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Relationships Between Identities for Quantum Bernstein Bases and Formulas for Hypergeometric Series

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Abstract. Two seemingly disparate mathematical entities – quantum Bernstein bases and hypergeometric series – are revealed to be intimately related. The partition of unity property for quantum Bernstein bases is shown to be equivalent to the Chu-Vandermonde formula for hypergeometric series, and the Marsden identity for quantum Bernstein bases is shown to be equivalent to the Pfaff-Saalschütz formula for hypergeometric series. The equivalence of the q-versions of these formulas and identities is also demonstrated.

1. Introduction

We are going to investigate connections between two ostensibly very different, but nevertheless deeply interrelated, theories: quantum Bernstein bases and hypergeometric series. Quantum Bernstein bases were introduced first in Approximation Theory and later in Computer-Aided Geometric Design [3, 5, 9, 11, 12, 14, 16–19] as an extension of the classical Bernstein bases [4], to study approximation methods for curves and surfaces. Hypergeometric series were initiated by Euler and Gauss in the early 19th century. The first important use of hypergeometric series was in (as solutions of) differential equations, but later hypergeometric series have become prominent in number theory, combinatonics, orthogonal polynomials, approximation theory, physics, and other fields [1, 2, 6, 8].

Quantum Bernstein bases come in two flavors: the h-Bernstein bases and the q-Bernstein bases. The h-Bernstein bases were studied first by Stancu [18, 19], later by Goldman and Barry [3, 5], and more recently by Simeonov et al. [16]. The q-Bernstein bases were introduced by Phillips and his collaborators [9–14] for the interval $[0, 1]$ and extended to arbitrary intervals $[a, b]$ by Lewanowicz and Woźny [7] and Simeonov et al. [17]. Two fundamental identities are valid for both types of quantum Bernstein bases: the partition of unity property and a Marsden identity. The partition of unity property ensures that approximations using these bases are affine invariant, that is, these approximations are independent of the underlying coordinate system. The importance of the Marsden identity is that this identity can be used to derive many other

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identities for these bases, as well as to represent the monomials in terms of these bases. The quantum versions of the Marsden identity are equivalent to the quantum forms of the blossom, which, in analogy to the classical blossom for classical Bernstein bases, provide the dual functionals for the quantum Bernstein bases [16, 17]. For classical Bernstein bases, these two formulas – the partition of unity and the Marsden identity – are both direct consequences of the binomial theorem, but for the quantum Bernstein bases these formulas have deeper origins.

Hypergeometric series also come in two types: the classical versions and the q-versions (basic hypergeometric series) [1, 2, 6, 8]. The Chu-Vandermonde formula and the Pfaff-Saalschütz formula are among the few examples of hypergeometric series that have closed forms (are summable) and are widely used to establish many other identities [1, 2, 6, 8].

Our results reveal that there is a strong connection between quantum Bernstein bases and hypergeometric series, even though these two theories grew out of two very different mathematical traditions. We shall show that:

1. the partition of unity property for the h-Bernstein bases is equivalent to the classical Chu-Vandermonde formula for hypergeometric series;
2. the Marsden identity for the h-Bernstein bases is equivalent to the classical Pfaff-Saalschütz formula for hypergeometric series;
3. the partition of unity property for the q-Bernstein bases is equivalent to the q-Chu-Vandermonde formula for basic hypergeometric series;
4. the Marsden identity for the q-Bernstein bases is equivalent to the q-Pfaff-Saalschütz formula for basic hypergeometric series.

We take our inspiration from the paper by Simeonov and Goldman [15], who introduced and investigated an Askey-Wilson analogue of the Bernstein bases. Simeonov and Goldman proved that:

1. the partition of unity property for the Askey-Wilson Bernstein bases is equivalent to the terminating version of Rogers' ${}_6\phi_5$ sum [2];
2. the Marsden identity for the Askey-Wilson Bernstein bases is equivalent to a summation formula for a very-well poised terminating ${}_8W_7$ series (Jackson's q-analogue of Dougall's ${}_7F_6$ sum [2]).

This paper is organized as follows. In Section 2, we investigate the relationships between identities for the h-Bernstein bases and hypergeometric series. In Section 2.1 we establish the notation and a few simple identities for shifted factorials. In Section 2.2, we recall the definition of hypergeometric series and state the Chu-Vandermonde and Pfaff-Saalschütz formulas. In Section 2.3 we briefly review the h-Bernstein bases, the h-partition of unity property, and the h-Marsden identity. In Section 2.4 we prove the first two of our main results: items 1 and 2 listed above.

Section 3 parallels Section 2, but here we investigate the relationships between identities for the q-Bernstein bases and formulas for basic (q-)hypergeometric series. In Section 3.1 we establish the notation and a few simple identities for q-shifted factorials. In Section 3.2, we recall the definition of basic hypergeometric series and state the q-Chu-Vandermonde and q-Pfaff-Saalschütz formulas. In Section 3.3 we briefly review the q-Bernstein bases, the q-partition of unity property, and the q-Marsden identity. In Section 3.4 we prove the other two of our main results: items 3 and 4 listed above. We close in Section 4 with a brief discussion about the future potential of our intuitions.

2. h-Bernstein basis functions and hypergeometric series

2.1. Shifted factorials

Throughout this section, we shall adopt the following standard definitions and notation for the shifted factorials and the multiple shifted factorials [1, 2, 8].

Shifted Factorials

$$(A)_0 = 1, \quad (A)_n = \prod_{i=0}^{n-1}(A+i), \quad n = 1, 2, \dots . \tag{2.1}$$

Multiple Shifted Factorials

$$(a_1, a_2, \dots, a_m)_n = (a_1)_n (a_2)_n \cdots (a_m)_n. \tag{2.2}$$

The following straightforward identities for shifted factorials will be used in Section 2.4.

$$\binom{A}{n} = (-1)^n \frac{(-A)_n}{n!}, \tag{2.3}$$

$$(A)_{n-k} = \frac{(-1)^k (A)_n}{(-A-n+1)_k}, \tag{2.4}$$

$$(A-n+1)_n = (-1)^n (-A)_n. \tag{2.5}$$

In (2.3), $\binom{A}{n}$ denotes the generalized binomial coefficient [6].

2.2. Hypergeometric series

The ${}_rF_s$ hypergeometric series is defined by [8, (1.3.11)]

$${}_rF_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r)_k}{(b_1, \dots, b_s)_k} \frac{z^k}{k!}. \tag{2.6}$$

If $a_j = -n$ for some $n \in \mathbb{N}$, then $(a_j)_k = (-n)_k = 0$ for all $k \geq n+1$. Therefore, in this case, the right-hand side of (2.6) reduces to a finite sum:

$${}_rF_s\left(\begin{matrix} -n, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}\middle| z\right) = \sum_{k=0}^{n} \frac{(-n, a_2, \dots, a_r)_k}{(b_1, \dots, b_s)_k} \frac{z^k}{k!}. \tag{2.7}$$

We shall focus on the following two classical hypergeometric formulas:
The Chu-Vandermonde summation formula [8, (1.4.3)]

$${}_2F_1\left(\begin{matrix} -n, B \\ C \end{matrix}\middle| 1\right) = \frac{(C-B)_n}{(C)_n}. \tag{2.8}$$

The Pfaff-Saalschütz summation formula [8, (1.4.5)]

$${}_3F_2\left(\begin{matrix} -n, A, B \\ C, D \end{matrix}\middle| 1\right) = \frac{(C-A)_n (C-B)_n}{(C)_n (C-A-B)_n}, \tag{2.9}$$

where $D = A + B - C + 1 - n$.

2.3. h-Bernstein basis functions

The h-Bernstein basis functions over the interval $[a, b]$ are defined by [16, (2.1)]

$$B_k^n(t; [a, b]; h) = \binom{n}{k} \frac{\prod_{i=0}^{k-1}(t-a+ih) \prod_{i=0}^{n-k-1}(b-t+ih)}{\prod_{i=0}^{n-1}(b-a+ih)}, \quad k = 0, \dots, n. \tag{2.10}$$

These functions were introduced by Stancu [18, 19] to define and study generalized Bernstein approximating operators. We shall focus on the following two identities for the h-Bernstein bases:
The Partition of Unity Property [16, (5.2)]

$$\sum_{k=0}^{n} B_k^n(t;[a,b];h) = 1. \tag{2.11}$$

The h-Marsden Identity [16, (5.1)]

$$\frac{\prod_{i=0}^{n-1}(x-t+ih)}{\prod_{i=0}^{n-1}(b-a+ih)} = \sum_{k=0}^{n}(-1)^k \frac{B_{n-k}^n(x;[a-(n-1)h,b];-h)}{\binom{n}{k}} B_k^n(t;[a,b];h). \tag{2.12}$$

2.4. Equivalences between identities for h-Bernstein bases and hypergeometric summation formulas

We begin with an alternative representation for the h-Bernstein basis functions $B_k^n(t;[a,b];h)$ in terms of the shifted factorials.

Proposition 2.1.

$$B_k^n(t;[a,b];h) = \frac{\left(\frac{b-t}{h}\right)_n}{\left(\frac{b-a}{h}\right)_n} \frac{\left(-n, \frac{t-a}{h}\right)_k}{k!\left(\frac{t-b}{h}-n+1\right)_k}, \qquad k = 0,\dots,n. \tag{2.13}$$

Proof. Factoring out h from the products on the right-hand side of (2.10) and using (2.3), we can rewrite the h-Bernstein basis functions in the form

$$B_k^n(t;[a,b];h) = \frac{(-1)^k(-n)_k}{k!} \frac{h^k\left(\frac{t-a}{h}\right)_k h^{n-k}\left(\frac{b-t}{h}\right)_{n-k}}{h^n\left(\frac{b-a}{h}\right)_n}, \qquad k = 0,\dots,n.$$

Now (2.13) follows from this equation after simplifying and applying (2.4) with $A = \frac{b-t}{h}$. □

Next we will show that the hypergeometric formulas (2.8) and (2.9) can be derived from the identities (2.11) and (2.12) for the h-Bernstein basis functions.

Corollary 2.2. *The hypergeometric form of the partition of unity property is*

$${}_2F_1\left(\begin{matrix}-n, \frac{t-a}{h} \\ \frac{t-b}{h}-n+1\end{matrix}\middle| 1\right) = \frac{\left(\frac{b-a}{h}\right)_n}{\left(\frac{b-t}{h}\right)_n}. \tag{2.14}$$

Proof. Substituting (2.13) into the partition of unity property (2.11) yields

$$\sum_{k=0}^{n} \frac{\left(\frac{b-t}{h}\right)_n}{\left(\frac{b-a}{h}\right)_n} \frac{(-n)_k\left(\frac{t-a}{h}\right)_k}{k!\left(\frac{t-b}{h}-n+1\right)_k} = 1.$$

Multiplying both sides by $\frac{\left(\frac{b-a}{h}\right)_n}{\left(\frac{b-t}{h}\right)_n}$ and invoking (2.7), we obtain (2.14). □

Proposition 2.3.

$$B_{n-k}^n(x;[a-(n-1)h,b];-h) = (-1)^k\binom{n}{k}\frac{\left(\frac{x-a}{h}\right)_n\left(\frac{x-b}{h}\right)_k}{\left(\frac{b-a}{h}\right)_n\left(\frac{x-a}{h}\right)_k}. \tag{2.15}$$

Proof. Using (2.10) and factoring out $-h$ from the products on the right-hand side, we can rewrite $B^n_{n-k}(x;[a-(n-1)h,b];-h)$ in the form

$$B^n_{n-k}(x;[a-(n-1)h,b];-h)=\frac{\binom{n}{n-k}\left(\frac{a-x}{h}-n+1\right)_{n-k}\left(\frac{x-b}{h}\right)_k}{\left(\frac{a-b}{h}-n+1\right)_n}.$$

Applying (2.4) with $A=\frac{a-x}{h}-n+1$, this equation reduces to

$$B^n_{n-k}(x;[a-(n-1)h,b];-h)=(-1)^k\binom{n}{k}\frac{\left(\frac{a-x}{h}-n+1\right)_n\left(\frac{x-b}{h}\right)_k}{\left(\frac{a-b}{h}-n+1\right)_n\left(\frac{x-a}{h}\right)_k}.$$

Invoking (2.5) with $A=\frac{a-x}{h}$ and with $A=\frac{a-b}{h}$ yields (2.15). □

Corollary 2.4. *The hypergeometric form of the h-Marsden identity is*

$${}_3F_2\left(\begin{matrix}-n,\frac{x-b}{h},\frac{t-a}{h}\\ \frac{x-a}{h},\frac{t-b}{h}-n+1\end{matrix}\middle|1\right)=\frac{\left(\frac{x-t}{h}\right)_n\left(\frac{b-a}{h}\right)_n}{\left(\frac{x-a}{h}\right)_n\left(\frac{b-t}{h}\right)_n}. \tag{2.16}$$

Proof. Using (2.15) and (2.13), we can rewrite the right-hand side of the h-Marsden identity (2.12) in the form

$$\sum_{k=0}^{n}\frac{\left(\frac{x-a}{h}\right)_n\left(\frac{x-b}{h}\right)_k}{\left(\frac{b-a}{h}\right)_n\left(\frac{x-a}{h}\right)_k}\frac{\left(\frac{b-t}{h}\right)_n}{\left(\frac{b-a}{h}\right)_n}\frac{(-n)_k\left(\frac{t-a}{h}\right)_k}{k!\left(\frac{t-b}{h}-n+1\right)_k}=\frac{\left(\frac{x-a}{h}\right)_n\left(\frac{b-t}{h}\right)_n}{\left(\frac{b-a}{h}\right)_n^2}\sum_{k=0}^{n}\frac{(-n)_k\left(\frac{x-b}{h}\right)_k\left(\frac{t-a}{h}\right)_k}{k!\left(\frac{x-a}{h}\right)_k\left(\frac{t-b}{h}-n+1\right)_k}.$$

Rewriting the left-hand side of the h-Marsden identity (2.12) in terms of shifted factorials, (2.12) becomes

$$\frac{\left(\frac{x-t}{h}\right)_n}{\left(\frac{b-a}{h}\right)_n}=\frac{\left(\frac{x-a}{h}\right)_n\left(\frac{b-t}{h}\right)_n}{\left(\frac{b-a}{h}\right)_n^2}\sum_{k=0}^{n}\frac{(-n)_k\left(\frac{x-b}{h}\right)_k\left(\frac{t-a}{h}\right)_k}{k!\left(\frac{x-a}{h}\right)_k\left(\frac{t-b}{h}-n+1\right)_k},$$

which by (2.7) is equivalent to (2.16). □

Theorem 2.5. *The Partition of Unity property for the h-Bernstein basis functions is equivalent to the Chu-Vandermonde summation formula for hypergeometric series.*

Proof. We first derive the Chu-Vandermonde formula (2.8) from the hypergeometric form (2.14) of the partition of unity property. In (2.14) set $\frac{t-a}{h}=B$ and $\frac{t-b}{h}-n+1=C$. With these substitutions, the left-hand side of (2.14) becomes the left-hand side of (2.8). The right-hand side of (2.14) becomes $\frac{(B-C-n+1)_n}{(-C-n+1)_n}=\frac{(C-B)_n}{(C)_n}$ after applying (2.5), which is the right-hand side of (2.8).

Conversely, the hypergeometric form (2.14) of the partition of unity property can be derived from the Chu-Vandermonde formula (2.8) by setting in (2.8) $B=\frac{t-a}{h}$ and $C=\frac{t-b}{h}-n+1$. With these substitutions, (2.8) becomes

$${}_2F_1\left(\begin{matrix}-n,\frac{t-a}{h}\\ \frac{t-b}{h}-n+1\end{matrix}\middle|1\right)=\frac{\left(\frac{a-b}{h}-n+1\right)_n}{\left(\frac{t-b}{h}-n+1\right)_n}=\frac{\left(\frac{b-a}{h}\right)_n}{\left(\frac{b-t}{h}\right)_n},$$

where we used (2.5) with $A=\frac{a-b}{h}$ and with $A=\frac{t-b}{h}$, which is the hypergeometric form (2.14) of the partition of unity property. □

Theorem 2.6. *The Marsden Identity for the h-Bernstein basis functions is equivalent to the Pfaff-Saalschütz summation formula for hypergeometric series.*

Proof. First we derive the Pfaff-Saalschütz formula (2.9) from the hypergeometric form (2.16) of the h-Marsden identity. In (2.16) set $\frac{x-b}{h} = A$, $\frac{t-a}{h} = B$, $\frac{x-a}{h} = C$, and $\frac{t-b}{h} - n + 1 = D$. With these substitutions $D = A + B - C + 1 - n$, and (2.16) becomes (2.9).

Conversely, the hypergeometric form (2.16) of the h-Marsden identity can be derived from the Pfaff-Saalschütz formula (2.9). Set $A = \frac{x-b}{h}$, $B = \frac{t-a}{h}$, $C = \frac{x-a}{h}$, and $D = \frac{t-b}{h} - n + 1$ in (2.9). With these substitutions, (2.9) becomes (2.16), which is the hypergeometric form of the h-Marsden identity. □

3. q-Bernstein basis functions and basic hypergeometric series

3.1. q-Shifted factorials

Throughout this section we will adopt the following standard definitions and notation for the q-integers, q-factorials, q-shifted factorials, and the q-binomial coefficients [1, 2, 8].

q-Integers

$$[0]_q = 1, \qquad [n]_q = \begin{cases} (1-q^n)/(1-q), & q \neq 1, \\ n, & q = 1, \end{cases} \qquad n = 1, 2, \dots . \tag{3.1}$$

q-Factorials

$$[0]_q! = 1, \qquad [n]_q! = \prod_{k=1}^{n} [k]_q, \qquad n = 1, 2, \dots . \tag{3.2}$$

q-Shifted Factorials

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \dots . \tag{3.3}$$

Multiple q-Shifted Factorials

$$(a_1, \dots, a_m; q)_n = \prod_{k=0}^{m} (a_k; q)_n, \quad n = 0, 1, \dots . \tag{3.4}$$

q-Binomial Coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \qquad k = 0, \dots, n. \tag{3.5}$$

We will use the following straightforward identities for q-shifted factorials in Section 3.4

$$(q^{-n}E; q)_n = (-E)^n q^{-\binom{n+1}{2}} (q/E; q)_n, \tag{3.6}$$

$$(E; q)_{n-k} = (E; q)_n (-E)^{-k} q^{\binom{n-k}{2} - \binom{n}{2}} / (q^{1-n}/E; q)_k. \tag{3.7}$$

3.2. Basic hypergeometric series

The ${}_r\phi_s$ basic hypergeometric series is defined by [8, (12.1.6)]

$${}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} (-q^{(k-1)/2})^{k(s+1-r)} z^k. \tag{3.8}$$

If $r = s + 1$, (3.8) reduces to

$${}_{s+1}\phi_s\left(\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix} \middle| q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_{s+1}; q)_k}{(q, b_1, \dots, b_s; q)_k} z^k. \tag{3.9}$$

In addition, if $a_j = q^{-n}$ for some $n \in \mathbb{N}$, then $(a_j;q)_k = (q^{-n};q)_k = 0$ for all $k \geq n+1$. Therefore, in this case, the right-hand side of (3.9) reduces to a finite sum:

$$ {}_{s+1}\phi_s\left(\begin{matrix} q^{-n}, a_2, \ldots, a_{s+1} \\ b_1, b_2, \ldots, b_s \end{matrix}\middle| q, z\right) = \sum_{k=0}^{n} \frac{(q^{-n}, a_2, \ldots, a_{s+1}; q)_k}{(q, b_1, \ldots, b_s; q)_k} z^k. \tag{3.10} $$

We shall focus on the following two basic hypergeometric formulas:
The q-Chu-Vandermonde summation formula [8, (12.2.17)]

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, A \\ C \end{matrix}\middle| q, q\right) = A^n \frac{(C/A;q)_n}{(C;q)_n}. \tag{3.11} $$

The q-Pfaff-Saalschütz summmation formula [8, (12.2.15)]

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, A, B \\ C, D \end{matrix}\middle| q, q\right) = \frac{(C/A, C/B;q)_n}{(C, C/(AB);q)_n}, \tag{3.12} $$

where $CD = q^{1-n}AB$.

3.3. q-Bernstein basis functions

The q-Bernstein basis functions over the interval $[a,b]$ are defined by [17, (6.1)]

$$ B_k^n(t;[a,b];q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(t-aq^i)\prod_{i=0}^{n-k-1}(b-tq^i)}{\prod_{i=0}^{n-1}(b-aq^i)}, \qquad k = 0, \ldots, n. \tag{3.13} $$

Once again we shall focus on the following two identities for the q-Bernstein bases:
The Partition of Unity Property [17, (7.2)]

$$ \sum_{k=0}^{n} B_k^n(t;[a,b];q) = 1. \tag{3.14} $$

The q-Marsden Identity [17, (7.1)]

$$ \frac{\prod_{i=0}^{n-1}(x-tq^i)}{\prod_{i=0}^{n-1}(b-aq^i)} = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \frac{B_{n-k}^n(x;[q^{n-1}a,b];1/q)}{\begin{bmatrix} n \\ k \end{bmatrix}_{1/q}} B_k^n(t;[a,b];q). \tag{3.15} $$

3.4. Equivalences between identities for q-Bernstein basis functions and basic hypergeometric sums

Here we follow the approach in Section 2.4 for the h-Bernstein basis functions and hypergeometric series to derive relationships between identities for the q-Bernstein basis functions and basic hypergeometric formulas.

We begin with an alternative representation for the q-Bernstein basis functions $B_k^n(t;[a,b];q)$ in terms of the q-shifted factorials.

Proposition 3.1.

$$ B_k^n(t;[a,b];q) = \frac{(t/b;q)_n}{(a/b;q)_n} \frac{(q^{-n}, a/t;q)_k}{(q, q^{1-n}b/t;q)_k} q^k. \tag{3.16} $$

Proof. Using q-shifted factorials, we can rewrite (3.13) as

$$B_k^n(t;[a,b];q) = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} t^k (a/t;q)_k b^{n-k} (t/b;q)_{n-k} \frac{b^{-n}}{(a/b;q)_n}.$$

Applying (3.7) with $E = t/b$ and with $E = q$, we obtain

$$\begin{aligned} B_k^n(t;[a,b];q) &= \frac{(q;q)_n}{(q;q)_k} t^k (a/t;q)_k b^{-k} \frac{1}{(a/b;q)_n} \frac{(t/b;q)_n (q^{-n};q)_k}{(q;q)_n (q^{1-n}b/t;q)_k} (qb/t)^k \\ &= \frac{(t/b;q)_n}{(a/b;q)_n} \frac{(q^{-n};q)_k (a/t;q)_k}{(q;q)_k (q^{1-n}b/t;q)_k} q^k. \end{aligned}$$

□

Next we will show that the basic hypergeometric formulas (3.11) and (3.12) can be derived from the identities (3.14) and (3.15) for the q-Bernstein basis functions.

Corollary 3.2. *The basic hypergeometric form of the partition of unity property is*

$${}_2\phi_1\left(\begin{matrix} q^{-n}, a/t \\ q^{1-n}b/t \end{matrix} \middle| q, q\right) = \frac{(a/b;q)_n}{(t/b;q)_n}. \tag{3.17}$$

Proof. Substituting (3.16) into the partition of unity property (3.14) yields

$$\frac{(t/b;q)_n}{(a/b;q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, a/t;q)_k}{(q, q^{1-n}b/t;q)_k} q^k = 1.$$

Multiplying both sides by $\frac{(a/b;q)_n}{(t/b;q)_n}$ and invoking (3.10), we obtain (3.17). □

Proposition 3.3.

$$B_{n-k}^n(x;[q^{n-1}a,b];1/q) = \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} (-1)^k q^{-\binom{k}{2}} (x/b)^n \frac{(a/x;q)_n (b/x;q)_k}{(a/b;q)_n (a/x;q)_k}. \tag{3.18}$$

Proof. Using (3.13), we can rewrite $B_{n-k}^n(x;[q^{n-1}a,b];1/q)$ in the form

$$B_{n-k}^n(x;[q^{n-1}a,b];1/q) = \begin{bmatrix} n \\ n-k \end{bmatrix}_{1/q} x^{n-k} (q^{n-1}a/x;1/q)_{n-k} b^k (x/b;1/q)_k \frac{b^{-n}}{(q^{n-1}a/b;1/q)_n}. \tag{3.19}$$

From (3.3), it follows easily that

$$\begin{aligned} &(q^{n-1}a/x;1/q)_{n-k} = \frac{(a/x;q)_n}{(a/x;q)_k}, \\ &(x/b;1/q)_k = (-x/b)^k q^{-\binom{k}{2}} (b/x;q)_k, \\ &(q^{n-1}a/b;1/q)_n = (a/b;q)_n. \end{aligned}$$

Substituting these identities into (3.19), we obtain (3.18). □

Corollary 3.4. *The basic hypergeometric form of the q-Marsden identity is*

$${}_3\phi_2\left(\begin{matrix} q^{-n}, b/x, a/t \\ a/x, q^{1-n}b/t \end{matrix} \middle| q, q\right) = \frac{(a/b, t/x;q)_n}{(a/x, t/b;q)_n}. \tag{3.20}$$

Proof. Substituting (3.18) and (3.16) into the q-Marsden identity (3.15), we get

$$\frac{x^n}{b^n}\sum_{k=0}^{n}\frac{(a/x;q)_n(t/b;q)_n}{(a/b;q)_n^2}\frac{(q^{-n};q)_k(a/t;q)_k(b/x;q)_k}{(q;q)_k(a/x;q)_k(q^{1-n}b/t;q)_k}q^k=\frac{x^n}{b^n}\frac{(t/x;q)_n}{(a/b;q)_n}.$$

After simplifying, this equation reduces to

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k(a/t;q)_k(b/x;q)_k}{(q;q)_k(a/x;q)_k(q^{1-n}b/t;q)_k}q^k=\frac{(t/x;q)_n(a/b;q)_n}{(t/b;q)_n(a/x;q)_n}.$$

By (3.10) the basic hypergeometric form of this sum is (3.20). □

Theorem 3.5. *The Partition of Unity property for the q-Bernstein basis functions is equivalent to the q-Chu-Vandermonde summation formula for basic hypergeometric series.*

Proof. We first derive the q-Chu-Vandermonde summation formula (3.11) from the basic hypergeometric form (3.17) of the partition of unity property for the q-Bernstein basis functions. Set $a/t = A$ and $q^{1-n}b/t = C$ in (3.17). Then the left-hand side of (3.17) becomes the left-hand side of (3.11). The right-hand side of (3.17) becomes $\frac{(q^{1-n}A/C;q)_n}{(q^{1-n}/C;q)_n}$, which reduces to the right-hand side of (3.11) after applying (3.6) with $E = qA/C$ and with $E = q/C$.

Conversely, the basic hypergeometric form (3.17) of the partition of unity property can be derived from the q-Chu-Vandermonde formula (3.11) by setting in (3.11) $A = a/t$ and $C = q^{1-n}b/t$. With these substitutions, (3.11) becomes

$${}_2\phi_1\left(\begin{matrix}q^{-n}, a/t\\ q^{1-n}b/t\end{matrix}\middle|q, q\right)=(a/t)^n\frac{(q^{1-n}b/a;q)_n}{(q^{1-n}b/t;q)_n}=\frac{(a/b;q)_n}{(t/b;q)_n},$$

after applying (3.6) with $E = qb/a$ and with $E = qb/t$, which is (3.17). □

Theorem 3.6. *The Marsden Identity for the q-Bernstein basis functions is equivalent to the q-Pfaff-Saalschütz summation formula for basic hypergeometric series.*

Proof. First we derive the q-Pfaff-Saalschütz summation formula (3.12) from the basic hypergeometric form (3.20) of the q-Marsden identity. In (3.20) set $b/x = A$, $a/t = B$, $a/x = C$, and $q^{1-n}b/t = D$. With these substitutions $CD = q^{1-n}AB$, and (3.20) becomes (3.12).

Conversely, the basic hypergeometric form (3.20) of the q-Marsden identity can be derived from the q-Pfaff-Saalschütz formula (3.12). Set $A = b/x$, $B = a/t$, $C = a/x$, and $D = q^{1-n}b/t$ in (3.12). With these substitutions, (3.12) becomes

$${}_3\phi_2\left(\begin{matrix}q^{-n}, b/x, a/t\\ a/x, q^{1-n}b/t\end{matrix}\middle|q, q\right)=\frac{(a/b, t/x;q)_n}{(a/x, t/b;q)_n},$$

which is the basic hypergeometric form (3.20) of the q-Marsden identity. □

4. Conclusion

We have shown that the theories of quantum Bernstein bases and hypergeometric series are intimately related. This new insight about two theories that grew out of two very different, seemingly unrelated, mathematical traditions, allows us to use methods and techniques (e.g. blossoming [16, 17]) from one theory to derive important results from the other theory. In this paper we have used four identities for quantum Bernstein bases to give new proofs of four standard hypergeometric sums. We expect that this newly established connection between these two fields will lead in the future to new results in both disciplines.

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