



Digital Co-Hopf Spaces

Ozgur Ege^a, Ismet Karaca^a

^aDepartment of Mathematics, Ege University, Bornova, Izmir, 35100 Turkey

Abstract. In this work, we deal with co-Hopf space structure of digital images. We prove that a pointed digital image having the same digital homotopy type as a digital co-Hopf space is itself a digital co-Hopf space. We conclude that a κ -deformation retract of a digital co-Hopf space is a digital co-Hopf space. We also show that the digital equivalences are digital co-Hopf homomorphisms.

1. Introduction

The theory of digital topology has been developing since the 1980s. It has been described by methods from topology, geometric topology and especially algebraic topology. Since it is used in many fields such as image analysis, computer graphics, geometric design, etc., this topic has managed to attract the attention of many researchers. Significant and useful results have been obtained by using some notions from classical topology, algebraic topology and pure mathematics. The main purpose of digital topology is to determine topological properties of discrete objects.

In recent years, there have been many progresses in digital topology. In [3] and [5], the digital versions of some notions in topology and digital continuous functions are studied. Digital H-spaces are introduced in [7] and their properties are given in [8].

A co-Hopf space (X, ϕ) consists of a topological space X and a continuous function $\phi : X \rightarrow X \vee X$, called co-multiplication, such that

$$\phi_1 \circ \phi \simeq 1_X \simeq \phi_2 \circ \phi,$$

where ϕ_1, ϕ_2 are the projections $X \vee X \rightarrow X$ onto the first and second summands of the wedge product. Since a binary operation can be defined via co-Hopf spaces on $[(X, x_0), (Y, y_0)]$ which is a homotopy class, co-Hopf spaces have great importance in algebraic topology. Hopf-spaces are usually called H-spaces, and likewise co-Hopf spaces are often called co-H spaces. For detailed knowledge about co-Hopf spaces, see [1].

This paper is organized as follows: The section of preliminaries provides required information about digital images. In the next section, we introduce the notion of digital co-Hopf space and prove some significant theorems related to this new space. Lastly, we state some conclusions about our study.

2010 *Mathematics Subject Classification.* Primary 55Q05; Secondary 68R01

Keywords. Digital topology, digital co-Hopf space, digital homotopy

Received: 24 August 2019; Revised: 31 January 2020; Accepted: 02 February 2020

Communicated by Ljubiša D.R. Kočinac

Email addresses: ozgur.ege@ege.edu.tr (Ozgur Ege), ismet.karaca@ege.edu.tr (Ismet Karaca)

2. Preliminaries

Let \mathbb{Z}^n be the set of lattice points in the n -dimensional Euclidean space where \mathbb{Z} is the set of integers. We say that (X, κ) is a digital image where X is a finite set in \mathbb{Z}^n and κ is an adjacency relation for the members of X .

Definition 2.1. ([6]) For a positive integer l with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$, p and q are c_l -adjacent, if

- (1) there are at most l indices i such that $|p_i - q_i| = 1$, and
- (2) for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

The value of l determines the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$. Thus, in \mathbb{Z} , we have $c_1 = 2$ -adjacency, in \mathbb{Z}^2 , we have $c_1 = 4$ -adjacency and $c_2 = 8$ -adjacency, in \mathbb{Z}^3 , we have $c_1 = 6$ -adjacency, $c_2 = 18$ -adjacency, and $c_3 = 26$ -adjacency.

Given a natural number l in conditions (1) and (2) with $1 \leq l \leq n$, l determines each of the κ -adjacency relations of \mathbb{Z}^n in terms of (1) and (2) [10] as follows:

$$\kappa \in \left\{ 2n \ (n \geq 1), 3^n - 1 \ (n \geq 2), 3^n - \sum_{t=0}^{r-2} C_t^r 2^{n-t} - 1 \ (2 \leq r \leq n-1, n \geq 3) \right\}$$

where $C_t^n = \frac{n!}{(n-t)!t!}$. For the most advanced version of a generalized adjacency relations of \mathbb{Z}^n , see [13].

A digital interval [3] is a set of the form $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b, a, b \in \mathbb{Z}\}$.

For the cartesian product of two digital images X_1 and X_2 , the adjacency relation [2] is defined as follows: Two points $x_i, y_i \in (X_i, \kappa_i)$, (x_0, y_0) and (x_1, y_1) are adjacent in $X_1 \times X_2$ if and only if one of the following is satisfied:

- $x_0 = x_1$ and y_0 and y_1 are κ_1 -adjacent; or
- x_0 and x_1 are κ_0 -adjacent and $y_0 = y_1$; or
- x_0 and x_1 are κ_0 -adjacent and y_0 and y_1 are κ_1 -adjacent.

Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. A function $f : X \rightarrow Y$ is (κ_0, κ_1) -continuous [4] if and only if for every $\{x_0, x_1\} \subset X$ such that x_0 and x_1 are κ_0 -adjacent, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1 -adjacent.

Definition 2.2. ([4]) Let X and Y be digital images. Let $f, g : X \rightarrow Y$ be (κ, λ) -continuous functions and suppose there is a positive integer m and a function $H : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$ such that

- for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$;
- for all $x \in X$, $H_x : [0, m]_{\mathbb{Z}} \rightarrow Y$ defined by $H_x(t) = H(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is $(2, \lambda)$ -continuous;
- for all $t \in [0, m]_{\mathbb{Z}}$, $H_t : X \rightarrow Y$ defined by $H_t(x) = H(x, t)$ for all $x \in X$ is (κ, λ) -continuous.

Then H is a digital (κ, λ) -homotopy between f and g , and f and g are (κ, λ) -homotopic in Y .

A pointed digital image with κ -adjacency is a triple (X, x_0, κ) , where x_0 is the base point of X . A pointed digitally continuous function $f : (X, x_0, \kappa) \rightarrow (Y, y_0, \lambda)$ is a (κ, λ) -continuous function from X to Y such that $f(x_0) = y_0$. A digital homotopy $H : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$ between f and g is called a *pointed digital homotopy* [4] between f and g if for all $t \in [0, m]_{\mathbb{Z}}$, $H(x_0, t) = y_0$. Digital homotopy and pointed homotopy are equivalence relations among digitally continuous functions.

A pointed digital image (X, x_0, κ) is said to be κ -contractible [3] if the identity map $1_X : X \rightarrow X$ is (κ, κ) -homotopic to the constant map $c_{x_0} : X \rightarrow X$.

Let $\emptyset \neq (A, \kappa) \subset (X, \kappa)$. We say that A is a κ -retract of X [3] if and only if there is a (κ, κ) -continuous function $r : X \rightarrow A$ such that $r(a) = a$ for all $a \in A$. The function r is called a κ -retraction of X onto A . Let $i : A \rightarrow X$ be the inclusion function. A is called a κ -deformation retract of X [5] if there exists a κ -homotopy $H : X \times [0, m]_{\mathbb{Z}} \rightarrow X$ between the identity map 1_X and $i \circ r$, for some κ -retraction r of X onto A .

Let $f : X \rightarrow Y$ be a (κ, λ) -continuous function and $g : Y \rightarrow X$ be a (λ, κ) -continuous function such that

$$f \circ g \simeq_{(\lambda, \lambda)} 1_Y \quad \text{and} \quad g \circ f \simeq_{(\kappa, \kappa)} 1_X,$$

where 1_X and 1_Y are identity maps on X and Y , respectively. Then X and Y have the same (κ, λ) -homotopy type [5, 9] and f is called a *digital homotopy equivalence*.

The notion of digital wedge was firstly introduced in [11]. As a more advanced version was given as follows:

Definition 2.3. ([12]) For pointed digital images (X, k_0, x_0) in \mathbb{Z}^{n_0} and (Y, k_1, y_0) in \mathbb{Z}^{n_1} , the digital wedge of (X, k_0) and (Y, k_1) , written $(X \vee Y, (x_0, y_0))$, is the digital image in \mathbb{Z}^n

$$\{(x, y) \in X \times Y \mid x = x_0 \text{ or } y = y_0\} \tag{1}$$

with the following compatible $k(m, n)$ (or k)-adjacency relative to both (X, k_0) and (Y, k_1) , and the only one point (x_0, y_0) in common such that

(W1) the $k(m, n)$ (or k)-adjacency is determined by the numbers m and n with $n = n_0 + n_1$, $m = m_0 + m_1$ which satisfies (W 1 – 1) below, where the numbers m_i are taken from the k_i (or $k(m_i, n_i)$)-adjacency relations of the given digital images (X, k_0, x_0) and (Y, k_1, y_0) , $i \in \{0, 1\}$.

(W 1 – 1) In view of (1), it can be considered that the projection maps from $X \vee Y$ onto X and Y , respectively denoted by

$$W_X : (X \vee Y, (x_0, y_0)) \rightarrow (X, x_0) \quad \text{and} \quad W_Y : (X \vee Y, (x_0, y_0)) \rightarrow (Y, y_0),$$

where $W_X(x, y) = x$ and $W_Y(x, y) = y$.

In relation to the establishment of a compatible k -adjacency of the digital wedge $(X \vee Y, (x_0, y_0))$, the following restriction maps of W_X and W_Y on $(X \times \{y_0\}, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ and $(\{x_0\} \times Y, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ satisfy the following properties, respectively:

- (1) $W_X|_{X \times \{y_0\}} : (X \times \{y_0\}, k) \rightarrow (X, k_0)$ is a (k, k_0) – isomorphism; and
- (2) $W_Y|_{\{x_0\} \times Y} : (\{x_0\} \times Y, k) \rightarrow (Y, k_1)$ is a (k, k_1) – isomorphism.

(W2) Any two distinct elements $(x, y_0) \in X \times \{y_0\}$ and $(x_0, y) \in \{x_0\} \times Y$ of $X \vee Y$, such that $x \neq x_0$ and $y \neq y_0$, are not $k(m, n)$ (or k)-adjacent to each other.

3. Digital Co-Hopf Spaces

In this section, we first define the notion of digital co-Hopf space. Then we give some important properties of this new space.

Definition 3.1. Let (X, p, κ) be a pointed digital image, where $p \in X$. If there is a (κ, κ_v) -continuous function $\phi : X \rightarrow X \vee X$, called digital co-multiplication, such that

$$\pi_1 \circ \phi \simeq_{(\kappa, \kappa)} 1_X \simeq_{(\kappa, \kappa)} \pi_2 \circ \phi,$$

where κ_v is the adjacency relation on $X \vee X$ and $\pi_1, \pi_2 : X \vee X \rightarrow X$ are the first and second projections, respectively, then (X, p, κ) is called a digital co-Hopf space.

Definition 3.2. Consider three digital co-Hopf spaces (A, a, κ_1) , (B, b, κ_2) , (X, x, κ_3) and digital continuous functions $f : A \rightarrow X$ and $g : B \rightarrow X$. The $(\kappa_v, \kappa_{v'})$ -continuous function

$$f \vee g : A \vee B \rightarrow X \vee X$$

is defined by $(f \vee g)(a, b) = (f(a), g(b))$, where κ_v and $\kappa_{v'}$ are adjacency relations on $A \vee B$ and $X \vee X$, respectively.

Example 3.3. Let $X = \{p\}$ be a single point digital image with κ -adjacency. Since

$$\pi_1 \circ \phi(p) = \pi_1(p, p) = p = 1_X(p) \quad \text{and} \quad \pi_2 \circ \phi(p) = \pi_2(p, p) = p = 1_X(p),$$

we have

$$\pi_1 \circ \phi \simeq_{(\kappa, \kappa)} 1_X \simeq_{(\kappa, \kappa)} \pi_2 \circ \phi.$$

So (X, p, κ) is a digital co-Hopf space.

Theorem 3.4. *A pointed digital image having the same digital homotopy type as a digital co-Hopf space is itself a digital co-Hopf space.*

Proof. Let (X, p, κ) be a digital co-Hopf space with the digital continuous co-multiplication ϕ and (Y, q, κ') has the same (κ, κ') -homotopy type as (X, p, κ) . Then there are two functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that

$$f \circ g \simeq_{(\kappa', \kappa')} 1_Y \quad \text{and} \quad g \circ f \simeq_{(\kappa, \kappa)} 1_X.$$

Since (X, p, κ) is a digital co-Hopf space, the following equivalences exist:

$$\pi_1 \circ \phi \simeq_{(\kappa, \kappa)} 1_X \simeq_{(\kappa, \kappa)} \pi_2 \circ \phi,$$

where $\pi_1, \pi_2 : X \vee X \rightarrow X$. Consider the following maps:

$$\phi' : Y \rightarrow Y \vee Y \quad \text{and} \quad \pi'_1, \pi'_2 : Y \vee Y \rightarrow Y.$$

We will show that

$$\pi'_1 \circ \phi' \simeq_{(\kappa', \kappa')} 1_Y \simeq_{(\kappa', \kappa')} \pi'_2 \circ \phi'.$$

If we define $\phi' = (f \vee f) \circ \phi \circ g$ and $\pi'_1 = f \circ \pi_1 \circ (g \vee g)$, then we get

$$\begin{aligned} \pi'_1 \circ \phi' &= f \circ \pi_1 \circ (g \vee g) \circ (f \vee f) \circ \phi \circ g \\ &= f \circ \pi_1 \circ ((g \circ f) \vee (g \circ f)) \circ \phi \circ g \\ &\simeq_{(\kappa, \kappa)} f \circ \pi_1 \circ 1_{X \vee X} \circ \phi \circ g \\ &= f \circ \pi_1 \circ \phi \circ g \\ &\simeq_{(\kappa, \kappa)} f \circ 1_X \circ g \\ &= f \circ g \\ &\simeq_{(\kappa', \kappa')} 1_Y. \end{aligned}$$

On the other hand, if we define $\pi'_2 = f \circ \pi_2 \circ (g \vee g)$, then we obtain the following:

$$\begin{aligned} \pi'_2 \circ \phi' &= f \circ \pi_2 \circ (g \vee g) \circ (f \vee f) \circ \phi \circ g \\ &= f \circ \pi_2 \circ ((g \circ f) \vee (g \circ f)) \circ \phi \circ g \\ &\simeq_{(\kappa, \kappa)} f \circ \pi_2 \circ 1_{X \vee X} \circ \phi \circ g \\ &= f \circ \pi_2 \circ \phi \circ g \\ &\simeq_{(\kappa, \kappa)} f \circ 1_X \circ g \\ &= f \circ g \\ &\simeq_{(\kappa', \kappa')} 1_Y. \end{aligned}$$

Thus, we have the required result. \square

Theorem 3.5. *Let (X, p, κ) be a digital co-Hopf space. A κ -deformation retract of (X, p, κ) is a digital co-Hopf space.*

Proof. Let $A \subset X$ be a κ -retract with a κ -retraction $r : X \rightarrow A$. So we have

$$r \circ i \simeq_{(\kappa, \kappa)} 1_A \quad \text{and} \quad i \circ r \simeq_{(\kappa, \kappa)} 1_X,$$

where $i : A \rightarrow X$ is an inclusion map. Since (X, p, κ) is a digital co-Hopf space, it has a digital co-multiplication map $\phi : X \rightarrow X \vee X$. Defining $\phi_A = (r \vee r) \circ \phi \circ i$ and $\pi_{1A} = r \circ \pi_1 \circ (i \vee i)$, we obtain the following:

$$\begin{aligned} \pi_{1A} \circ \phi_A &= r \circ \pi_1 \circ (i \vee i) \circ (r \vee r) \circ \phi \circ i \\ &= r \circ \pi_1 \circ ((i \circ r) \vee (i \circ r)) \circ \phi \circ i \\ &\simeq_{(\kappa, \kappa)} r \circ \pi_1 \circ (1_{X \vee X}) \circ \phi \circ i \\ &= r \circ \pi_1 \circ \phi \circ i \\ &\simeq_{(\kappa, \kappa)} r \circ i \\ &\simeq_{(\kappa, \kappa)} 1_A. \end{aligned}$$

In a similar way, if we define $\pi_{2_A} = r \circ \pi_2 \circ (i \vee i)$, then we have $\pi_{2_A} \circ \phi_A \simeq_{(\kappa, \kappa)} 1_A$. So we conclude that (A, q, κ) is a digital co-Hopf space, where $r(p) = q \in A$. \square

Theorem 3.6. *Suppose that (X, p, κ) and (Y, q, λ) are two digital co-Hopf spaces. Then $(X \vee Y, (p, q), \kappa_v)$ is also a digital co-Hopf space.*

Proof. Let ϕ and ψ be digital co-multiplications on X and Y , respectively. We can define the digital co-multiplication θ on $X \vee Y$ as follows: $\theta = T \circ (\phi \vee \psi)$, where

$$T : (X \vee X) \vee (Y \vee Y) \rightarrow (X \vee Y) \vee (X \vee Y)$$

is a switching map defined by $T((u, v), (w, z)) = ((u, w), (v, z))$ where $(u, v) \in X \vee X$ and $(w, z) \in Y \vee Y$. Since

$$\begin{aligned} \pi_1 \circ \theta(p, q) &= \pi_1 \circ (T \circ (\phi \vee \psi))(p, q) \\ &= \pi_1 \circ T(\phi(p), \psi(q)) \\ &= \pi_1 \circ T((p, a^*), (q, b^*)) \\ &= \pi_1((p, q), (a^*, b^*)) \\ &= (p, q) \\ &= 1_{X \vee Y}(p, q), \end{aligned}$$

where $(p, q) \in X \vee Y$, $a^* \in X$, $b^* \in Y$, and $\pi_1, \pi_2 : (X \vee Y) \vee (X \vee Y) \rightarrow (X \vee Y)$, we have $\pi_1 \circ \theta \simeq_{(\kappa_v, \kappa_v)} 1_{X \vee Y}$. Similarly, we can conclude that $\pi_2 \circ \theta \simeq_{(\kappa_v, \kappa_v)} 1_{X \vee Y}$. So $(X \vee Y, (p, q), \kappa_v)$ is a digital co-Hopf space. \square

Definition 3.7. A pointed digital image (X, p, κ) is called a digital co-Hopf group if there exists a (κ, κ_v) -continuous function $\phi : X \rightarrow X \vee X$ such that $\pi_1 \circ \phi = 1_X$ and $\pi_2 \circ \phi = 1_X$.

Lemma 3.8. *Let (X, p, κ) be a pointed digital image. Every digital co-Hopf group of (X, p, κ) is a set which consists of a single element as the base point.*

Proof. By Definition 3.7, (X, p, κ) has a (κ, κ_v) -continuous function ϕ such that

$$\pi_1 \circ \phi = 1_X \quad \text{and} \quad \pi_2 \circ \phi = 1_X.$$

Consider an arbitrary point x in X . For the element $\phi(x)$ in $X \vee X$, there are two possible cases such that (p, x^*) or (x^*, p) for some $x^* \in X$.

Case 1: If $\phi(x) = (p, x^*)$, then we have

$$\pi_1 \circ \phi(x) = \pi_1(p, x^*) = p.$$

Since $\pi_1 \circ \phi = 1_X$, we conclude that $x = p$.

Case 2: If $\phi(x) = (x^*, p)$, then we obtain

$$\pi_2 \circ \phi(x) = \pi_2(x^*, p) = p.$$

By the equation $\pi_2 \circ \phi = 1_X$, we see that $x = p$. In both cases, it is shown that every digital co-Hopf group of (X, p, κ) is a set which consists of a single element as the base point. \square

Definition 3.9. Let (X, κ) and (Y, κ') be digital co-Hopf spaces with the digital continuous co-multiplication γ and γ' , respectively. A function $f : X \rightarrow Y$ is called a digital co-Hopf homomorphism if

$$(f \vee f) \circ \gamma \simeq_{(\kappa, \kappa'_v)} \gamma' \circ f,$$

where κ'_v is the adjacency relation in $Y \vee Y$.

Theorem 3.10. Let (X, p, κ) , (Y, q, κ') and (Z, r, κ'') are pointed digital co-Hopf spaces. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are digital co-Hopf homomorphisms, then $g \circ f : X \rightarrow Z$ is also a digital co-Hopf homomorphism.

Proof. Let γ , γ' and γ'' be digital co-multiplications of X , Y and Z , respectively. Since f and g are digital co-Hopf homomorphisms, we have the following digital homotopies:

$$\begin{aligned} (f \vee f) \circ \gamma &\simeq_{(\kappa, \kappa'_v)} \gamma' \circ f, \\ (g \vee g) \circ \gamma' &\simeq_{(\kappa', \kappa''_v)} \gamma'' \circ g, \end{aligned}$$

where κ'_v and κ''_v are adjacency relations in $Y \vee Y$ and $Z \vee Z$, respectively. Using the last two statements, we obtain

$$\begin{aligned} ((g \circ f) \vee (g \circ f)) \circ \gamma &= (g \vee g) \circ (f \vee f) \circ \gamma \\ &\simeq_{(\kappa, \kappa'_v)} (g \vee g) \circ \gamma' \circ f \\ &\simeq_{(\kappa, \kappa''_v)} \gamma'' \circ (g \circ f). \end{aligned}$$

By Definition 3.9, we conclude that $g \circ f$ is a digital co-Hopf homomorphism. \square

Theorem 3.11. Let the pointed two digital images (X, κ) and (Y, κ') be digital co-Hopf spaces with the same (κ, κ') -homotopy type. Then the digital equivalences are digital co-Hopf homomorphisms.

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be digital homotopy equivalences, i.e.,

$$f \circ g \simeq_{(\kappa', \kappa')} 1_Y \quad \text{and} \quad g \circ f \simeq_{(\kappa, \kappa)} 1_X.$$

Let γ and γ' be digital continuous co-multiplications of X and Y , respectively. We can define γ as the composition of the following functions

$$X \xrightarrow{f} Y \xrightarrow{\gamma'} Y \vee Y \xrightarrow{g \vee g} X \vee X,$$

i.e., $\gamma = (g \vee g) \circ \gamma' \circ f$. We can say the following:

$$f \circ g \simeq_{(\kappa', \kappa')} 1_Y \Rightarrow (f \circ g) \vee (f \circ g) \simeq_{(\kappa'_v, \kappa'_v)} 1_{Y \vee Y},$$

where κ'_v is the adjacency relation in $Y \vee Y$. Then we obtain

$$\begin{aligned} (f \vee f) \circ \gamma &= (f \vee f) \circ (g \vee g) \circ \gamma' \circ f \\ &= ((f \circ g) \vee (f \circ g)) \circ \gamma' \circ f \\ &\simeq_{(\kappa, \kappa'_v)} 1_{Y \vee Y} \circ \gamma' \circ f \\ &= \gamma' \circ f. \end{aligned}$$

Therefore, f is a digital co-Hopf homomorphism.

Similarly, we can define γ' as the composition of the following functions

$$Y \xrightarrow{g} X \xrightarrow{\gamma} X \vee X \xrightarrow{f \vee f} Y \vee Y,$$

i.e., $\gamma' = (f \vee f) \circ \gamma \circ g$. Thus we have the following:

$$g \circ f \simeq_{(\kappa, \kappa)} 1_X \Rightarrow (g \circ f) \vee (g \circ f) \simeq_{(\kappa_v, \kappa_v)} 1_{X \vee X},$$

where κ_v is the adjacency relation in $X \vee X$. So we get

$$\begin{aligned} (g \vee g) \circ \gamma' &= (g \vee g) \circ (f \vee f) \circ \gamma \circ g \\ &= ((g \circ f) \vee (g \circ f)) \circ \gamma \circ g \\ &\simeq_{(\kappa', \kappa_v)} 1_{X \vee X} \circ \gamma \circ g \\ &= \gamma \circ g. \end{aligned}$$

As a result, we conclude that g is also a digital co-Hopf homomorphism. \square

4. Conclusion

The principal goal of this study is to introduce digital co-Hopf spaces and to show some important results. The idea of associating of digital topology and co-Hopf spaces is very interesting. We believe that acquired all results will draw significant interest from the community of digital topology.

Acknowledgments

We would like to express our gratitude to the anonymous referees for their helpful suggestions and corrections.

References

- [1] M. Arkowitz, Co-H-Spaces, *Handbook of Algebraic Topology*, North-Holland, New York, NY, USA, 1995.
- [2] C. Berge, *Graphs and Hypergraphs*, (2nd edition), North-Holland, Amsterdam, 1976.
- [3] L. Boxer, Digitally continuous functions, *Pattern Recogn. Lett.* 15 (1994) 833–839.
- [4] L. Boxer, A classical construction for the digital fundamental group, *J. Math. Imaging Vision* 10 (1999) 51–62.
- [5] L. Boxer, Properties of digital homotopy, *J. Math. Imaging Vision* 22 (2005) 19–26.
- [6] L. Boxer, Homotopy properties of sphere-like digital images, *J. Math. Imaging Vision* 24 (2006) 167–175.
- [7] O. Ege, I. Karaca, Digital H-spaces, In: *ISCSE 2013 3rd International Symposium on Computing in Science and Engineering*, 24-25 October 2013, Kusadasi, Izmir, Turkey, Gediz University Publications, 2013, 133–138.
- [8] O. Ege, I. Karaca, Some properties of digital H-spaces, *Turk. J. Elec. Eng. Comp. Sci.* 24 (2016) 1930–1941.
- [9] S.E. Han, On the classification of the digital images up to a digital homotopy equivalence, *J. Comput. Commun. Res.* 10 (2000) 194–207.
- [10] S.E. Han, An extended digital (k_0, k_1) -continuity, *J. Appl. Math. Comput.* 16 (2004) 445–452.
- [11] S.E. Han, Non-product property of the digital fundamental group, *Inform. Sciences* 171 (2005) 73–91.
- [12] S.E. Han, Non-ultra regular digital covering spaces with nontrivial automorphism groups, *Filomat* 27 (2013) 1205–1218.
- [13] S.E. Han, Estimation of the complexity of a digital image form the viewpoint of fixed point theory, *Appl. Math. Comput.* 347 (2019) 236–248.