y'\big)'.
\end{aligned}$$

Hence, $\theta(x) \wedge y' \in L_{(\theta,\mathcal{G})}$, which implies that $\mathfrak{B}(\theta, \mathcal{G}, L) \subseteq L_{(\theta,\mathcal{G})}$. Similarly, $L \subseteq \mathfrak{B}(\mathcal{G}, L) \subseteq L_{\mathcal{G}}$.

(2) Let $u \in L_{(\theta,\mathcal{G})} \setminus \{\perp\}$ and $x \in At(B)$ with $x \leq u$ be given. Then, $u' = \psi^L_{(\theta,\mathcal{G})}(u') \geq \varphi^L_{(\theta,\mathcal{G})}(u')$. If $x \leq \varphi^L_{(\theta,\mathcal{G})}(u')$, then

$$\begin{aligned}
x \leq \varphi^L_{(\theta,\mathcal{G})}(u') \wedge u &\Rightarrow x \vee u' \leq (u \vee u') \wedge (\varphi^L_{(\theta,\mathcal{G})}(u') \vee u') = u' \\
&\Rightarrow x \leq u' \\
&\Rightarrow x = \perp,
\end{aligned}$$

which is a contradiction. Hence, $x \not\leq \varphi^L_{(\theta,\mathcal{G})}(u')$ and there exists an element v of $L \cap \uparrow x$ such that $\theta(v) \wedge u' \notin \mathcal{G}$. We put $a = \theta(v) \wedge u'$. It is clear that $\theta(v) \wedge a' \in \mathfrak{B}(\theta, \mathcal{G}, L)$ and $x \leq \theta(v) \wedge u = \theta(v) \wedge a' \leq u$. So, this equality shows that every element u of $L_{(\theta,\mathcal{G})}$ can be written as the joint of $\mathfrak{B}(\theta, \mathcal{G}, L)$.

(3) Similar to the proof of statement (2).

(4) Let $a_1, a_2 \notin \mathcal{G}$ and $v_1, v_2 \in L$. Then $a_1 \vee a_2 \notin \mathcal{G}$ and $v_1 \wedge v_2 \in L$, which implies that

$$(v_1 \wedge a_1') \wedge (v_2 \wedge a_2') = (v_1 \wedge v_2) \wedge (a_1 \vee a_2)' \in \mathfrak{B}(\mathcal{G}, L).$$

Hence, $\mathfrak{B}(\mathcal{G}, L)$ is closed under finite meets.

(5) Let $a_1, a_2 \notin \mathcal{G}$ and $\theta(v_1), \theta(v_2) \in L$. Then, $a_1 \vee a_2 \notin \mathcal{G}$ and $\theta(v_1 \wedge v_2) = \theta(v_1) \wedge \theta(v_2) \in L$, which implies that

$$\theta(v_1) \wedge a_1' \wedge \theta(v_2) \wedge a_2' = \theta(v_1 \wedge v_2) \wedge (a_1 \vee a_2)' \in \mathfrak{B}(\theta, \mathcal{G}, L).$$

Hence, $\mathfrak{B}(\theta, \mathcal{G}, L)$ is closed under finite meets. $\square$

Proposition 5.9. *Let $\mathcal{G} \in \theta - RG(B)$ and $b \in B$. If $b \leq \varphi^L_{\mathcal{G}}(b)$, then $\mathrm{cl}_L(b) = \mathrm{cl}_L\Big(\varphi^L_{\mathcal{G}}(b)\Big)$.*

Proof. Since $b \leq \varphi^L_{\mathcal{G}}(b)$, we infer that $\mathrm{cl}_L(b) \leq \mathrm{cl}_L\Big(\varphi^L_{\mathcal{G}}(b)\Big)$. On the other hand, by Proposition 4.5, we have $\mathrm{cl}_L\Big(\varphi^L_{\mathcal{G}}(b)\Big) \leq \mathrm{cl}_L\Big(\mathrm{cl}_L(b)\Big) = \mathrm{cl}_L(b)$, then $\mathrm{cl}_L(b) = \mathrm{cl}_L\Big(\varphi^L_{\mathcal{G}}(b)\Big)$. $\square$

Proposition 5.10. *Let B be an atomic complete Boolean algebra. For every $\mathcal{G} \in \theta - RG(B)$ and every $b \in B$, the following statements hold.*

1. *If $b \leq \varphi^L_{\mathcal{G}}(b)$, then $\mathrm{cl}_L(b) = \mathrm{cl}_{L_{\mathcal{G}}}(b)$.*
2. *If $b \leq \varphi^L_{(\theta,\mathcal{G})}(b)$, then $\mathrm{cl}_L(b) = \mathrm{cl}_{L_{(\theta,\mathcal{G})}}(b)$.*
3. $\mathrm{cl}_{L_{\mathcal{G}}}\Big(\varphi^L_{\mathcal{G}}(b)\Big) = \varphi^L_{\mathcal{G}}(b)$.
4. $\mathrm{cl}_{L_{\mathcal{G}}}(b) = \psi^L_{\mathcal{G}}(b)$.

Proof. (1). It is clear that $C_L(b) \subseteq C_{L_{\mathcal{G}}}(b)$, then $\mathrm{cl}_{L_{\mathcal{G}}}(b) \leq \mathrm{cl}_L(b)$. Let $x \in At(B)$ with $x \nleq \mathrm{cl}_{L_{\mathcal{G}}}(b)$ be given. Then, there exists an element a of $C_{L_{\mathcal{G}}}(b)$ such that $x \nleq a$, which implies that $x \leq a'$. Since $a' \in L_{\mathcal{G}}$, we conclude from Proposition 5.8 that there exists a subset $\{x_\lambda \wedge y'_\lambda\}_{\lambda\in\Lambda}$ of $\mathfrak{B}(\mathcal{G}, L)$ such that $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq L$, $\{y_\lambda\}_{\lambda\in\Lambda} \cap \mathcal{G} = \emptyset$ and $a' = \bigvee_{\lambda\in\Lambda}(x_\lambda \wedge y'_\lambda)$, which implies that $a = \bigwedge_{\lambda\in\Lambda}(x'_\lambda \vee y_\lambda)$ and there exists an element λ_0 of Λ such that $x \nleq x'_{\lambda_0} \vee y_{\lambda_0}$. Hence, $x \leq x_{\lambda_0}$ and

$$b \leq a \leq x'_{\lambda_0} \vee y_{\lambda_0} \Rightarrow b \wedge (x_{\lambda_0} \wedge y'_{\lambda_0}) = \bot.$$

Also, we have

$$\begin{array}{rcll} x_{\lambda_0} \wedge b & \leq & x_{\lambda_0} \wedge \varphi^L_{\mathcal{G}}(b), & \text{by hypothesis} \\ & = & x_{\lambda_0} \wedge \varphi^L_{\mathcal{G}}(x_{\lambda_0} \wedge b), & \text{by Proposition 4.11} \\ & = & x_{\lambda_0} \wedge \varphi^L_{\mathcal{G}}(x_{\lambda_0} \wedge b \wedge y'_{\lambda_0}), & \text{by Proposition 4.13} \\ & = & x_{\lambda_0} \wedge \varphi^L_{\mathcal{G}}(\bot) & \\ & = & x_{\lambda_0} \wedge \bot, & \text{by Proposition 4.3} \\ & = & \bot. & \end{array}$$

Hence, $x'_{\lambda_0} \in C_L(b)$ and $x \nleq x'_{\lambda_0}$, which implies that $x \nleq \mathrm{cl}_L(b)$. Therefore, $\mathrm{cl}_L(b) \leq \mathrm{cl}_{L_{\mathcal{G}}}(b)$.

(2) It is clear that $C_L(b) \subseteq C_{L_{(\theta,\mathcal{G})}}(b)$, then $\mathrm{cl}_{L_{(\theta,\mathcal{G})}}(b) \leq \mathrm{cl}_L(b)$. Let $x \in At(B)$ with $x \nleq \mathrm{cl}_{L_{(\theta,\mathcal{G})}}(b)$ be given. Then, there exists an element a of $C_{L_{(\theta,\mathcal{G})}}(b)$ such that $x \nleq a$, which implies that $x \leq a'$. Since $a \in L_{(\theta,\mathcal{G})}$, we conclude from Proposition 5.8(2) that there exists a subset $\{\theta(x_\lambda) \wedge y'_\lambda\}_{\lambda\in\Lambda}$ of $\mathfrak{B}(\theta, \mathcal{G}, L)$ such that $\{\theta(x_\lambda)\}_{\lambda\in\Lambda} \subseteq L$, $\{y_\lambda\}_{\lambda\in\Lambda} \cap \mathcal{G} = \emptyset$ and $a' = \bigvee_{\lambda\in\Lambda}\Big(\theta(x_\lambda) \wedge y'_\lambda\Big)$, which implies that $a = \bigwedge_{\lambda\in\Lambda}\Big((\theta(x)'_\lambda \vee y_\lambda)\Big)$ and there exists an element λ_0 of Λ such that $x \nleq \Big(\theta(x)\Big)'_{\lambda_0} \vee y_{\lambda_0}$. Hence, $x \leq \theta(x_{\lambda_0})$ and

$$b \leq a \leq \theta(x_{\lambda_0})' \vee y_{\lambda_0} \Rightarrow b \wedge \theta(x_{\lambda_0}) \wedge y'_{\lambda_0} = \bot.$$

Also, we have

$$\begin{array}{rcll} \theta(x_{\lambda_0}) \wedge b & \leq & \theta(x_{\lambda_0}) \wedge \varphi^L_{(\theta,\mathcal{G})}(b), & \text{by hypothesis} \\ & = & \theta(x_{\lambda_0}) \wedge \varphi^L_{(\theta,\mathcal{G})}\Big(\theta(x_{\lambda_0}) \wedge b\Big), & \text{by Proposition 4.11} \\ & = & \theta(x_{\lambda_0}) \wedge \varphi^L_{(\theta,\mathcal{G})}\Big(\theta(x_{\lambda_0}) \wedge b \wedge y'_{\lambda_0}\Big), & \text{by Proposition 4.13} \\ & = & \theta(x_{\lambda_0}) \wedge \varphi^L_{(\theta,\mathcal{G})}(\bot) & \\ & = & \theta(x_{\lambda_0}) \wedge \bot, & \text{by Proposition 4.3} \\ & = & \bot. & \end{array}$$

Hence, $\theta(x_{\lambda_0}) \in C_L(b)$ and $x \nleq \theta(x_{\lambda_0})'$, which implies that $x \nleq \mathrm{cl}_L(b)$. Therefore, $\mathrm{cl}_L(b) \leq \mathrm{cl}_{L_{(\theta,\mathcal{G})}}(b)$.

(3) Since

$$\mathrm{cl}_{L_\mathcal{G}}\Big(\varphi^L_\mathcal{G}(b)\Big) = \bigwedge \Big\{ x \in B \colon \varphi^L_\mathcal{G}(b) \vee \varphi^L_\mathcal{G}(x) \leq x \Big\}$$

and, by Proposition 4.5(3),

$$\varphi^L_\mathcal{G}(b) \vee \varphi^L_\mathcal{G}\Big(\varphi^L_\mathcal{G}(b)\Big) = \varphi^L_\mathcal{G}(b),$$

we conclude that $\mathrm{cl}_{L_\mathcal{G}}\Big(\varphi^L_\mathcal{G}(b)\Big) \leq \varphi^L_\mathcal{G}(b)$, which implies that

$$\mathrm{cl}_{L_\mathcal{G}}\Big(\varphi^L_\mathcal{G}(b)\Big) = \varphi^L_\mathcal{G}(b).$$

(4) Since $\mathrm{cl}_{L_\mathcal{G}}(b) = \bigwedge \Big\{ x \in B \colon b \vee \varphi^L_\mathcal{G}(x) \leq x \Big\}$ and

$$b \vee \varphi^L_\mathcal{G}\Big(b \vee \varphi^L_\mathcal{G}(b)\Big) = b \vee \varphi^L_\mathcal{G}(b) \vee \varphi^L_\mathcal{G}\Big(\varphi^L_\mathcal{G}(b)\Big) = b \vee \varphi^L_\mathcal{G}(b),$$

we conclude that $\mathrm{cl}_{L_\mathcal{G}}(b) \leq b \vee \varphi^L_\mathcal{G}(b) = \psi^L_\mathcal{G}(b)$. Let $x \nleq \mathrm{cl}_{L_\mathcal{G}}(b)$ and $x \in At(B)$. Similar to the proof of the statement (1), there exists an element x_{λ_0} of L and $y_{\lambda_0} \notin \mathcal{G}$ such that $x \leq x_{\lambda_0} \wedge y'_{\lambda_0}$ and $b \wedge x_{\lambda_0} \wedge y'_{\lambda_0} = \perp$. Hence, by Proposition 4.13, we have

$$\varphi^L_\mathcal{G}(b \wedge x_{\lambda_0}) = \varphi^L_\mathcal{G}(b \wedge x_{\lambda_0} \wedge y'_{\lambda_0}) = \perp.$$

Also, by Proposition 4.11,

$$x_{\lambda_0} \wedge \varphi^L_\mathcal{G}(b) = x_{\lambda_0} \wedge \varphi^L_\mathcal{G}(b \wedge x_{\lambda_0}) = \perp.$$

Now, if $x \leq \varphi^L_\mathcal{G}(b)$, then

$$\perp \neq x = x \wedge \varphi^L_\mathcal{G}(b) \leq x_{\lambda_0} \wedge \varphi^L_\mathcal{G}(b) = \perp,$$

which is a contradiction. So that $x \nleq \varphi^L_\mathcal{G}(b)$. Therefore, $\varphi^L_\mathcal{G}(b) \leq \mathrm{cl}_{L_\mathcal{G}}(b)$, which implies that $\psi^L_\mathcal{G}(b) = b \vee \varphi^L_\mathcal{G}(b) \leq \mathrm{cl}_{L_\mathcal{G}}(b)$. Hence, $\psi^L_\mathcal{G}(b) = \mathrm{cl}_{L_\mathcal{G}}(b)$. □

6. Frame Suitable for a Grill

In this section, we consider grills and θ-grills satisfying a certain condition and give some properties of them.

Definition 6.1. A frame L is said to be suitable for a grill $\mathcal{G}$ (a θ-grill $\mathcal{G}$) if for every $b \in B$, $b \wedge \Big(\varphi^L_\mathcal{G}(b)\Big)' \notin \mathcal{G}$ $\Big(b \wedge \Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big)' \notin \mathcal{G}\Big)$.

It is easy to see that, if L is suitable for the θ-grill $\mathcal{G}$, then L is suitable for grill $\mathcal{G}$.

Proposition 6.2. *Let L be suitable for the θ- grill $\mathcal{G}$ and $b \in B$. If $b \wedge \varphi^L_{(\theta,\mathcal{G})}(b) = \perp$, then $b \notin \mathcal{G}$.*

Proof. We have

$$\begin{aligned} b &= b \wedge \Big(\varphi^L_{(\theta,\mathcal{G})}(b) \vee \Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big)'\Big) \\ &= \Big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\Big) \vee \Big(b \wedge \Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big)'\Big) \\ &= b \wedge \Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big)' \notin \mathcal{G}. \end{aligned}$$

□

Lemma 6.3. *Let B be an atomic complete Boolean algebra, $\mathcal{G} \in \theta - RG(B)$ and $b \in B$. Then,*

$$\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) \wedge \varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) = \perp.$$

Proof. Let $b \in B$ and $\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) \wedge \varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) \neq \perp$. Then, there exists an atom element x of B such that

$$x \leq \Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) \wedge \varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big),$$

which implies that

$$\begin{array}{rcl} x \leq \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)' & \Rightarrow & x \nleq \varphi^L_{(\theta,\mathcal{G})}(b) \\ & \Rightarrow & x \notin H^b_{(\theta,\mathcal{G})} \\ & \Rightarrow & \theta(b_x) \wedge b \notin \mathcal{G} \ \text{ for some } b_x \in L \cap \uparrow x. \end{array}$$

Since $\mathcal{G}$ is a θ-grill on B and

$$\theta\Big(\theta(b_x) \wedge b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) \quad \leq \quad \theta\Big(\theta(b_x) \wedge b\Big),$$

we conclude that $\theta(b_x) \wedge b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)' \notin \mathcal{G}$, which implies that $x \notin H^{b \wedge \left(\varphi^L_{(\theta,\mathcal{G})}(b)\right)'}_{(\theta,\mathcal{G})}$. Therefore $x \nleq \varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big)$. This is a contradiction. □

Proposition 6.4. *Consider the following statements for a θ-grill $\mathcal{G}$ on B:*

1. *For every $b \in B$, if $b \wedge \varphi^L_{(\theta,\mathcal{G})}(b) = \perp$, then $\varphi^L_{(\theta,\mathcal{G})}(b) = \perp$.*
2. *For every $b \in B$, $\varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) = \perp$.*
3. *For every $b \in B$, $\varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\Big) = \varphi^L_{(\theta,\mathcal{G})}(b)$.*

Then the statement (2) implies the statement (3) and the statement (3) implies the statement (1). If B is the atomic complete Boolean algebra, then statements (1), (2) and (3) are equivalent with L is suitable for the θ-grill $\mathcal{G}$ on B.

Proof. (2) ⇒ (3) Let $b \in B$, then we have

$$\begin{array}{rcl} \varphi^L_{(\theta,\mathcal{G})}(b) & = & \varphi^L_{(\theta,\mathcal{G})}\Big(\Big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\Big) \vee \Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big)\Big) \\ & = & \varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\Big) \vee \varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) \\ & = & \varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\Big), \qquad \text{by the statement (2).} \end{array}$$

(3) ⇒ (1) Let $b \in B$ and $b \wedge \varphi^L_{(\theta,\mathcal{G})}(b) = \perp$, then, by Proposition 4.3 (1) and statement (3), $\varphi^L_{(\theta,\mathcal{G})}(b) = \varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\Big) = \perp$.

Now, suppose that B is the atomic complete Boolean algebra and we show that the statement (1) implies the statement (2). By Lemma 6.3,

$$b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)' \wedge \varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) = \perp.$$

Hence, by statement (1), for every $b \in B$, $\varphi^L_{(\theta,\mathcal{G})}\Big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\Big) = \perp$. □

Proposition 6.5. *Let L be suitable for a θ-grill $\mathcal{G}$ on B, then the following statements hold.*

1. *For every $b \in B$, $\varphi^L_{(\theta,\mathcal{G})}(b) \leq \varphi^L_{(\theta,\mathcal{G})}\Big(\varphi^L_{(\theta,\mathcal{G})}(b)\Big)$.*

2. *For every $b \in B$, $\varphi^L_{\mathcal{G}}(b) \leq \varphi^L_{\mathcal{G}}\big(\varphi^L_{\mathcal{G}}(b)\big)$.*
3. *If B is the atomic complete Boolean algebra, then $\varphi^L_{\mathcal{G}}$ is an idempotent operator, i.e., $\varphi^L_{\mathcal{G}}\big(\varphi^L_{\mathcal{G}}(b)\big) = \varphi^L_{\mathcal{G}}(b)$ for any $b \in B$.*

Proof. (1) Let $b \in B$, then we have

$$\begin{aligned}
\varphi^L_{(\theta,\mathcal{G})}(b) &= \varphi^L_{(\theta,\mathcal{G})}\Big(\big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\big) \vee \big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\big)\Big)\\
&= \varphi^L_{(\theta,\mathcal{G})}\big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\big) \vee \varphi^L_{(\theta,\mathcal{G})}\big(b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)'\big)\\
&= \varphi^L_{(\theta,\mathcal{G})}\big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\big) \vee \perp, \qquad \text{as } b \wedge \big(\varphi^L_{(\theta,\mathcal{G})}(b)\big)' \notin \mathcal{G}\\
&= \varphi^L_{(\theta,\mathcal{G})}\big(b \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\big)\\
&\leq \varphi^L_{(\theta,\mathcal{G})}\big(\varphi^L_{(\theta,\mathcal{G})}(b)\big), \qquad \text{by Proposition 4.3(2).}
\end{aligned}$$

(2) It is similar to (1).
(3). By Proposition 4.5(2), it is clear. □

In the following example, we show that $\varphi^L_{(\theta,\mathcal{G})}$ is not necessary an idempotent operator.

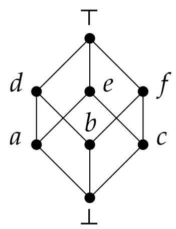
Example 6.6. Consider the Boolean algebra, subframe L and θ-grill $\mathcal{G}$ as in Example 4.6. It is easy to see that $\varphi^L_{(\theta,\mathcal{G})}(5) = 10$ and $\varphi^L_{(\theta,\mathcal{G})}(10) = 70$. Then, $\varphi^L_{(\theta,\mathcal{G})}\big(\varphi^L_{(\theta,\mathcal{G})}(5)\big) = 70$, and so $\varphi^L_{(\theta,\mathcal{G})}(5) \neq \varphi^L_{(\theta,\mathcal{G})}\big(\varphi^L_{(\theta,\mathcal{G})}(5)\big)$.

Proposition 6.7. *Let B be an atomic complete Boolean algebra. If L is suitable for a grill $\mathcal{G}$ on B, then $\mathfrak{B}(\mathcal{G}, L) = L_{\mathcal{G}}$.*

Proof. (1) Let $b \in L_{\mathcal{G}}$, then $b' = \varphi^L_{\mathcal{G}}(b') \vee \big(b' \wedge \big(\varphi^L_{\mathcal{G}}(b')\big)'\big)$, which implies that $b = \big(\varphi^L_{\mathcal{G}}(b')\big)' \wedge \big(b \vee \varphi^L_{\mathcal{G}}(b')\big)$. Since L is suitable for the θ-grill $\mathcal{G}$, we conclude that $b' \wedge \big(\varphi^L_{\mathcal{G}}(b')\big)' \notin \mathcal{G}$. Also, by Proposition 4.5(2), $\big(\varphi^L_{\mathcal{G}}(b')\big)' \in L$. Hence $b \in \mathfrak{B}(\mathcal{G}, L)$, that is $L_{\mathcal{G}} \subseteq \mathfrak{B}(\mathcal{G}, L)$. By Proposition 5.8(2), the proof is then complete. □

The following example shows that Proposition 6.7 is not true for a θ-grill $\mathcal{G}$.

Example 6.8. Take $B = \{\perp, a, b, c, d, e, f, \top\}$. We define the binary relation $\leq$ on B in the following figure.



The function $\theta\colon B \to B$ by

$$\theta = \begin{pmatrix} \perp & a & b & c & d & e & f & \top \\ \perp & d & b & f & d & \top & f & \top \end{pmatrix},$$

is an approximation on B. Consider subframe $L = \{\perp, a, c, e, \top\}$ and θ-grill $\mathcal{G} = \{c, f, e, \top\}$ on B. It is easy to check that L is suitable for θ-grill $\mathcal{G}$. Moreover, $L_{(\theta,\mathcal{G})} = \{\perp, a, c, e, f, \top\}$ and $B(\theta, \mathcal{G}, L) = \{\perp, c, e, f, \top\}$. Then $B(\theta, \mathcal{G}, L) \subset L_{(\theta,\mathcal{G})}$.

Proposition 6.9. *Let B be an atomic complete Boolean algebra, and let L be suitable for a grill $\mathcal{G}$ on B. If $a \in L$, $b \in B$, then*

$$\varphi^L_{\mathcal{G}}(a \wedge b) = \varphi^L_{\mathcal{G}}\big(a \wedge \varphi^L_{\mathcal{G}}(b)\big) = \mathrm{cl}_L\big(a \wedge \varphi^L_{\mathcal{G}}(b)\big).$$

Proof. In view of Proposition 4.11(1), Proposition 6.5(2) and Proposition 4.3(2), which implies that

$$\begin{aligned} a \in L \quad &\Rightarrow \quad a \wedge \varphi^L_{\mathcal{G}}(b) \leq \varphi^L_{\mathcal{G}}(a \wedge b) \\ &\Rightarrow \quad \varphi^L_{\mathcal{G}}\big(a \wedge \varphi^L_{\mathcal{G}}(b)\big) \leq \varphi^L_{\mathcal{G}}(\varphi^L_{\mathcal{G}}\big(a \wedge b)\big) = \varphi^L_{\mathcal{G}}(a \wedge b). \end{aligned}$$

Also, we have $\varphi^L_{\mathcal{G}}\Big(a \wedge b \wedge \big(\varphi^L_{\mathcal{G}}(b)\big)'\Big) \leq \varphi^L_{\mathcal{G}}\Big(b \wedge \big(\varphi^L_{\mathcal{G}}(b)\big)'\Big) = \perp$, as $b \wedge \big(\varphi^L_{\mathcal{G}}(b)\big)' \notin \mathcal{G}$. On the other hand, by Proposition 4.13(1), we have

$$\begin{aligned} \varphi^L_{\mathcal{G}}(a \wedge b) \wedge \Big(\varphi^L_{\mathcal{G}}\big(a \wedge \varphi^L_{\mathcal{G}}(b)\big)\Big)' \quad &\leq \quad \varphi^L_{\mathcal{G}}\Big(a \wedge b \wedge \big(a \wedge \varphi^L_{\mathcal{G}}(b)\big)'\Big) \\ &= \quad \varphi^L_{\mathcal{G}}\Big(a \wedge b \wedge \big(\varphi^L_{\mathcal{G}}(b)\big)'\Big) \\ &= \quad \perp. \end{aligned}$$

Hence, $\varphi^L_{\mathcal{G}}(a \wedge b) \leq \varphi^L_{\mathcal{G}}\big(a \wedge \varphi^L_{\mathcal{G}}(b)\big)$, which implies that $\varphi^L_{\mathcal{G}}(a \wedge b) = \varphi^L_{\mathcal{G}}\big(a \wedge \varphi^L_{\mathcal{G}}(b)\big) \leq \mathrm{cl}_L\big(\varphi^L_{\mathcal{G}}(a \wedge b)\big)$, by Proposition 4.5(2). Now by using Propositions 4.5(2) and 4.11(1) we obtain,

$$a \wedge \varphi^L_{\mathcal{G}}(b) \leq \varphi^L_{\mathcal{G}}(a \wedge b) \Rightarrow \mathrm{cl}_L\big(a \wedge \varphi^L_{\mathcal{G}}(b)\big) \leq \mathrm{cl}_L\big(\varphi^L_{\mathcal{G}}(a \wedge b)\big) = \varphi^L_{\mathcal{G}}(a \wedge b).$$

The proof is now complete. □

Corollary 6.10. *Let B be an atomic complete Boolean algebra, and let L be suitable for a grill $\mathcal{G}$ on B. If $a \in L \setminus \mathcal{G}$, then $a \leq \big(\varphi^L_{\mathcal{G}}(\top)\big)'$.*

Proof. Let $a \in L \setminus \mathcal{G}$. In view of Propositions 4.8(2) and 6.9, we obtain

$$\perp = \varphi^L_{\mathcal{G}}(a) = \varphi^L_{\mathcal{G}}(a \wedge \top) = \varphi^L_{\mathcal{G}}\big(a \wedge \varphi^L_{\mathcal{G}}(\top)\big) = \mathrm{cl}_L\big(a \wedge \varphi^L_{\mathcal{G}}(\top)\big).$$

Hence, $a \wedge \varphi^L_{\mathcal{G}}(\top) = \perp$, which implies that $a \leq \big(\varphi^L_{\mathcal{G}}(\top)\big)'$. □

Corollary 6.11. *Let B be an atomic complete Boolean algebra, and let L be suitable for a grill $\mathcal{G}$ on B. $\varphi^L_{\mathcal{G}}(\top) = \top$ if and only if $L \setminus \{\perp\} \subseteq \mathcal{G}$.*

Proof. By Proposition 4.12(1) and Corollary 6.10, it is clear. □

In the following example, we show that Proposition 6.9 is not true for a θ-grill $\mathcal{G}$ on a atomic complete Boolean algebra.

Example 6.12. Consider the Boolean algebra, subframe L and θ-grill $\mathcal{G}$ as in Example 4.6. Then, it is easy to see that L is a suitable for θ-grill $\mathcal{G}$. For $a = 7$ and $b = 2$, we have $\varphi^L_{(\theta,\mathcal{G})}(7) = 7$ and $\varphi^L_{(\theta,\mathcal{G})}(2) = 70$. Then $\varphi^L_{(\theta,\mathcal{G})}(a \wedge b) = \varphi^L_{(\theta,\mathcal{G})}(1) = 1$ and $\varphi^L_{(\theta,\mathcal{G})}\big(7 \wedge \varphi^L_{(\theta,\mathcal{G})}(2)\big) = \varphi^L_{(\theta,\mathcal{G})}(7) = 7$. Therefore $\varphi^L_{(\theta,\mathcal{G})}(a \wedge b) \neq \varphi^L_{(\theta,\mathcal{G})}\big(a \wedge \varphi^L_{(\theta,\mathcal{G})}(b)\big)$.

Proposition 6.13. *Let B be an atomic complete Boolean algebra , and let L be suitable for a grill $\mathcal{G}$ on B. Suppose that $x \in L$, $y \notin \mathcal{G}$ and $z = x \wedge y'$. If $L \setminus \{\perp\} \subseteq \mathcal{G}$, then*

$$\mathrm{cl}_{L_{\mathcal{G}}} z = \mathrm{cl}_L z = \varphi^L_{\mathcal{G}}(z) = \varphi^L_{\mathcal{G}}(x) = \mathrm{cl}_L x = \mathrm{cl}_{L_{\mathcal{G}}} x.$$

Proof. By Proposition 4.12(2), $x \leq \varphi^L_{\mathcal{G}}(x)$, which implies that

$$\mathrm{cl}_{L_{\mathcal{G}}}(x) = \mathrm{cl}_{L_{\mathcal{G}}}\big(\varphi^L_{\mathcal{G}}(x)\big) = \varphi^L_{\mathcal{G}}(x) = \mathrm{cl}_L\big(\varphi^L_{\mathcal{G}}(x)\big) = \mathrm{cl}_L x,$$

by Proposition 5.10. In view of Proposition 4.13(2), $\varphi^L_{\mathcal{G}}(z) = \varphi^L_{\mathcal{G}}(x)$. By Proposition 4.12(1) and 4.13(1), we have

$$\big(\varphi^L_{\mathcal{G}}(z)\big)' = \varphi^L_{\mathcal{G}}(\top) \wedge \big(\varphi^L_{\mathcal{G}}(z)\big)' = \varphi^L_{\mathcal{G}}(\top \wedge z') \wedge \big(\varphi^L_{\mathcal{G}}(z)\big)' = \varphi^L_{\mathcal{G}}(z') \wedge \big(\varphi^L_{\mathcal{G}}(z)\big)',$$

which implies that $\left(\varphi^L_{\mathcal{G}}(z)\right)' \leq \varphi^L_{\mathcal{G}}(z')$. Also, we have

$$\begin{array}{rcll} \varphi^L_{\mathcal{G}}(z') & = & \varphi^L_{\mathcal{G}}(x' \vee y) & \\ & = & \varphi^L_{\mathcal{G}}(x') \vee \varphi^L_{\mathcal{G}}(y), & \text{by Proposition 4.10} \\ & = & \varphi^L_{\mathcal{G}}(x') \vee \perp, & \text{by Proposition 4.8(2)} \\ & = & \varphi^L_{\mathcal{G}}(x') & \\ & \leq & x', & \text{by Proposition 4.11(2)} \\ & \leq & x' \vee y & \\ & = & z'. & \end{array}$$

Hence, $z \leq \varphi^L_{\mathcal{G}}(z)$. Then, by proposition 5.10, we have

$$\mathrm{cl}_{L_{\mathcal{G}}}(z) = \mathrm{cl}_{L_{\mathcal{G}}}\left(\varphi^L_{\mathcal{G}}(z)\right) = \mathrm{cl}_L\left(\varphi^L_{\mathcal{G}}(z)\right) = \varphi^L_{\mathcal{G}}(z) = \mathrm{cl}_L(z).$$

The proof is now complete. □

Example 4.6 shows that the above proposition is not true for a θ-grill $\mathcal{G}$.

7. Conclusion

This paper has addressed a pointfree version of grills. In the present paper, we defined an approximation θ on a bounded lattice. Also, by an approximation θ, we introduce new concept θ-grill on a bounded lattice. Moreover, we discussed some properties of grills and θ-grills. This may be a part of our future research. There are still a number of fields that can be explored using θ-grills. Due to the fact that the radical ideals and z-ideals of a ring form a frame. So we can expand the concept of grill and θ-grill to algebra.

Acknowledgements

The authors thank the anonymous referees for their valuable comments and suggestions for improving the paper.

References

[1] A. Al-Omari, T. Noiri, Decompositions of continuity via grills, Jord. J. Math. Stat. 4 (2011) 33–46.
[2] A. Al-Omari, T. Noiri, Weak forms of $\mathcal{G} - \alpha$-open sets and decompositions of continuity via grills, Bol. Soc. Paran. Mat. 31 (2013) 19–29.
[3] K.C. Chattopadhyay, W.J. Thron, Extensions of closure spaces, Can. J. Math. 29 (1977) 1277–1286.
[4] G. Choquet, Sur les notions de filtre et grille, Comptes Rendus Acad. Sci. Paris. 224 (1947) 171–173.
[5] A.M.A. El-Latif, R.A. Hosny, Some soft sets via soft grills, South Asian J. Math. 8 (2018) 18–30.
[6] S. Givant, P. Halmos, Introduction to Boolean Algebras, Undergraduate Texts in Mathematics, Springer, 2009.
[7] E. Hatir, S. Jafari, On some new classes of sets and a new decomposition of continuity via grills, J. Adv. Math. Studies. 3 (2010) 33–40.
[8] R.A. Hosny, Remarks on soft topological spaces with soft grill, Far East J. Math. Sci. 86 (2014) 111–128.
[9] P.T. Johnstone, Stone Spaces, Cambridge Studies in Advanced Mathematics 3, Cambridge University Press, Cambridge, 1982.
[10] B. Roy, M.N. Mukherjee, On a typical topology induced by a grill, Soochow J. Math. 33 (2007) 771–786.