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Inequalities for the Generalized Tsallis Relative Operator Entropy

Ismail Nikoufar[a], Zahra Yousefzadeh[a]

[a] *Department of Mathematics, Payame Noor University, P.O. Box 19395-3697 Tehran, Iran.*

Abstract. In this paper, we present some inequalities for the generalized relative operator entropy according to the generalized Tsallis relative operator entropy. Our results are generalizations of some existing inequalities.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space and let A and B be two invertible positive operators acting on $\mathcal{H}$. Fujii and Kamei [7] introduced the relative operator entropy of invertible positive operators A and B by

$$S(A|B) = A^{\frac{1}{2}} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

Furuta [4] defined the generalized relative operator entropy by

$$S_\lambda(A|B) = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\lambda \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}},$$

where $\lambda \in [0,1]$. It is obvious that $S_0(A|B) = S(A|B)$. In [13], Yanagi, Kuriyama and Furuichi introduced the Tsallis relative operator entropy defined by

$$T_\lambda(A|B) = \frac{A\#_\lambda B - A}{\lambda}$$

for $\lambda \in [0,1]$, where the operator λ-geometric mean $A\#_\lambda B$ is defined by

$$A\#_\lambda B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}.$$

Fujii and Kamei [7] obtained the following inequality as a relationship between the relative operator entropy and the Tsallis relative operator entropy:

$$T_{-\lambda}(A|B) \le S(A|B) \le T_\lambda(A|B). \tag{1}$$

Yanagi et. al [13] generalized the definition of the Tsallis relative operator entropy. They defined the generalized Tsallis relative operator entropy by

$$\widetilde{T}_{\mu,k,\lambda}(A|B) = \frac{A\natural_{\mu+k\lambda} B - A\natural_{\mu+(k-1)\lambda} B}{\lambda},$$

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Email addresses: nikoufar@pnu.ac.ir (Ismail Nikoufar), yousefzadehzahra@yahoo.com (Zahra Yousefzadeh)

where $\lambda, \mu \in \mathbb{R}$, $\lambda \neq 0$, $k \in \mathbb{Z}$ and $A\natural_\mu B$ denotes the extended operator μ-geometric mean for $\mu \in \mathbb{R}$, see [4] for instance. The extended operator μ-geometric mean $A\natural_\mu B$ is defined by

$$A\natural_\mu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu A^{\frac{1}{2}}.$$

In particular, for $\lambda \in [0, 1]$,

$$\widetilde{T}_{0,1,\lambda}(A|B) = \frac{A\natural_\lambda B - A\natural_0 B}{\lambda} = \frac{A\#_\lambda B - A}{\lambda} = T_\lambda(A|B).$$

As a generalization of (1) we will give the relationship between the generalized relative operator entropy and the generalized Tsallis relative operator entropy in (10), (14), and (17). Dragomir gave some inequalities for relative operator entropy in [1, 2].

Zou [14] proved some operator inequalities related to the Tsallis relative operator entropy and generalized some existing inequalities. Then, he presented some inequalities in [15] for the relative operator entropy generalizing some results obtained in [14] and presented some new lower and upper bounds for the Tsallis relative operator entropy and relative operator entropy. For more information on the Tsallis relative entropy and relative operator entropy the reader is referred to [5, 6, 8, 10–12] and the references therein.

In this paper, we prove some inequalities for the generalized relative operator entropy according to the generalized Tsallis relative operator entropy. Meanwhile, we also give some lower and upper bounds for the generalized relative operator entropy and generalized Tsallis relative operator entropy. Our results are generalizations of some existing inequalities, cf. [14, 15].

2. Main Results

In this section, we achieve the upper bound of the generalized relative operator entropy according to the generalized Tsallis relative operator entropy and extended operator geometric mean. We begin this section with the following lemma.

Lemma 2.1. *Let $\lambda \in (0, 1]$ and $t > 0$. Then*

(i) $\log t \leq \frac{t^\lambda - 1}{\lambda}$,

(ii) $1 - \frac{1}{t} \leq \frac{t^{-\lambda} - 1}{-\lambda}$.

Proof. (i) Writing the right inequality in (1) in the scalar case, the inequality

$$a \log \frac{b}{a} \leq \frac{a^{1-\lambda} b^\lambda - a}{\lambda}$$

holds for any $a, b > 0$ and $\lambda \in (0, 1]$. Taking $a = 1$ and $b = t$ we obtain (i).

(ii) The arithmetic-geometric mean inequality

$$a^\lambda \leq \lambda a + 1 - \lambda$$

holds for any $a > 0$. Taking $a = \frac{1}{t}$ we obtain (ii) after a simple manipulation. □

Theorem 2.2. *Let $a > 0$, $\lambda \in (0, 1]$, $q \geq 0$, $k \in \mathbb{N}$, and $\nu \in [0, 1]$. For any invertible positive operators A and B,*

$$\begin{aligned} S_{q+(k-1)\lambda}(A|B) \leq\ & \frac{a^\lambda - 1}{\lambda}(\nu A\natural_{q+k\lambda} B + (1-\nu)A\natural_{q+(k-1)\lambda} B) \\ & + (\nu + (1-\nu)a^\lambda)\widetilde{T}_{q,k,\lambda}(A|B) - (\log a)A\natural_{q+(k-1)\lambda} B. \end{aligned} \tag{2}$$

Proof. By a simple calculation we get

$$\frac{(ax)^{\lambda}-1}{\lambda}x^{q+(k-1)\lambda}=(\frac{a^{\lambda}-1}{\lambda}x^{\lambda}+\frac{x^{\lambda}-1}{\lambda})x^{q+(k-1)\lambda}, \tag{3}$$

$$\frac{(ax)^{\lambda}-1}{\lambda}x^{q+(k-1)\lambda}=(\frac{x^{\lambda}-1}{\lambda}a^{\lambda}+\frac{a^{\lambda}-1}{\lambda})x^{q+(k-1)\lambda}. \tag{4}$$

It follows from the convex combination of (3) and (4) that

$$\begin{aligned}
&\frac{(ax)^{\lambda}-1}{\lambda}x^{q+(k-1)\lambda}\\
&=\nu(\frac{a^{\lambda}-1}{\lambda}x^{\lambda}+\frac{x^{\lambda}-1}{\lambda})x^{q+(k-1)\lambda}+(1-\nu)(\frac{x^{\lambda}-1}{\lambda}a^{\lambda}+\frac{a^{\lambda}-1}{\lambda})x^{q+(k-1)\lambda}\\
&=\frac{a^{\lambda}-1}{\lambda}(\nu x^{q+k\lambda}+(1-\nu)x^{q+(k-1)\lambda})+(\nu+(1-\nu)a^{\lambda})\frac{x^{q+k\lambda}-x^{q+(k-1)\lambda}}{\lambda}.
\end{aligned} \tag{5}$$

For a positive real number x one can easily find that

$$x^{q+(k-1)\lambda}\log ax\leq\frac{(ax)^{\lambda}-1}{\lambda}x^{q+(k-1)\lambda}.$$

Hence,

$$x^{q+(k-1)\lambda}\log x\leq\frac{(ax)^{\lambda}-1}{\lambda}x^{q+(k-1)\lambda}-(\log a)x^{q+(k-1)\lambda}. \tag{6}$$

Combining (5) and (6), we yield

$$\begin{aligned}
x^{q+(k-1)\lambda}\log x\leq&\frac{a^{\lambda}-1}{\lambda}(\nu x^{q+k\lambda}+(1-\nu)x^{q+(k-1)\lambda})\\
&+(\nu+(1-\nu)a^{\lambda})\frac{x^{q+k\lambda}-x^{q+(k-1)\lambda}}{\lambda}-(\log a)x^{q+(k-1)\lambda}.
\end{aligned} \tag{7}$$

The desired result follows from (7) by applying the functional calculus $x=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and then multiplying both sides with $A^{\frac{1}{2}}$. □

Corollary 2.3. *[15, Theorem 2.1] Let $a>0$, $\lambda\in(0,1]$, and $\nu\in[0,1]$. For any invertible positive operators A and B,*

$$S(A|B)\leq\frac{a^{\lambda}-1}{\lambda}(\nu A\sharp_{\lambda}B+(1-\nu)A)+(\nu+(1-\nu)a^{\lambda})T_{\lambda}(A|B)-(\log a)A.$$

Proof. By letting $q=0$ and $k=1$ in Theorem 2.2, we reach the result. □

Remark 2.4. *Putting $\nu=1$ in (2), we get*

$$S_{q+(k-1)\lambda}(A|B)\leq\frac{a^{\lambda}-1}{\lambda}A\sharp_{q+k\lambda}B+\widetilde{T}_{q,k,\lambda}(A|B)-(\log a)A\sharp_{q+(k-1)\lambda}B \tag{8}$$

and putting $\nu=0$ in (2), we obtain

$$S_{q+(k-1)\lambda}(A|B)\leq a^{\lambda}\widetilde{T}_{q,k,\lambda}(A|B)+(\frac{a^{\lambda}-1}{\lambda}-\log a)A\sharp_{q+(k-1)\lambda}B. \tag{9}$$

Moreover, setting $a=1$ in (8) or (9) one can find that

$$S_{q+(k-1)\lambda}(A|B)\leq\widetilde{T}_{q,k,\lambda}(A|B). \tag{10}$$

We emphasis the inequality (10) discovers the relation of ordering between generalized Tsallis relative operator entropy and generalized relative operator entropy. In particular, this inequality recovers the upper bound obtained in (1) by Fujii and Kamei.

We find the lower bound of the generalized relative operator entropy according to the generalized Tsallis relative operator entropy and extended operator geometric mean as follows.

Theorem 2.5. *Let $a > 0$, $q > 0$, $\lambda \in (0, 1]$ and $\nu \in [0, 1]$. For any invertible positive operators A and B,*

$$\begin{aligned} S_{-q-(k-1)\lambda}(A|B) \geq{} & \frac{a^\lambda - 1}{\lambda a^\lambda}(\nu A\natural_{-q-k\lambda}B + (1-\nu)A\natural_{-q-(k-1)\lambda}B) \\ & + (\nu + (1-\nu)a^{-\lambda})\widetilde{T}_{-q,k,-\lambda}(A|B) - (\log a)A\natural_{-q-(k-1)\lambda}B. \end{aligned} \tag{11}$$

Proof. Substituting x by x^{-1} and a by a^{-1} in (7), respectively, we have

$$\begin{aligned} x^{-q-(k-1)\lambda}\log x^{-1} \leq{} & \frac{a^{-\lambda}-1}{\lambda}(\nu x^{-q-k\lambda} + (1-\nu)x^{-q-(k-1)\lambda}) \\ & + (\nu + (1-\nu)a^{-\lambda})\frac{x^{-q-k\lambda} - x^{-q-(k-1)\lambda}}{\lambda} \\ & - (\log a^{-1})x^{-q-(k-1)\lambda} \end{aligned}$$

for all $x > 0$. Hence

$$\begin{aligned} -x^{-q-(k-1)\lambda}\log x \leq{} & \frac{1-a^\lambda}{\lambda a^\lambda}(\nu x^{-q-k\lambda} + (1-\nu)x^{-q-(k-1)\lambda}) \\ & + (\nu + (1-\nu)a^{-\lambda})\frac{x^{-q-k\lambda} - x^{-q-(k-1)\lambda}}{\lambda} \\ & + (\log a)x^{-q-(k-1)\lambda} \end{aligned}$$

for all $x > 0$. From this we observe that

$$\begin{aligned} x^{-q-(k-1)\lambda}\log x \geq{} & \frac{a^\lambda - 1}{\lambda a^\lambda}(\nu x^{-q-k\lambda} + (1-\nu)x^{-q-(k-1)\lambda}) \\ & + (\nu + (1-\nu)a^{-\lambda})\frac{x^{-q+k(-\lambda)} - x^{-q+(k-1)(-\lambda)}}{-\lambda} \\ & - (\log a)x^{-q-(k-1)\lambda} \end{aligned}$$

for all $x > 0$. The desired result follows from (7) by applying the functional calculus $x = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and then multiplying both sides with $A^{\frac{1}{2}}$. □

Corollary 2.6. *[15, Theorem 2.3] Let $a > 0$, $\lambda \in (0, 1]$, and $\nu \in [0, 1]$. For any invertible positive operators A and B,*

$$S(A|B) \geq \frac{a^\lambda - 1}{\lambda a^\lambda}(\nu A\natural_{-\lambda}B + (1-\nu)A) + (\nu + (1-\nu)a^{-\lambda})T_{-\lambda}(A|B) - (\log a)A.$$

Proof. By letting $q = 0$ and $k = 1$ in Theorem 2.5, we find the result. □

Remark 2.7. *Setting $\nu = 1$ in (11), we get*

$$\begin{aligned} S_{-q-(k-1)\lambda}(A|B) \geq{} & \widetilde{T}_{-q,k,-\lambda}(A|B) + \frac{a^\lambda - 1}{\lambda a^\lambda}A\natural_{-q-k\lambda}B \\ & - (\log a)A\natural_{-q-(k-1)\lambda}B \end{aligned} \tag{12}$$

and setting $\nu = 0$ in (11), *we obtain*

$$S_{-q-(k-1)\lambda}(A|B) \geq a^{-\lambda}\widetilde{T}_{-q,k,-\lambda}(A|B) + (\frac{a^{\lambda}-1}{\lambda a^{\lambda}} - \log a)A\sharp_{-q-(k-1)\lambda}B, \tag{13}$$

which is a lower bound for $S_{-q-(k-1)\lambda}(A|B)$. Moreover, letting $a = 1$ in (12) *or* (13), *we deduce*

$$S_{-q-(k-1)\lambda}(A|B) \geq \widetilde{T}_{-q,k,-\lambda}(A|B). \tag{14}$$

In particular, this inequality recovers the lower bound obtained in (1) *by Fujii and Kamei.*

Corollary 2.8. *Let $a > 0$, $\lambda \in (0,1]$, $q \geq 0$, $k \in \mathbb{N}$. For any invertible positive operators A and B,*

$$\begin{aligned}\widetilde{T}_{-q,k,-\lambda}(A|B) &- \frac{1-a^{\lambda}}{\lambda a^{\lambda}}A\sharp_{-q-k\lambda}B - (\log a)A\sharp_{-q-(k-1)\lambda}B\\ &\leq S_{-q-(k-1)\lambda}(A|B)\\ &\leq S(A|B)\\ &\leq S_{q+(k-1)\lambda}(A|B)\\ &\leq a^{\lambda}\widetilde{T}_{q,k,\lambda}(A|B) + (\frac{a^{\lambda}-1}{\lambda} - \log a)A\sharp_{q+(k-1)\lambda}B.\end{aligned} \tag{15}$$

Proof. The function $\Gamma_A(x) = A^x \log A$ is an increasing function of x for every $A > 0$. On the other hand, $S_x(A|B) = A^{1/2}\Gamma_{A^{-1/2}BA^{-1/2}}(x)A^{1/2}$ and so the generalized relative operator entropy is an increasing function of x for every $A, B > 0$ [9, Corollary 1]. Hence, the second and third inequalities in (15) are hold. The first and forth inequalities follow from (12) and (9), respectively. □

By letting $q = 1$ and $k = 0$ in (15) we obtain [15, Remark 2.5] as follows:

$$\begin{aligned}T_{-\lambda}(A|B) &- \frac{1-a^{\lambda}}{\lambda a^{\lambda}}A\sharp_{-\lambda}B - (\log a)A\\ &\leq S(A|B)\\ &\leq a^{\lambda}T_{\lambda}(A|B) + (\frac{a^{\lambda}-1}{\lambda} - \log a)A.\end{aligned}$$

In view of Remark 2.4, 2.7 and inequalities (13), (8) one can deduce

$$\begin{aligned}a^{-\lambda}\widetilde{T}_{-q,k,-\lambda}(A|B) &+ (\frac{a^{\lambda}-1}{\lambda a^{\lambda}} - \log a)A\sharp_{-q-(k-1)\lambda}B\\ &\leq S_{-q-(k-1)\lambda}(A|B)\\ &\leq S(A|B)\\ &\leq S_{q+(k-1)\lambda}(A|B)\\ &\leq \widetilde{T}_{q,k,\lambda}(A|B) + \frac{a^{\lambda}-1}{\lambda}A\sharp_{q+k\lambda}B - (\log a)A\sharp_{q+(k-1)\lambda}B.\end{aligned} \tag{16}$$

Putting $a = 1$ in (15) or (16), we have

$$\begin{aligned}\widetilde{T}_{-q,k,-\lambda}(A|B) &\leq S_{-q-(k-1)\lambda}(A|B)\\ &\leq S(A|B)\\ &\leq S_{q+(k-1)\lambda}(A|B)\\ &\leq \widetilde{T}_{q,k,\lambda}(A|B),\end{aligned} \tag{17}$$

which is a refinement and generalization of (1). We emphasis the inequalities (17) totalize the relation of ordering between generalized Tsallis relative operator entropy and generalized relative operator entropy.

We now verify the lower and upper bound of the generalized Tsallis relative operator entropy with respect to the extended operator geometric mean.

Theorem 2.9. *Let $a > 0$, $\lambda \in (0,1]$, $q \geq 0$, $k \in \mathbb{N}$, and $\nu \in [0,1]$. For any invertible positive operators A and B,*

$$\begin{aligned}A\sharp_{-q-(k-1)\lambda}B - \frac{1}{a}A\sharp_{-q-(k-1)\lambda-1}B + \frac{a^{-\lambda}-1}{\lambda}(\nu A\sharp_{-q-k\lambda}B + (1-\nu)A\sharp_{-q-(k-1)\lambda}B)\\ \leq (\nu + (1-\nu)a^{-\lambda})\widetilde{T}_{-q,k,-\lambda}(A|B).\end{aligned}$$

Proof. By replacing $-\lambda$ and $-q$ instead of λ and q, respectively in (5) we find that

$$\begin{aligned}\frac{(ax)^{-\lambda}-1}{-\lambda}x^{-q-(k-1)\lambda} &= \frac{a^{-\lambda}-1}{-\lambda}(\nu x^{-q-k\lambda} + (1-\nu)x^{-q-(k-1)\lambda})\\ &\quad + (\nu + (1-\nu)a^{-\lambda})\frac{x^{-q-k\lambda} - x^{-q-(k-1)\lambda}}{-\lambda}.\end{aligned} \tag{18}$$

Due to Lemma 2.1 (ii) we have

$$(1-\frac{1}{ax})x^{-q-(k-1)\lambda} \leq \frac{(ax)^{-\lambda}-1}{-\lambda}x^{-q-(k-1)\lambda}. \tag{19}$$

Combining (18) and (19), we get

$$\begin{aligned}x^{-q-(k-1)\lambda} - \frac{1}{a}x^{-q-(k-1)\lambda-1} &\leq \frac{a^{-\lambda}-1}{-\lambda}(\nu x^{-q-k\lambda} + (1-\nu)x^{-q-(k-1)\lambda})\\ &\quad + (\nu + (1-\nu)a^{-\lambda})\frac{x^{-q-k\lambda} - x^{-q-(k-1)\lambda}}{-\lambda}\end{aligned}$$

and so

$$\begin{aligned}x^{-q-(k-1)\lambda} - \frac{1}{a}x^{-q-(k-1)\lambda-1} &+ \frac{a^{-\lambda}-1}{\lambda}(\nu x^{-q-k\lambda} + (1-\nu)x^{-q-(k-1)\lambda})\\ &\leq (\nu + (1-\nu)a^{-\lambda})\frac{x^{-q-k\lambda} - x^{-q-(k-1)\lambda}}{-\lambda}.\end{aligned}$$

The desired result follows from (7) by applying the functional calculus $x = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and then multiplying both sides with $A^{\frac{1}{2}}$. $\square$

Corollary 2.10. *[15, Theorem 2.6.] Let $a > 0$, $\lambda \in (0,1]$, and $\nu \in [0,1]$. For any invertible positive operators A and B,*

$$A - \frac{1}{a}AB^{-1}A + \frac{a^{-\lambda}-1}{\lambda}(\nu A\sharp_{-\lambda}B + (1-\nu)A) \leq (\nu + (1-\nu)a^{-\lambda})T_{-\lambda}(A|B).$$

Proof. By setting $q = 0$ and $k = 1$ in Theorem 2.9, we conclude the result. $\square$

Corollary 2.11. *Let $a > 0$, $\lambda \in (0,1]$, $q \geq 0$, $k \in \mathbb{N}$, and $\nu \in [0,1]$. For any invertible positive operators A and B,*

$$\begin{aligned}&(1-\log a)A\sharp_{-q-(k-1)\lambda}B - \frac{1}{a}A\sharp_{-q-(k-1)\lambda-1}B\\ &\quad\leq (\nu + (1-\nu)a^{-\lambda})\widetilde{T}_{-q,k,-\lambda}(A|B)\\ &\quad - (\log a)A\sharp_{-q-(k-1)\lambda}B + \frac{a^{\lambda}-1}{\lambda a^{\lambda}}(\nu A\sharp_{-q-k\lambda}B + (1-\nu)A\sharp_{-q-(k-1)\lambda}B)\\ &\quad\leq S_{-q-(k-1)\lambda}(A|B).\end{aligned} \tag{20}$$

Proof. The upper bound of $(\nu + (1-\nu)a^{-\lambda})\widetilde{T}_{-q,k,-\lambda}(A|B)$ follows from Theorem 2.5. So,

$$\begin{aligned}(\nu + (1-\nu)a^{-\lambda})\widetilde{T}_{-q,k,-\lambda}(A|B) \le\ & S_{-q-(k-1)\lambda}(A|B) \\ & + (\log a)A\sharp_{-q-(k-1)\lambda}B \\ & - \frac{a^{\lambda}-1}{\lambda a^{\lambda}}(\nu A\sharp_{-q-k\lambda}B + (1-\nu)A\sharp_{-q-(k-1)\lambda}B).\end{aligned} \tag{21}$$

Combining this by the lower bound of $(\nu + (1-\nu)a^{-\lambda})\widetilde{T}_{-q,k,-\lambda}(A|B)$ in Theorem 2.9, we obtain

$$\begin{aligned} & A\sharp_{-q-(k-1)\lambda}B - \frac{1}{a}A\sharp_{-q-(k-1)\lambda-1}B + \frac{a^{-\lambda}-1}{\lambda}(\nu A\sharp_{-q-k\lambda}B + (1-\nu)A\sharp_{-q-(k-1)\lambda}B) \\ & \quad \le (\nu + (1-\nu)a^{-\lambda})\widetilde{T}_{-q,k,-\lambda}(A|B) \\ & \quad \le S_{-q-(k-1)\lambda}(A|B) \\ & \quad + (\log a)A\sharp_{-q-(k-1)\lambda}B \\ & \quad - \frac{a^{\lambda}-1}{\lambda a^{\lambda}}(\nu A\sharp_{-q-k\lambda}B + (1-\nu)A\sharp_{-q-(k-1)\lambda}B).\end{aligned} \tag{22}$$

Adding the operator

$$-(\log a)A\sharp_{-q-(k-1)\lambda}B + \frac{a^{\lambda}-1}{\lambda a^{\lambda}}(\nu A\sharp_{-q-k\lambda}B + (1-\nu)A\sharp_{-q-(k-1)\lambda}B)$$

to both sides of the inequalities in (22), we find the desired inequalities. □

Corollary 2.12. *Let $a > 0$, $\lambda \in (0,1]$, $q \ge 0$, $k \in \mathbb{N}$, and $\nu \in [0,1]$. For any invertible positive operators A and B,*

$$\begin{aligned} S_{q+(k-1)\lambda}(A|B) & \le ((1-\nu) + \nu a^{-\lambda})\widetilde{T}_{q,k,\lambda}(A|B) \\ & + \frac{a^{-\lambda}-1}{\lambda}((1-\nu)A\sharp_{q+k\lambda}B + \nu A\sharp_{q+(k-1)\lambda}B) + (\log a)A\sharp_{q+(k-1)\lambda}B \\ & \le \frac{1}{a}A\sharp_{q+(k-1)\lambda+1}B + (\log a - 1)A\sharp_{q+(k-1)\lambda}B.\end{aligned} \tag{23}$$

Proof. Since $\frac{(ax)^{\lambda}-1}{\lambda} \le ax - 1$, by multiplying its both sides with $x^{q+(k-1)\lambda}$ and using (5) it follows from that

$$\begin{aligned} & \frac{a^{\lambda}-1}{\lambda}(\nu x^{q+k\lambda} + (1-\nu)x^{q+(k-1)\lambda}) \\ & \quad + (\nu + (1-\nu)a^{\lambda})\frac{x^{q+k\lambda} - x^{q+(k-1)\lambda}}{\lambda} \le (ax-1)x^{q+(k-1)\lambda}.\end{aligned}$$

By applying the functional calculus $x = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and then multiplying both sides with $A^{\frac{1}{2}}$ we get

$$\begin{aligned} (\nu + (1-\nu)a^{\lambda})\widetilde{T}_{q,k,\lambda}(A|B) & + \frac{a^{\lambda}-1}{\lambda}(\nu A\sharp_{q+k\lambda}B + (1-\nu)A\sharp_{q+(k-1)\lambda}B) \\ & \le aA\sharp_{q+(k-1)\lambda+1}B - A\sharp_{q+(k-1)\lambda}B.\end{aligned} \tag{24}$$

The lower bound of the left side of (24) is $S_{q+(k-1)\lambda}(A|B) + (\log a)A\sharp_{q+(k-1)\lambda}B$ by (2). So, the inequalities (2) and (24) entail that

$$\begin{aligned} & S_{q+(k-1)\lambda}(A|B) + (\log a)A\sharp_{q+(k-1)\lambda}B \\ & \quad \le (\nu + (1-\nu)a^{\lambda})\widetilde{T}_{q,k,\lambda}(A|B) \\ & \quad + \frac{a^{\lambda}-1}{\lambda}(\nu A\sharp_{q+k\lambda}B + (1-\nu)A\sharp_{q+(k-1)\lambda}B) \\ & \quad \le aA\sharp_{q+(k-1)\lambda+1}B - A\sharp_{q+(k-1)\lambda}B.\end{aligned} \tag{25}$$

By replacing a with a^{-1} and $1-\nu$ with ν, respectively in (25) and adding the term $(\log a)A\sharp_{q+(k-1)\lambda}B$ to both sides of the inequalities in (25), we get the result. □

Corollary 2.13. *Let $a>0$, $\lambda\in(0,1]$, $q\geq 0$, $k\in\mathbb{N}$, and $\nu\in[0,1]$. For any invertible positive operators A and B,*

$$\begin{aligned}
&(1-\log a)A\sharp_{-q-(k-1)\lambda}B-\frac{1}{a}A\sharp_{-q-(k-1)\lambda-1}B\\
&\quad\leq(\nu+(1-\nu)a^{-\lambda})\widetilde{T}_{-q,k,-\lambda}(A|B)\\
&\qquad-(\log a)A\sharp_{-q-(k-1)\lambda}B+\frac{a^{\lambda}-1}{\lambda a^{\lambda}}(\nu A\sharp_{-q-k\lambda}B+(1-\nu)A\sharp_{-q-(k-1)\lambda}B)\\
&\quad\leq S_{-q-(k-1)\lambda}(A|B)\\
&\quad\leq S(A|B)\\
&\quad\leq S_{q+(k-1)\lambda}(A|B)\\
&\quad\leq((1-\nu)+\nu a^{-\lambda})\widetilde{T}_{q,k,\lambda}(A|B)\\
&\quad+\frac{a^{-\lambda}-1}{\lambda}((1-\nu)A\sharp_{q+k\lambda}B+\nu A\sharp_{q+(k-1)\lambda}B)+(\log a)A\sharp_{q+(k-1)\lambda}B\\
&\quad\leq\frac{1}{a}A\sharp_{q+(k-1)\lambda+1}B+(\log a-1)A\sharp_{q+(k-1)\lambda}B.
\end{aligned}$$

Proof. The first and second inequalities follow from Corollary 2.11. The third and forth inequalities are a consequence of Corollary 2.8 and the fifth and sixth inequalities are based on Corollary 2.12. □

We remark that [14, Theorem 2.1] follows from the first inequality of above corollary by setting $q=0$, $k=1$, and $\nu=0$. Indeed, we observe

$$A-\frac{1}{a}AB^{-1}A+\frac{1-a^{\lambda}}{\lambda a^{\lambda}}A\leq a^{-\lambda}T_{-\lambda}(A|B).$$

Set $q=0$, $k=1$, and $\nu=0$ in the second inequality of (20) to obtain

$$-(\log a+\frac{1-a^{\lambda}}{\lambda a^{\lambda}})A+a^{-\lambda}T_{-\lambda}(A|B)\leq S(A|B) \tag{26}$$

and put $q=0$, $k=1$, and $\nu=1$ in the first inequality of (25) to reach

$$S(A|B)\leq T_{\lambda}(A|B)-\frac{1-a^{\lambda}}{\lambda}A\sharp_{\lambda}B-(\log a)A. \tag{27}$$

By combining (26) and (27), we conclude [14, Theorem 2.2].

Corollary 2.14. *[14, Theorem 2.3] Let $a>0$, $\lambda\in(0,1]$. For any invertible positive operators A and B,*

$$\begin{aligned}
(1-\log a)A-\frac{1}{a}AB^{-1}A&\leq-(\log a+\frac{1-a^{\lambda}}{\lambda a^{\lambda}})A+a^{-\lambda}T_{-\lambda}(A|B)\\
&\leq S(A|B)\\
&\leq(\log a)A+T_{\lambda}(A|B)+\frac{1-a^{\lambda}}{\lambda a^{\lambda}}A\sharp_{\lambda}B\\
&\leq(\log a-1)A+\frac{1}{a}B.
\end{aligned}\tag{28}$$

Proof. By setting $q=0$, $k=1$, and $\nu=0$ in Corollary 2.13 we obtain the result. □

Note that Corollary 2.13 is a generalization of Corollary 2.14 and a refinement of the following inequality:

$$(1 - \log a)A - \frac{1}{a}AB^{-1}A \leq S(A|B) \leq (\log a - 1)A + \frac{1}{a}B, \tag{29}$$

which is due to Furuta [3].

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