Filomat 34:8 (2020), 2693–2704
https://doi.org/10.2298/FIL2008693C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Oscillations in Difference Equations With Continuous Time Caused by Several Deviating Arguments

**George E. Chatzarakis[a], Hajnalka Péics[b], Andrea Rožnjik[b]**

[a]*Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE), Marousi 15122, Athens, Greece*
[b]*University of Novi Sad, Faculty of Civil Engineering, Kozaračka 2/A, Subotica, Serbia*

**Abstract.** Sufficient conditions for the oscillation of all solutions of difference equations with continuous time caused by several deviating arguments are presented. Examples are provided to illustrate the results and compare them to relevant results in the literature.

## 1. Introduction

In the last few decades, the oscillatory behavior of the solutions of difference equations has been researched, see, for example, [1, 3–21] and references therein. Studies on the oscillatory properties of the solutions of difference equations with continuous variables include the work of Golda and Werbowski [5], Shen and Stavroulakis [16], Nowakowska and Werbowski [9–14] and Zhang and Choi [18]. These papers deal with linear functional equations. Karpuz and Öcalan [6], Ladas, Pakula and Wang [7], Zhang and Yan [19], Zhang, Yan and Zhao [21], Zhang, Yan and Choi [20] have investigated the oscillatory behavior of the solutions of constant delay difference equations with continuous time.

In this paper we consider the delay difference equation with continuous time

$$\Delta x(t) + \sum_{i=1}^{m} p_i(t)x(t - k_i(t)) = 0, \quad t \geq t_0, \tag{1}$$

and the (dual) advanced difference equation

$$\nabla x(t) - \sum_{i=1}^{m} p_i(t)x(t + k_i(t)) = 0, \quad t \geq t_0. \tag{2}$$

Equations (1) and (2) are studied under the following assumptions: everywhere $t_0$ is a real number, $m \geq 1$ is an integer, $p_i : [t_0, \infty) \to \mathbf{R}^+$, $1 \leq i \leq m$, are continuous functions and $k_i : [t_0, \infty) \to \{1, 2, \ldots\}$ are piecewise

2010 *Mathematics Subject Classification*. Primary 39A11; Secondary 39B72
*Keywords*. Functional Equations, Difference Equations with Continuous Time, Oscillatory Solutions
Received: 22 August 2019; Revised: 11 November 2019; Accepted: 21 June 2020
Communicated by Jelena Manojlović
The research of H. Péics is supported by the Serbian Ministry of Science, Technology and Development for Scientific Research Grant No. III44006.
*Email addresses:* geaxatz@otenet.gr (George E. Chatzarakis), peics@gf.uns.ac.rs (Hajnalka Péics), andrea@gf.uns.ac.rs (Andrea Rožnjik)

constant functions, $1 \le i \le m$. Throughout this paper, we are going to use the following notation:

$$\alpha_n = \liminf_{t\to\infty} \sum_{\tau=t-k_n(t)}^{t-1} p_n(\tau), \qquad \beta_n = \liminf_{t\to\infty} \sum_{\tau=t+1}^{t+k_n(t)} p_n(\tau), \quad 1 \le n \le m.$$

As usual, $\Delta$ denotes the *forward difference operator* $\Delta x(t) = x(t+1) - x(t)$ and $\nabla$ denotes the *backward difference operator* $\nabla x(t) = x(t) - x(t-1)$.
Set

$$t_{-1}(t_0) = \min_{1\le i\le m} \{\inf\{s - k_i(s) - 1 : s \ge t_0\}\}.$$

It is clear that $t_{-1}(t_0) \le t_0 - 2 < t_0 - 1$. A real-valued function $x : [t_0, \infty) \to \mathbf{R}$ is called a *solution* of the difference equation (1) (or (2)) if it is defined on the interval $[t_{-1}(t_0), \infty)$ (or such that $\sup\{|x(s)| : s \ge t^*\} > 0$ for any $t^* \ge t_0$) and $x$ satisfies equation (1) (or (2)) for any $t \ge t_0$. Such solution is called *oscillatory* if there exists a sequence of points $\{t_n\}_{n=1}^{\infty}$, $t_n \in [t_0, \infty)$, such that $\lim_{n\to\infty} t_n = \infty$ and $x(t_n) \cdot x(t_{n+1}) \le 0$ for $n = 1, 2, \ldots$. Otherwise, the solution is *non-oscillatory*.

The delay difference equation with constant delays

$$\Delta x(t) + \sum_{i=1}^{m} p_i(t) x(t - k_i) = 0, \quad t \ge t_0 \tag{3}$$

is special case of equation (1).

**Remark 1.1.** *Replacing $t + 1$ by $t$ in equation* (3), *this becomes*

$$x(t) - x(t-1) + \sum_{i=1}^{m} P_i(t) x(t - K_i) = 0, \quad t \ge t_0 + 1, \tag{4}$$

*where $P_i(t) = p_i(t-1)$ and $K_i = k_i + 1$ for $i = 1, 2, \ldots, m$.*

The research in this article advances the investigation of delay difference equations with continuous time in Chatzarakis, Győri, Péics and Stavroulakis [2]. It is motivated by the study of the oscillatory properties of the solutions of discrete difference equations with variable coefficients by Chatzarakis, Péics and Stavroulakis [4] as analogues of the oscillatory behavior of the solutions of corresponding differential equations observed by Ladas and Stavroulakis [8]. We establish sufficient conditions which ensure that all solutions of equations (1) and (2) are oscillatory. We illustrate our results through examples in which we compare them to relevant results in the literature.

## 2. Preliminaries

In 1982, Ladas and Stavroulakis [8], studied the differential equation with variable coefficients and with retarded (advanced) arguments of the form

$$y'(t) + \sum_{i=1}^{n} p_i(t) y(t - \tau_i) = 0 \quad \left( y'(t) - \sum_{i=1}^{n} p_i(t) y(t + \tau_i) = 0 \right), \tag{5}$$

and established the following theorem.

**Theorem A ([8], Theorems 5.1-5.2).** *Assume that*

$$\liminf_{t\to\infty} \int_{t-\tau_i/2}^{t} p_i(s)ds > 0 \quad \left( \liminf_{t\to\infty} \int_{t}^{t+\tau_i/2} p_i(s)ds > 0 \right), \quad i = 1, 2, \ldots, n, \quad and$$

$$\left[\prod_{i=1}^{n}\left(\sum_{j=1}^{n}\liminf_{t\to\infty}\int_{t-\tau_j}^{t}p_i(s)ds\right)\right]^{1/n} > \frac{1}{e}\quad\left(\left[\prod_{i=1}^{n}\left(\sum_{j=1}^{n}\liminf_{t\to\infty}\int_{t}^{t+\tau_j}p_i(s)ds\right)\right]^{1/n} > \frac{1}{e}\right).$$

*Then all solutions of* (5) *are oscillatory.*

In 2014, Chatzarakis, Peics and Stavroulakis [4], investigated the difference equation with several retarded arguments

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \tag{6}$$

and established the following theorem.

**Theorem B ([4], Theorem 2.1).** *Assume that $(\tau_i(n))$ are increasing sequences of integers such that $\tau_i(n) \le n-1$, $n \in \mathbf{N}_0$, $\lim_{n\to\infty}\tau_i(n) = \infty$, $1 \le i \le m$, and $(p_i(n))$, $1 \le i \le m$, are sequences of positive real numbers. Define $a_i$, $1 \le i \le m$, by $a_i = \liminf_{n\to\infty}\sum_{j=\tau_i(n)}^{n-1} p_i(j)$. If $a_i > 0$, $1 \le i \le m$, and*

$$\left[\prod_{i=1}^{m}\left(a_i + \sum_{j=1,j\neq i}^{m}\liminf_{n\to\infty}\sum_{k=\tau_j(n)}^{n-1}p_i(k)\right)\right]^{1/m} > \frac{1}{e},$$

*then all solutions of* (6) *oscillate.*

In the same paper, the authors investigated the difference equation with several advanced arguments

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0, \tag{7}$$

and established the following theorem.

**Theorem C ([4], Theorem 3.1).** *Assume that $(p_i(n))$, $1 \le i \le m$, are sequences of positive real numbers and $(\sigma_i(n))$ are increasing sequences of the integers such that $\sigma_i(n) \ge n+1$, $n \in \mathbf{N}$, $1 \le i \le m$. Define $b_i$, $1 \le i \le m$, by $b_i = \liminf_{n\to\infty}\sum_{j=n+1}^{\sigma_i(n)} p_i(j)$. If*

$$\left[\prod_{i=1}^{m}\left(b_i + \sum_{j=1,j\neq i}^{m}\liminf_{n\to\infty}\sum_{k=n+1}^{\sigma_j(n)}p_i(k)\right)\right]^{1/m} > \frac{1}{e},$$

*then all solutions of* (7) *oscillate.*

Beside the mentioned literatures, which have been the motivation for our research, we state comparable results from the literature to our results.

Equation (4) is a special case of the linear functional equation of the form

$$x(g^s(t)) = \sum_{i=0}^{s-1} Q_i(t)x(g^i(t)) + \sum_{i=s+1}^{M+1} Q_i(t)x(g^i(t)), \tag{8}$$

where $M \ge 1$, $s \in \{1,2,\dots,M\}$, $g^0(t) = t$, $g^{i+1}(t) = g(g^i(t))$, $i = 0,1,\dots$ and $Q_i$ are nonnegative real-valued functions for $i = 0,1,\dots,s-1,s+1,\dots,M+1$, that was studied by Nowakowska and Werbowski [10–12]. Nowakowska and Werbowski in [9] considered equation (8) for $s = 1$. Equation (8) yields (4) making the following choices for the parameters and coefficients: $M = m$, $s = 1$, $g(t) = t-1$, $Q_0(t) \equiv 1$ and $Q_i(t) = P_{i-1}(t)$, $i = 2,3,\dots,M+1$. Hence, the sufficient conditions for oscillation developed by Nowakowska and Werbowski [10], for equation (4), have been stated in the following theorem.

**Theorem D ([10], Theorems 1-3).** *Let $B_1(t) = \sum_{i=1}^{m-1} P_i(t)P_{m-i}(t-i) + P_m(t)$. If*

$$\liminf_{t\to\infty} \sum_{i=0}^{m-1} B_1(t-i) > \left(\frac{m}{m+1}\right)^{m+1} \tag{9}$$

*or*

$$\limsup_{t\to\infty} \sum_{i=0}^{m} B_1(t-i) \prod_{j=1}^{m} \left(1 + \sum_{k=1}^{i} B_1(t-(k+m))\right) > 1 \tag{10}$$

*or*

$$\begin{gathered} \sum_{i=0}^{m-1} B_1(t-i) \geq \delta, \quad \delta < \left(\frac{m}{m+1}\right)^{m+1}, \\ \limsup_{t\to\infty} \sum_{i=0}^{m} B_1(t-i) \prod_{j=1}^{m} \left(1 + \sum_{k=1}^{i} B_1(t-(k+m))\right) > 1 - \delta^{m+1}, \end{gathered} \tag{11}$$

*then all solutions of* (4) *are oscillatory.*

The sufficient conditions for the oscillation obtained by Nowakowska and Werbowski [11] formulated for the equation (4) have a form formulated in terms of the following theorem.

**Theorem E ([11], Theorems 2, 4).** *Let*

$$B_2(t) = \sum_{i=1}^{m} P_i(t). \tag{12}$$

*If*

$$\liminf_{t\to\infty} B_2(t) > \frac{1}{4} \tag{13}$$

*or*

$$\limsup_{t\to\infty} \left(B_2(t) + B_2(t-1) + B_2(t-1)B_2(t-2)\right) > 1 \tag{14}$$

*or*

$$\begin{gathered} B_2(t) \geq \delta > 0, \quad \delta < \frac{1}{4}, \\ \limsup_{t\to\infty} \left(B_2(t) + B_2(t-1) + B_2(t-1)B_2(t-2)\right) > 1 - \delta^2, \end{gathered} \tag{15}$$

*then all solutions of* (4) *are oscillatory.*

The sufficient condition for the oscillation of the solutions of equation (4) presented in Theorem 1 in Nowakowska and Werbowski [12] is same as (13), but conditions in Theorems 2 and 3 have the following form.

**Theorem F ([12], Theorems 2-3).** *Let $B_2$ be defined as* (12). *If*

$$\limsup_{t\to\infty} \left(B_2(t) + B_2(t+1) + B_2(t+1)B_2(t+2)\right) > 1 \tag{16}$$

*or*

$$\begin{aligned} B_2(t) \geq \delta > 0, \quad \delta < \frac{1}{4}, \\ \limsup_{t\to\infty} \left(B_2(t) + B_2(t+1) + B_2(t+1)B_2(t+2)\right) > 1 - \delta^2, \end{aligned} \tag{17}$$

*then all solutions of* (4) *are oscillatory.*

The oscillatory conditions obtained by Nowakowska and Werbowski [9] for equation (4) have the form of conditions (9) and (13).

Nowakowska and Werbowski in [13] considered the iterative equation

$$x(g(t)) = \sum_{i=1}^{M} A_i(t)x(g^{i+1}(t)) + \sum_{j=0}^{l} \bar{A}_j(t)x(g^{-i}(t)), \tag{18}$$

where $M \geq 1$, $l \geq 0$, $g^0(t) = t$, $g^{i+1}(t) = g(g^i(t))$, $i = 0, 1, \dots$, $g^{-1}$ is the inverse function of $g$, $A_i$, $i = 1, 2, \dots, M$, and $\bar{A}_i$, $j = 1, 2, \dots, l$, are nonnegative real-valued functions and $\bar{A}_0$ is positive real-valued function. For $M = m$, $l = 0$, $g(t) = t - 1$, $\bar{A}_0(t) \equiv 1$ and $A_i(t) = P_i(t)$, $i = 1, 2, \dots, M$, equation (18) yields (4). Therefore, Theorems 1 and 3 in [13] provide conditions which ensure that equation (4) has only oscillatory solutions and are equivalent to conditions stated in Theorem D.

## 3. Delay Difference Equations

The following two results are the discrete analogues of the results obtained by Ladas and Stavroulakis and were stated in Theorem A for delay differential equations. At the same time, those results are the continuous analogues of the results in Theorem B, formulated by Chatzarakis, Péics and Stavroulakis.

**Theorem 3.1.** *The condition*

$$\liminf_{t\to\infty} \sum_{\tau = t - \left[\frac{k_n(t)}{2}\right]}^{t-1} p_n(\tau) > 0 \quad \textit{for } k_n(t) \geq 2 \textit{ eventually}, \quad n = 1, 2, \dots, m$$

*in conjunction with the condition*

$$\left( \prod_{i=1}^{m} \left( \alpha_i + \sum_{n=1, n\neq i}^{m} \liminf_{t\to\infty} \sum_{\tau = t - k_n(t)}^{t-1} p_i(\tau) \right) \right)^{\frac{1}{m}} > \frac{1}{e} \tag{19}$$

*or*

$$\frac{1}{m} \left( \sum_{i=1}^{m} \sqrt{\alpha_i} \right)^2 = \frac{1}{m} \left( \sum_{i=1}^{m} \left( \liminf_{t\to\infty} \sum_{\tau = t - k_i(t)}^{t-1} p_i(\tau) \right)^{\frac{1}{2}} \right)^2 > \frac{1}{e} \tag{20}$$

*imply that all solutions of* (1) *are oscillatory.*

*Proof.* It suffices to show that (1) does not have an eventually positive solution. To this end suppose that $x(t)$ is a solution of (1) such that $x(t) > 0$ for $t \geq t_1 \geq t_{-1}(t_0)$. Choose a $t_2 > t_1$ such that $x(t - k_i(t)) > 0$, $i = 1, 2, \dots, m$, for $t \geq t_2$. Then from (1) we get that $\Delta x(t) < 0$, i.e. $x(t+1) < x(t)$ for $t \geq t_2$. Next choose a $t_3 > t_2$ such that $x(t) < x(t - k_i(t))$, $i = 1, 2, \dots, m$, for $t \geq t_3$.
Set

$$\omega_i(t) = \frac{x(t - k_i(t))}{x(t)}, \quad i = 1, 2, \dots, m, \quad \text{for} \quad t \geq t_3 \tag{21}$$

and $\ell_i = \liminf_{t\to\infty} \omega_i(t)$, $i = 1, 2, \ldots, m$. Then $\omega_i(t) > 1$, $\ell_i \geq 1$ for $i = 1, 2, \ldots, m$. Dividing both sides of equation (1) by $x(t)$ for $t \geq t_3$, using (21) and summing both sides of the obtained equation from $t - k_n(t)$ to $t - 1$ for $n = 1, 2, \ldots, m$, we get

$$\sum_{\tau=t-k_n(t)}^{t-1} \frac{\Delta x(\tau)}{x(\tau)} + \sum_{i=1}^{m} \sum_{\tau=t-k_n(t)}^{t-1} p_i(\tau)\omega_i(\tau) = 0, \quad n = 1, 2, \ldots, m. \tag{22}$$

In similar way as in the proof of the Theorem B, we can get that

$$\sum_{\tau=t-k_n(t)}^{t-1} \frac{\Delta x(\tau)}{x(\tau)} \geq -\ln \omega_n(t), \quad n = 1, 2, \ldots, m. \tag{23}$$

Combining (22) and (23), we obtain

$$\ln \omega_n(t) \geq \sum_{i=1}^{m} \sum_{\tau=t-k_n(t)}^{t-1} p_i(\tau)\omega_i(\tau), \quad n = 1, 2, \ldots, m. \tag{24}$$

Taking the limit inferior on both sides of inequalities (24), we obtain

$$\ln \ell_n \geq \sum_{i=1}^{m} \ell_i \left( \liminf_{t\to\infty} \sum_{\tau=t-k_n(t)}^{t-1} p_i(\tau) \right), \quad n = 1, 2, \ldots, m. \tag{25}$$

Case 1. $\ell_n < \infty$ for $n = 1, 2, \ldots, m$.

First, assume that condition (19) holds. Summing up both sides of the inequality (25) from 1 to $m$ we get the inequality

$$\sum_{n=1}^{m} \ln \ell_n \geq \sum_{i=1}^{m} \ell_i \left( \sum_{n=1}^{m} \liminf_{t\to\infty} \sum_{\tau=t-k_n(t)}^{t-1} p_i(\tau) \right).$$

Set

$$F(\ell_1, \ell_2, \ldots, \ell_m) = \sum_{n=1}^{m} \ln \ell_n - \sum_{i=1}^{m} \ell_i \left( \sum_{n=1}^{m} \liminf_{t\to\infty} \sum_{\tau=t-k_n(t)}^{t-1} p_i(\tau) \right).$$

At the critical point

$$\left( \left( \sum_{n=1}^{m} \liminf_{t\to\infty} \sum_{\tau=t-k_n(t)}^{t-1} p_1(\tau) \right)^{-1}, \ldots, \left( \sum_{n=1}^{m} \liminf_{t\to\infty} \sum_{\tau=t-k_n(t)}^{t-1} p_m(\tau) \right)^{-1} \right)$$

the function $F$ has a maximum and

$$F_{max} = \sum_{i=1}^{m} \left( -\ln \left( \sum_{n=1}^{m} \liminf_{t\to\infty} \sum_{\tau=t-k_n(t)}^{t-1} p_i(\tau) \right) \right) - m \geq 0.$$

Similar argumentation as in the proof of the Theorem B leads to

$$\left( \prod_{i=1}^{m} \left( \alpha_i + \sum_{n=1, n\neq i}^{m} \liminf_{t\to\infty} \sum_{\tau=t-k_n(t)}^{t-1} p_i(\tau) \right) \right)^{\frac{1}{m}} \leq \frac{1}{e},$$

which contradicts hypothesis (19).

Now, assume that condition 20 holds. Using (25) and the fact that $\frac{1}{e} \geq \frac{\ln \ell_n}{\ell_n}$, $n = 1, 2, \ldots, m$, we get

$$\frac{1}{e} \geq \sum_{i=1}^{m} \frac{\ell_i}{\ell_n} \left( \liminf_{t\to\infty} \sum_{\tau=t-k_n(t)}^{t-1} p_i(\tau) \right), \quad n = 1, 2, \ldots, m.$$

Summing up the above inequalities, we obtain

$$\frac{m}{e} \geq \sum_{i=1}^{m} \alpha_i + \sum_{\substack{i<j \\ i,j=1}}^{m} \left( \frac{\ell_i}{\ell_j} \liminf_{t\to\infty} \sum_{\tau=t-k_j(t)}^{t-1} p_i(\tau) + \frac{\ell_j}{\ell_i} \liminf_{t\to\infty} \sum_{\tau=t-k_i(t)}^{t-1} p_j(\tau) \right).$$

Using the relation between arithmetic and geometric means we have

$$\frac{m}{e} \geq \sum_{i=1}^{m} \alpha_i + 2 \sum_{\substack{i<j \\ i,j=1}}^{m} \sqrt{\left( \liminf_{t\to\infty} \sum_{\tau=t-k_j(t)}^{t-1} p_i(\tau) \right) \left( \liminf_{t\to\infty} \sum_{\tau=t-k_i(t)}^{t-1} p_j(\tau) \right)}$$

and

$$\frac{m}{e} \geq \left( \sum_{i=1}^{m} \left( \liminf_{t\to\infty} \sum_{\tau=t-k_i(t)}^{t-1} p_i(\tau) \right)^{\frac{1}{2}} \right)^2 = \left( \sum_{i=1}^{m} (\alpha_i)^{\frac{1}{2}} \right)^2,$$

which contradicts hypothesis (20).

Case 2. $\ell_n = \infty$ for some $n$, $n = 1, 2, \ldots, m$. That is,

$$\lim_{t\to\infty} \frac{x(t - k_n(t))}{x(t)} = +\infty \quad \text{for some } n, \quad n = 1, 2, \ldots, m. \tag{26}$$

From equation (1) and for the value $n$ for which (26) holds, we have

$$\Delta x(t) + p_n(t) x(t - k_n(t)) \leq 0, \quad t \geq t_3. \tag{27}$$

If $k_n(t) = 1$, we have $x(t+1) - x(t) + p_n(t)x(t-1) \leq 0, t \geq t_3$. Dividing both sides of the last inequality by $x(t)$ we obtain

$$\frac{x(t+1)}{x(t)} - 1 + p_n(t) \frac{x(t-1)}{x(t)} \leq 0, \quad t \geq t_3. \tag{28}$$

On the other hand

$$\lim_{t\to\infty} \frac{x(t-1)}{x(t)} = \lim_{t\to\infty} \frac{x(t)}{x(t+1)} = +\infty,$$

which is contradiction to inequality (28).

If $k_n(t) \geq 2$, summing up both sides of the inequality (27) from $t - \left[\frac{k_n(t)}{2}\right]$ to $t - 1$, we obtain

$$\sum_{\tau=t-\left[\frac{k_n(t)}{2}\right]}^{t-1} \Delta x(\tau) + \sum_{\tau=t-\left[\frac{k_n(t)}{2}\right]}^{t-1} p_n(\tau) x(\tau - k_n(t)) \leq 0, \quad t \geq t_3.$$

Due to $\Delta x(t) < 0$, we have that

$$x(t) - x\left( t - \left[ \frac{k_n(t)}{2} \right] \right) + x(t - k_n(t)) \sum_{\tau=t-\left[\frac{k_n(t)}{2}\right]}^{t-1} p_n(\tau) \leq 0, \quad t \geq t_3.$$

Similarly as in the proof of Theorem 2.1 in [8], (26) leads to contradiction. The proof of the theorem is complete. □

**Advanced Difference Equations**

The following results are the discrete analogues of the results by Ladas and Stavroulakis stated in Theorem A, for advanced differential equations. At the same time, they are the continuous analogues of the results by Chatzarakis, Péics and Stavroulakis stated in Theorem C. The proofs follow a similar procedure as those in the previous section ant thus, are omitted.

**Theorem 3.2.** *If*

$$\liminf_{t\to\infty} \sum_{\tau=t+1}^{t+\left[\frac{k_n(t)}{2}\right]} p_n(\tau) > 0 \quad \text{for } k_n(t) \geq 2 \text{ eventually}, \quad n = 1, 2, \ldots, m$$

*and the condition*

$$\left(\prod_{i=1}^{m}\left(\beta_i + \sum_{n=1, n\neq i}^{m} \liminf_{t\to\infty} \sum_{\tau=t+1}^{t+k_n(t)} p_i(\tau)\right)\right)^{\frac{1}{m}} > \frac{1}{e}$$

*or*

$$\frac{1}{m}\left(\sum_{i=1}^{m}\sqrt{\beta_i}\right)^2 = \frac{1}{m}\left(\sum_{i=1}^{m}\left(\liminf_{t\to\infty}\sum_{\tau=t+1}^{t+k_i(t)} p_i(\tau)\right)^{\frac{1}{2}}\right)^2 > \frac{1}{e}$$

*holds, then all solutions of* (2) *are oscillatory.*

## 4. Examples and Comparisons

The independence of our conditions for delay difference equations from relevant conditions in literature is illustrated by considering the special case (4). Therefore, we reformulate the above presented results for it.

**Corollary 4.1.** *For constants $K_i \in \{2, 3, \ldots\}$, and positive and continuous functions $P_i$, $i = 1, 2, \ldots, m$, the condition*

$$\liminf_{t\to\infty} \sum_{\tau=t-\left[\frac{K_i-1}{2}\right]}^{t-1} P_i(\tau) > 0 \quad \text{for } K_i \geq 3, \quad i = 1, 2, \ldots, m \tag{29}$$

*in conjunction with the condition*

$$\left(\prod_{i=1}^{m}\left(\sum_{j=1}^{m}\liminf_{t\to\infty}\sum_{\tau=t-K_j+1}^{t-1} P_i(\tau)\right)\right)^{\frac{1}{m}} > \frac{1}{e}, \tag{30}$$

*or*

$$\frac{1}{m}\left(\sum_{i=1}^{m}\left(\liminf_{t\to\infty}\sum_{\tau=t-K_i+1}^{t-1} P_i(\tau)\right)^{\frac{1}{2}}\right)^2 > \frac{1}{e} \tag{31}$$

*imply that every solution of* (4) *oscillates.*

Let us first show that conditions (19) and (20) are independent. We illustrate it on equation (4) with constant coefficients, and therefore we reformulate the presented results for the constant coefficients and constant delays.

**Corollary 4.2.** *If, for positive constants $P_i$, $i = 1, 2, \ldots, m$, and constants $K_i \in \{2, 3, \ldots\}$,*

$$\left(\prod_{i=1}^{m} P_i\right)^{\frac{1}{m}} \left(\sum_{n=1}^{m}(K_n - 1)\right) > \frac{1}{e} \tag{32}$$

*or*

$$\frac{1}{m}\left(\sum_{i=1}^{m} \sqrt{P_i(K_i - 1)}\right)^2 > \frac{1}{e}, \tag{33}$$

*then all solutions of the equation*

$$x(t) - x(t-1) + \sum_{i=1}^{m} P_i x(t - K_i) = 0, \ t \geq t_0 + 1$$

*are oscillatory.*

Analogous statements as Remark 1.1 and Corolaries 4.1 and 4.2 can be formulated for advanced equations.

The following example illustrates that conditions (32) and (33) are independent, pointing out that conditions (19) and (20) are independent.

**Example 4.3.** *For the delay difference equation*

$$x(t) - x(t-1) + \frac{3}{25}x(t-2) + \frac{13}{100}x(t-3) = 0, \quad t \geq 0, \tag{34}$$

*the condition* (32) *is satisfied and* (33) *is not fulfilled, but for the delay difference equation*

$$x(t) - x(t-1) + \frac{1}{10}x(t-2) + \frac{3}{20}x(t-3) = 0, \quad t \geq 0, \tag{35}$$

*condition* (32) *is not fulfilled while* (33) *is.*
*Namely,*

| *Left side of conditions* (32), (33) | *Value for* (34) | *Value for* (35) |
|---|---|---|
| $\sqrt{P_1 P_2}\,((K_1 - 1) + ((K_2 - 1))$ | $= \frac{3\sqrt{39}}{50} \approx 0.3747 > \frac{1}{e}$ | $= \frac{3\sqrt{6}}{20} \approx 0.367423 < \frac{1}{e}$ |
| $\frac{1}{2}\left(\sqrt{P_1(K_1 - 1)} + \sqrt{P_2(K_2 - 1)}\right)^2$ | $\approx 0.366635 < \frac{1}{e}$ | $\approx 0.373205 > \frac{1}{e}$ |

The following example illustrates the independence of conditions (30) and (31) from the conditions in papers [9–13], i.e., from the conditions in Theorems D, E, F.

**Example 4.4.** *Consider the delay difference equation*

$$x(t) - x(t-1) + \frac{13}{100}\,x(t-2) + \frac{\sin 2t + 8}{60}\,x(t-3) = 0, \quad t \geq 0. \tag{36}$$

*Here $m = 2$, $P_1(t) \equiv \frac{13}{100}$, $P_2(t) = \frac{\sin 2t + 8}{60}$, $K_1 = 2$ and $K_2 = 3$, thus*

$$\liminf_{t\to\infty} P_2(t-1) = \frac{2}{15} - \frac{1}{60} = \frac{7}{60} > 0$$

*and condition* (29) *is fulfilled. Since,*

$$\liminf_{t\to\infty}\sum_{\tau=t-K_1+1}^{t-1}P_1(\tau)=\frac{13}{100},\qquad \liminf_{t\to\infty}\sum_{\tau=t-K_2+1}^{t-1}P_1(\tau)=\frac{13}{50},$$

$$\liminf_{t\to\infty}\sum_{\tau=t-K_1+1}^{t-1}P_2(\tau)=\frac{7}{60},\qquad \liminf_{t\to\infty}\sum_{\tau=t-K_2+1}^{t-1}P_2(\tau)=\frac{4}{15}-\frac{1}{30}\cos 1\approx 0.248657,$$

*condition* (30) *takes the form*

$$\left(\prod_{i=1}^{2}\left(\sum_{j=1}^{2}\liminf_{t\to\infty}\sum_{\tau=t-K_j+1}^{t-1}P_i(\tau)\right)\right)^{\frac{1}{2}}\approx 0.37746>\frac{1}{e}\approx 0.367879.$$

*That means that condition* (30) *is fulfilled, so every solution of* (36) *oscillates. Furthermore, condition* (31) *gives*

$$\frac{1}{2}\left(\sum_{i=1}^{2}\sqrt{\liminf_{t\to\infty}\sum_{\tau=t-K_i+1}^{t-1}P_i(\tau)}\right)^2\approx 0.369121>\frac{1}{e}$$

*and it is fulfilled, too.*

*On the other hand, concerning the conditions in Theorem D, we have that*

$$B_1(t)=P_1(t)P_1(t-1)+P_2(t)=\frac{4507}{30000}+\frac{1}{60}\sin 2t.$$

*Thus, conditions* (9) *and* (10) *give*

$$\liminf_{t\to\infty}\left(B_1(t)+B_1(t-1)\right)>\left(\frac{2}{3}\right)^3 \quad \textit{and}$$

$$\limsup_{t\to\infty}\left(B_1(t)+B_1(t-1)\left(1+B_1(t-3)\right)^2+B_1(t-2)\left(1+B_1(t-3)+B_1(t-4)\right)^2\right)>1.$$

*However,* $\left(\frac{2}{3}\right)^3\approx 0.296296,$

$$\liminf_{t\to\infty}\left(B_1(t)+B_1(t-1)\right)=\liminf_{t\to\infty}\left(\frac{4507}{15000}+\frac{1}{60}\left(\sin 2t+\sin\left(2t-2\right)\right)\right)\approx 0.282457$$

*and*

$$\limsup_{t\to\infty}\sum_{i=0}^{2}B_1(t-i)\prod_{j=1}^{2}\left(1+\sum_{k=1}^{i}B_1(t-(k+2))\right)\approx 0.604293, \tag{37}$$

*which means that* (9) *and* (10) *are not satisfied.*

*For the sake of showing that condition* (11) *is not fulfilled, as well, let us take* $\delta$ *such that*

$$B_1(t)+B_1(t-1)\geq\delta \quad \textit{and} \quad \delta<\frac{8}{27},$$

*i.e.,* $\delta\leq\frac{4507}{15000}-\frac{1}{30}\cos 1$. *But*

$$1-\delta^3\geq 1-\left(\frac{4507}{15000}-\frac{1}{30}\cos 1\right)^3\approx 0.977465$$

*and* (37). *Thus, the following inequality*

$$\limsup_{t\to\infty} \sum_{i=0}^{2} B_1(t-i) \prod_{j=1}^{2} \left(1 + \sum_{k=1}^{i} B_1(t-(k+2))\right) > 1 - \delta^3,$$

*is not valid and therefore, condition* (11) *is not satisfied.*

*Considering conditions in Theorems E and F, we have that*

$$B_2(t) = P_1(t) + P_2(t) = \frac{79}{300} + \frac{1}{60}\sin 2t.$$

*Since*

$$\liminf_{t\to\infty} B_2(t) = \frac{37}{150} \approx 0.246667 < \frac{1}{4},$$

*condition* (13) *is not fulfilled. Due to*

$$\limsup_{t\to\infty} \left(B_2(t) + B_2(t-1) + B_2(t-1)B_2(t-2)\right) \approx 0.612428, \tag{38}$$

$$\limsup_{t\to\infty} \left(B_2(t) + B_2(t+1) + B_2(t+1)B_2(t+2)\right) \approx 0.612428, \tag{39}$$

*conditions* (14) *and* (16) *are not fulfilled, as well.*

*Since, for* $\delta$ *such that* $B_2(t) \geq \delta > 0$ *and* $\delta < \frac{1}{4}$ *we have*

$$1 - \delta^2 \geq 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16} \approx 0.9375$$

*and* (38) *and* (39). *Thus, the inequalities*

$$\begin{aligned}\limsup_{t\to\infty} \left(B_2(t) + B_2(t-1) + B_2(t-1)B_2(t-2)\right) &> 1 - \delta^2,\\ \limsup_{t\to\infty} \left(B_2(t) + B_2(t+1) + B_2(t+1)B_2(t+2)\right) &> 1 - \delta^2\end{aligned}$$

*are not valid and hence, conditions* (15) *and* (17) *are not satisfied.*

## Acknowledgment

We are grateful to the reviewer whose comments helped us to improve this paper.

## References

[1] G. E. Chatzarakis, Sufficient oscillation conditions for deviating difference equations, Filomat **33** 11 (2019) 3291–3305.
[2] G. E. Chatzarakis, I. Győri, H. Péics, I. P. Stavroulakis, Existence of positive solutions of linear delay difference equations with continuous time, Electron. J. Qual. Theory Differ. Equ. **2015** 15 (2015) 1–23.
[3] G. E. Chatzarakis, I. Jadlovska, Oscillations in Difference Equations with Several Arguments Using an Iterative Method, Filomat **32** 1 (2018), 255–273.
[4] G. E. Chatzarakis, H. Péics, I. P. Stavroulakis, Oscillations in Difference Equations with Deviating Arguments and Variable Coefficients, Abstr. Appl. Anal. **2014** Art. ID 902616 (2014) 9 pp.
[5] W. Golda, J. Werbowski, Oscillation of Linear Functional Equations of the Second Order, Funkcial. Ekvac. **37** (1994) 221–227.
[6] B. Karpuz, Ö. Öcalan, *Discrete Approach on Oscillation of Difference Equations with Continuous Variable*, Adv. Dyn. Syst. Appl., **3** 2 (2008) 283–290.
[7] G. Ladas, L. Pakula, Z. Wang, Necessary and sufficient conditions for the oscillation of difference equations, Panamer. Math. J. **2** 1 (1922) 17–26.
[8] G. Ladas, I. P. Stavroulakis, Oscillation Caused by Several Retarded and Advanced Arguments, J. Differential Equations **44** 2 (1982) 134–152.

[9] W. Nowakowska, J. Werbowski, *Oscillation of Linear Functional Equations of Higher Order*, Arch. Math. (Brno) **31** 4 (1995) 251–258.
[10] W. Nowakowska, J. Werbowski, *Oscillatory behavior of solutions of functional equations*, Nonlinear Anal. **44** 6 Ser. A: Theory Methods (2001) 767–775.
[11] W. Nowakowska, J. Werbowski, *Conditions for the Oscillation of Solutions of Iterative Equations*, Abstr. Appl. Anal. 7 (2004) 543–550.
[12] W. Nowakowska, J. Werbowski, *Oscillatory Solutions of Linear Iterative Functional Equations*, Indian J. Pure Appl. Math., **35** 4 (2004) 429–439.
[13] W. Nowakowska, J. Werbowski, *Oscillation of all Solutions of Iterative Equations*, Nonlinear Oscil. **10** 3 (2007) 351–366.
[14] W. Nowakowska, J. Werbowski, *On Connections between Oscillatory Solutions of Functional, Difference and Differential Equations*, Fasc. Math. **44** (2010) 95–106.
[15] J. H. Shen, *Comparison and Oscillation Results for Difference Equations with Continuous Variable*, Indian J. Pure Appl. Math. **31**(12) (2000) 1633–1642.
[16] J. Shen, I. P. Stavroulakis, *An Oscillation Criteria for Second Order Functional Equations*, Acta Math. Sci . Ser. B Engl. Ed. **22** 1 (2002) 56–62.
[17] B. G. Zhang, *Oscillations of a Class of Difference Equations with Continuous Arguments*, Appl. Math. Lett. **14** (2001) 557–561.
[18] B. G. Zhang, S. K. Choi, *Oscillation and nonoscillation of a Class of Functional Equations*, Math. Nachr. **227** (2001) 159–169.
[19] Y. Z. Zhang, J. R. Yan, *Oscillation criteria for difference equations with continuous arguments*, Acta Math. Sinica **38** 3 (1995) 406–411. (in Chinese)
[20] B. G. Zhang, J. Yan, S. K. Choi, *Oscillation for Difference Equations with Continuous Variable*, Comput. Math. Appl. **36** 9 (1998) 11–18.
[21] Y. Zhang, J. Yan, A. Zhao, *Oscillation Criteria for a Difference Equation*, Indian J. Pure Appl. Math. **28** 9 (1997) 1241–1249.