



Fixed Point Theorems in WC–Banach Algebras and Their Applications to Infinite Systems of Integral Equations

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Abstract. The paper is devoted to prove a few fixed point theorems for operators acting in WC–Banach algebras and satisfying some conditions expressed in terms of a generalized Lipschitz continuity and measures of weak noncompactness. Moreover, the assumptions imposed on the mentioned operators are formulated with help of weak topology and weak sequential continuity. Our fixed point results will be illustrated by proving the existence of solutions of an infinite system of nonlinear integral equations.

1. Introduction

The principal aim of the paper is to present a few results on the existence of fixed points of operators dealing on Banach algebras. More precisely, we will work with operators having the form $Tx = Ax + Bx + Cx$, where A , B , C are defined on subsets of a given Banach algebra and satisfy some conditions expressed in terms of a generalized Lipschitz continuity or the contractivity with respect to the De Blasi measure of weak noncompactness. But the main assumption exploited in our considerations is imposed on a Banach space X . Namely, it is required that X is the so-called WC–Banach algebra i.e., X is a Banach algebra in which the product of two weakly compact subsets of X is weakly compact. This concept was introduced quite recently in [6] and it turned out to be very fruitful in investigations of some problems of operator theory under weak topology features [14].

In the present paper we will also use conditions describing some other properties of operators acting in Banach spaces. Those conditions are denoted by (H_1) and (H_2) and they allows us to distinguish some classes of operators which transform each weakly convergent sequence in a Banach space X into a sequence containing a subsequence being strongly or weakly convergent in X , respectively.

The mentioned conditions (H_1) and (H_2) were used earlier by other authors but they play a specially important role in our considerations conducted in this paper.

The investigations of the paper will be illustrated by some examples showing the applicability and the usefulness of our results. Moreover, we provide an example indicating that our results are applicable in the theory of nonlinear integral equations.

Finally, let us mention that results obtained in the paper generalize a few ones contained in the papers [2, 6, 12, 15], for example.

2010 *Mathematics Subject Classification.* Primary 47H10, 47H08; Secondary 47H09, 45G15

Keywords. Fixed point theorem; WC–Banach algebra; Condition (H_1) ; Condition (H_2) ; Measure of weak noncompactness; Infinite system of integral equations

Received: 03 September 2019; Accepted: 13 February 2020

Communicated by Dragan S. Djordjević

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2. Preliminaries and auxiliary facts

In this section we establish the notation used in the paper and we provide a few auxiliary facts which will be needed in our considerations. Moreover, we give here definitions of basic concepts applied in our study and we also indicated some essential properties of the concepts appearing in our reasonings.

Throughout the paper we denote by \mathbb{R} the set of real numbers. The symbol \mathbb{N} stands for the set of natural numbers (positive integers). By the symbol X we will denote a Banach space endowed with the norm $\|\cdot\|_X$ and with zero element θ . In general, we write $\|\cdot\|$ in place of $\|\cdot\|_X$. For $r > 0$ the symbol B_r denotes the closed ball centered at θ and with radius r and $D(A)$ denotes the domain of an operator A . By the symbol \mathfrak{M}_X we will denote the collection of all nonempty bounded subsets of X while \mathfrak{N}^W stands for its subfamily consisting of all relatively weakly compact sets. Moreover, for an arbitrary subset M of the space X the symbol \overline{M}^W will stand for the weak closure of M and the symbol $\text{conv}M$ denotes the convex hull of M . We will also write $\text{Conv}M$ to denote closed convex hull of M . Apart from this we use the standard notation $M_1 + M_2, \lambda M (\lambda \in \mathbb{R})$ for algebraic operations on sets.

We use the standard notation \rightarrow to denote the strong convergence and \rightharpoonup to denote the weak convergence in X .

Further, let us recall the concept of the De Blasi measure of weak noncompactness [9] being the function $\omega : \mathfrak{M}_X \rightarrow \mathbb{R}_+ = [0, \infty)$, defined in the following way

$$\omega(M) = \inf\{r > 0 : \text{there exists } W \in \mathfrak{N}^W \text{ such that } M \subset W + B_r\}.$$

For our purpose we recall some basic properties of the measure of weak noncompactness [4, 9].

Lemma 2.1. *The De Blasi measure of weak noncompactness ω satisfies the following conditions:*

- (i) $M_1 \subset M_2 \Rightarrow \omega(M_1) \leq \omega(M_2)$.
- (ii) $\omega(M) = 0 \Leftrightarrow M \in \mathfrak{N}^W$.
- (iii) $\omega(\overline{M}^W) = \omega(M)$.
- (iv) $\omega(\text{conv}M) = \omega(M)$.
- (v) $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$.
- (vi) $\omega(\lambda M) = |\lambda|\omega(M)$ for $\lambda \in \mathbb{R}$.
- (vii) $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$.
- (viii) *If (M_n) is a decreasing sequence of nonempty, bounded and weakly closed subsets of X such that $\lim_{n \rightarrow \infty} \omega(M_n) = 0$, then the set $M_\infty = \bigcap_{n=1}^\infty M_n$ is nonempty and weakly compact.*

It may be shown that $\omega(B_1) = 1$ provided the space X is nonreflexive [9]. Obviously $\omega(B_1) = 0$ in the case where X is a reflexive Banach space.

Now, we recall the definitions of some classes of operators acting in a Banach space X .

Definition 2.2. An operator $A : D(A) \rightarrow X$ ($D(A) \subset X$) is said to be ω -contraction (or ω -contractive) if it maps bounded sets into bounded sets and there exists a constant $k \in [0, 1)$ such that $\omega(AS) \leq k\omega(S)$ for any set $S \subset D(A)$ and such that $S \in \mathfrak{M}_X$.

An operator $A : D(A) \rightarrow X$ is called ω -condensing if it maps bounded sets into bounded sets and $\omega(AS) < \omega(S)$ for all sets $S \subset D(A)$ such that $S \in \mathfrak{M}_X$ and $\omega(S) > 0$.

Let us notice that every ω -contractive operator is ω -condensing.

In what follows we will always assume that $A : D(A) \subset X \rightarrow X$ is a given operator.

The basic concept concerning operators involved in our considerations are described in the below given definition.

Definition 2.3. We will say that the operator A satisfies *the condition* (H_1) if for any sequence $(x_n) \subset D(A)$ which is weakly convergent in X , the sequence (Ax_n) has a strongly convergent subsequence in X . We say that the operator A satisfies *the condition* (H_2) if for each weakly convergent sequence $(x_n) \subset D(A)$ the sequence (Ax_n) contains a weakly convergent subsequence in X .

Let us mention that the conditions (H_1) and (H_2) were considered in [1, 12, 14, 15]. We refer to monograph [14] and to review article [8] for some properties of operators satisfying the conditions (H_1) and (H_2) . Below we provide a few remarks concerning operators satisfying the conditions (H_1) and (H_2) (cf. [8,14]).

At first, let us assume that the operator $A : D(A) \subset X \rightarrow X$ satisfies the condition (H_1) . Then, the image AY of any relatively weakly compact subset Y of $D(A)$ is relatively compact.

Indeed, let us take a sequence $(y_n) \subset AY$. Then, there exists a sequence $(x_n) \subset Y$ such that $y_n = Ax_n$ for $n = 1, 2, \dots$. Since the set Y is relatively weakly compact, the sequence (x_n) contains a subsequence (x_{k_n}) being weakly convergent (to an element x). Next, consider the sequence (Ax_{k_n}) . In view of the condition (H_1) we infer that there exists a subsequence (y_{p_n}) of the sequence (Ax_{k_n}) (in other words, there exists a subsequence (y_{p_n}) of the sequence (y_n) which is strongly convergent in the space X). This proves our claim.

Next, let us observe that if the operator $A : D(A) \subset X \rightarrow X$ satisfies the condition (H_2) then the image of an arbitrary weakly compact set (relatively weakly compact) is also weakly compact (relatively weakly compact).

Obviously, the proof of this assertion can be conducted similarly as above.

It is easily seen that any operator A satisfying the condition (H_1) satisfies also the condition (H_2) . Apart from this, if X is a reflexive Banach space and the operator $A : D(A) \subset X \rightarrow X$ is bounded (i.e., A transforms any bounded subset of $D(A)$ into a bounded one) then the operator A satisfies the condition (H_2) .

Now, we recall other definitions which will be used in our study. These definitions are related to the Lipschitz continuity.

Definition 2.4. An operator $A : D(A) \subset X \rightarrow X$ is said to be *nonexpansive* if

$$\|Ax - Ay\| \leq \|x - y\|$$

for all $x, y \in D(A)$.

Definition 2.5. An operator $A : D(A) \subset X \rightarrow X$ is called *D-Lipschitzian* if there exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\phi(0) = 0$, and such that

$$\|Ax - Ay\| \leq \phi(\|x - y\|)$$

for $x, y \in D(A)$.

Obviously every Lipschitzian operator is *D-Lipschitzian*, but the converse may not be true. Moreover, if $\phi(r) < r$ for $r > 0$, then the operator A is referred to as a *nonlinear contraction* with a contraction function ϕ . For our further purposes the following lemma will be useful.

Lemma 2.6. Let $A : D(A) \subset X \rightarrow X$ be a *D-Lipschitzian operator* with the *D-function* ϕ on a Banach space X . Moreover, we assume that A satisfies the (H_2) condition. Then, for each set $S \in \mathfrak{M}_X$ such that $S \subset D(A)$ the following inequality

$$\omega(AS) \leq \phi(\omega(S))$$

is satisfied.

For the proof we refer to [1].

The basic concept exploited in our investigations is the concept of *WC-Banach algebra*. In order to introduce that concept let us assume that X is a Banach algebra with the operation of the multiplication of elements $x, y \in X$ denoted by xy . We will assume that for arbitrary $x, y \in X$ the inequality $\|xy\| \leq \|x\| \cdot \|y\|$ is satisfied. Such a Banach algebra is sometimes called the *normalized Banach algebra* [3].

Definition 2.7. A Banach algebra X will be called the *WC-Banach algebra* if the product $W \cdot W'$ of arbitrary weakly compact subsets W, W' of X is weakly compact.

Below we give a few examples of WC–Banach algebras.

Example 2.8. Let K be a Hausdorff compact space. For a given Banach algebra E denote by $C(K, E)$ the Banach algebra of all continuous functions acting from K into E and equipped with the supremum norm. It may be shown that $C(K, E)$ forms the WC–Banach algebra, provided E is a WC–Banach algebra. Indeed, utilizing the Dobrakov theorem [10] characterizing the weak convergence in the Banach space $C(K, E)$ we can show (cf. [6], for example) that $C(K, E)$ is the WC–Banach algebra.

Example 2.9. Now, let us consider the classical Banach sequence space c_0 consisting of all real (or complex) sequence converging to zero and equipped with the standard supremum (maximum) norm. Let us define the product of two sequences $x = (x_n), y = (y_n) \in c_0$ in the classical way:

$$x \cdot y = (x_n) \cdot (y_n) = (x_n y_n).$$

Observe that for $x, y \in c_0$ we have:

$$\begin{aligned} \|x \cdot y\| &= \sup\{|x_n y_n| : n \in \mathbb{N}\} = \sup\{|x_n| \cdot |y_n| : n \in \mathbb{N}\} \\ &\leq \sup\{\|x\| |y_n| : n \in \mathbb{N}\} = \|x\| \sup\{|y_n| : n \in \mathbb{N}\} \\ &= \|x\| \cdot \|y\|. \end{aligned}$$

Thus c_0 is a Banach algebra (normalized).

We show that c_0 is the WC–Banach algebra. To this end let us recall [11] that in the Banach space c_0 the sequence $(x_k) = ((x_k^n))$, where $x_k = (x_k^n) \in c_0$ for any $k = 1, 2, \dots$, is convergent to an element $x = (x^k) \in c_0$ if and only if the sequence (x_k) is bounded and $\lim_{n \rightarrow \infty} x_k^n = x^k$ for any $k = 1, 2, \dots$. In other words, the sequence $(x_k) = ((x_k^n))$ is weakly convergent to $x = (x^k) \in c_0$ if only if the sequence (x_k) is bounded in c_0 and is coordinatewise convergent to $x = (x^k)$.

Further, let us assume that W and W' are weakly compact subsets of the space c_0 . Consider the product $W \cdot W'$. Let us take an arbitrary sequence $(z_k) \subset W \cdot W', z_k = (z_k^n)$ for any $k = 1, 2, \dots$. This means that there exist two sequences $(x_k) \subset W, (y_k) \subset W'$ such that $z_k = x_k y_k$ for any $k = 1, 2, \dots$. Since the sets W and W' are weakly compact, without loss of generality we can assume that $x_k \rightarrow x$ and $y_k \rightarrow y$ as $k \rightarrow \infty$, where $x \in W$ and $y \in W'$. If we denote $x_k = (x_k^n), y_k = (y_k^n)$ for each $k = 1, 2, \dots$, and $x = (x^k), y = (y^k)$, then in view of the above quoted characterization of the weak convergence in c_0 we deduce that $\lim_{n \rightarrow \infty} x_k^n = x^k$ for any $k = 1, 2, \dots$. This implies that $\lim_{n \rightarrow \infty} z_k^n = \lim_{n \rightarrow \infty} x_k^n \cdot y_k^n = x^k y^k$ for $k = 1, 2, \dots$. Obviously, the sequence $(z_k) = (x_k) \cdot (y_k)$ is bounded in c_0 .

Thus we showed that the sequence (z_k) is weakly convergent to the element $z = x \cdot y \in W \cdot W'$. This allows us to infer that the set $W \cdot W'$ is weakly compact in the space c_0 .

3. Fixed point theorems in WC–Banach algebras

In this section we are going to present the main results of our paper. Those results are connected with new fixed point theorems for operators acting in WC–Banach algebras. At the beginning we gather some theorems which will be needed in our investigations.

Theorem 3.1 [16]. *Let S be a nonempty, closed and convex subset of a Banach space X and let $A : S \rightarrow S$ be a continuous operator satisfying the condition (H_1) . If AS is relatively weakly compact then A has at least one fixed point in S .*

The next result, coming from [2], presents a generalization of that containing in Theorem 3.1.

Theorem 3.2. *Let S be a nonempty, bounded, closed and convex subset of a Banach space X . Assume that $A : S \rightarrow S$ is a continuous map which satisfies the condition (H_1) . If A is ω -condensing then it has at least one fixed point in S .*

Lemma 3.3 [6]. Let M and M' be bounded subsets of a WC–Banach algebra X . Then, the following inequality is satisfied

$$\omega(M \cdot M') \leq \|M'\|\omega(M) + \|M\|\omega(M') + \omega(M)\omega(M'),$$

where the symbol $\|N\|$ denotes the norm of a bounded set N i.e., $\|N\| = \sup\{\|x\| : x \in N\}$.

Our first result is contained in the below formulated lemma.

Lemma 3.4. Let X be a Banach algebra and let S be a nonempty subset of X . Further, assume that $A, C : X \rightarrow X$ and $B : S \rightarrow X$ are operators satisfying the following conditions:

- (i) A and C are D -Lipschitzian with D -functions ϕ_A and ϕ_C , respectively.
- (ii) A is a regular operator on X i.e., A maps X into the set of invertible elements of X .
- (iii) BS is bounded with the bound equal to a constant L .
- (iv) For any $r > 0$ the inequality $L\phi_A(r) + \phi_C(r) < r$ is satisfied.

Then the operator $\left(\frac{I-C}{A}\right)^{-1} B$ exists on BS .

Proof. Let y be an arbitrarily fixed element of the set S . Define the mapping $\varphi_y : X \rightarrow X$ by putting

$$\varphi_y(x) = AxBy + Cx.$$

Then, in view of assumptions (i) and (iii), for arbitrary $x_1, x_2 \in X$ we get

$$\begin{aligned} \|\varphi_y(x_1) - \varphi_y(x_2)\| &\leq \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\ &\leq \|Ax_1 - Ax_2\| \|By\| + \|Cx_1 - Cx_2\| \\ &\leq L\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|) \\ &\leq \Psi\|x_1 - x_2\|, \end{aligned}$$

where $\Psi(r) = L\phi_A(r) + \phi_C(r)$. It is clear that Ψ is a continuous nondecreasing function from \mathbb{R}_+ into \mathbb{R}_+ satisfying $\Psi(r) < r$ (use hypothesis (iv)). This shows that φ_y is a Ψ -nonlinear contraction and therefore by Boyd-Wong fixed point theorem [7] we deduce that there exists a unique fixed point $x_y \in X$ of the operator φ_y i.e., the following equality is satisfied

$$\varphi_y(x_y) = x_y = Ax_yBy + Cx_y.$$

This implies that the element x_y satisfies the equation

$$(I - C)x_y = Ax_yBy.$$

Hence, in virtue of assumption (ii) we obtain

$$\left(\frac{I - C}{A}\right)x_y = By.$$

Thus, we can define the operator $\left(\frac{I-C}{A}\right)^{-1}$ on the set BS by putting

$$\left(\frac{I - C}{A}\right)^{-1} By = x_y.$$

This completes the proof. □

In what follows we state and prove our main result.

Theorem 3.5. Let X be a WC–Banach algebra and let S be a nonempty, bounded, closed and convex subset of X . Further, let A, B and C three operators such that $A, C : X \rightarrow X$ and $B : S \rightarrow X$, which satisfy the following conditions:

- (i) The operators A, C satisfy the condition (H_2) and are D -Lipschitzian with the D -functions ϕ_A and ϕ_C , respectively.
- (ii) A is a regular operator.
- (iii) The operator B is continuous on S , satisfies the condition (H_1) and the set BS is relatively weakly compact.
- (iv) For each $y \in S$ the following implication holds

$$x = AxBy + Cx \Rightarrow x \in S.$$

- (v) For any $r > 0$ the following inequality is satisfied

$$L\phi_A(r) + \phi_C(r) < r,$$

where $L = \|BS\|$.

Under the above assumptions the operator equation $x = AxBx + Cx$ has at least one solution in the set S .

Proof. At the beginning let us notice that according to Lemma 3.4 the operator $T = \left(\frac{I-C}{A}\right)^{-1} B : S \rightarrow X$ is well defined.

Next, we show that $TS \subset S$. To this end take $x \in S$. Denote $y = Tx$. This means that

$$y = \left(\frac{I-C}{A}\right)^{-1} Bx.$$

Hence we get

$$\frac{I-C}{A}y = Bx$$

or, equivalently

$$Bx = \frac{y - Cy}{Ay}.$$

Further, we derive that

$$AyBx = y - Cy$$

and, consequently

$$y = AyBx + Cy.$$

In view of assumption (iv) from the above equality we conclude that $y \in S$. This yields the desired inclusion $TS \subset S$.

In what follows we prove that the operator T is continuous on the set S . Indeed, taking into account assumption (iii) we see that it is sufficient to show that the operator $\left(\frac{I-C}{A}\right)^{-1}$ is continuous on the set BS . Thus, let us assume that (y_n) is a sequence contained in the set BS which is convergent to a point $y \in BS$. Let us denote:

$$x_n = \left(\frac{I-C}{A}\right)^{-1} y_n,$$

$$x = \left(\frac{I-C}{A}\right)^{-1} y.$$

We show that $x_n \rightarrow x$ as $n \rightarrow \infty$.

To prove the above announced fact let us observe that

$$\left(\frac{I-C}{A}\right)x_n = y_n.$$

Hence, we obtain

$$(I-C)x_n = y_nAx_n$$

and, consequently

$$x_n = y_n Ax_n + Cx_n. \tag{3.1}$$

Similarly we can show that

$$x = yAx + Cx. \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$\begin{aligned} \|x_n - x\| &\leq \|y_n Ax_n - yAx\| + \|Cx_n - Cx\| \\ &\leq \|y_n Ax_n - y_n Ax\| + \|y_n Ax - yAx\| + \|Cx_n - Cx\| \\ &\leq L\phi_A(\|x_n - x\|) + \|Ax\| \cdot \|y_n - y\| + \phi_C(\|x_n - x\|). \end{aligned} \tag{3.3}$$

Now, keeping in mind that $y \in S$, in view of assumption (vi) we deduce that $x \in S$. Taking into account that the set S is bounded and A is D-Lipschitzian we infer that AS is bounded. Thus there exists a constant $P > 0$ such that $\|Ax\| \leq P$ for each $x \in S$.

Next, taking limes superior in (3.3), we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|x_n - x\| &\leq L\phi_A(\limsup_{n \rightarrow \infty} \|x_n - x\|) + P \cdot 0 \\ &\quad + \phi_C(\limsup_{n \rightarrow \infty} \|x_n - x\|). \end{aligned} \tag{3.4}$$

Suppose that $r = \limsup_{n \rightarrow \infty} \|x_n - x\| > 0$. Then, from (3.4) we derive the inequality

$$r \leq L\phi_A(r) + \phi_C(r).$$

But this contradicts assumption (v) and yields that $r = 0$. Hence we infer that the operator $\left(\frac{I-C}{A}\right)^{-1}$ is continuous on the set BS .

Now, we are going to show that the operator T satisfies the condition (H_1) .

In order to prove this fact assume that $(x_n) \subset S$ is a weakly convergent sequence. Since, in view of assumption (iii) the operator B satisfies the condition (H_1) , this implies that the sequence (Bx_n) contains a strongly convergent subsequence (Bx_{k_n}) . In view of the continuity of the operator $\left(\frac{I-C}{A}\right)^{-1}$ on the set BS we obtain that the sequence

$$\left(\left(\frac{I-C}{A}\right)^{-1} Bx_{k_n}\right)$$

is strongly convergent i.e., the sequence (Tx_{k_n}) is strongly convergent. This means that the operator T satisfies the condition (H_1) .

In the next step of our proof we show that the set TS is relatively weakly compact. At first, let us notice that after converting of the equality

$$T = \left(\frac{I-C}{A}\right)^{-1} B$$

we obtain that $T = ATB + CT$.

Further, taking into account the properties of the De Blasi measure of weak noncompactness [9], we get

$$\omega(TS) \leq \omega(A(TS)BS) + \omega(C(TS)).$$

Hence, in view of Lemma 3.3, we obtain

$$\begin{aligned} \omega(TS) &\leq \|BS\|\omega(A(TS)) + \|A(TS)\|\omega(BS) \\ &\quad + \omega(C(TS)). \end{aligned} \tag{3.5}$$

In the above estimate we utilized the facts that the set BS is bounded (since BS is relatively weakly compact) and the set TS is also bounded (since $T(S) \subset S$). Moreover, keeping in mind the assumption that the operator A is D -Lipschitzian we deduce that the set $A(TS)$ is bounded.

Further, in virtue of assumption (i) the operator C is D -Lipschitzian and satisfies the condition (H_2) . Hence, in view of Lemma 2.6 we derive the estimate

$$\omega(C(TS)) \leq \phi_C(\omega(TS)). \tag{3.6}$$

Next, observe that according to assumption (iii) the set BS is relatively weakly compact. This yields

$$\omega(BS) = 0. \tag{3.7}$$

Now, let us notice that taking into account the boundedness of the set BS we can define the finite constant $L = \|BS\|$. Consequently, keeping in mind assumption (i) and Lemma 2.6, we obtain

$$\|BS\|\omega(A(TS)) \leq L\phi_A(\omega(TS)). \tag{3.8}$$

Further, combining (3.6)-(3.8) and taking into account estimate (3.5), we get

$$\omega(TS) \leq L\phi_A(\omega(TS)) + \phi_C(\omega(TS)). \tag{3.9}$$

According to the hypothesis (v), the last equations writes

$$\omega(TS) \leq L\phi_A(\omega(TS)) + \phi_C(\omega(TS)) < \omega(TS),$$

which is a contradiction and therefore $\omega(TS) = 0$.

Now the operator T satisfies the hypotheses of Theorem 3.1, so there exists $x \in S$ such that $x = Tx$. Hence, we obtain

$$x = \left(\frac{I - C}{A}\right)^{-1} Bx,$$

and consequently

$$\left(\frac{I - C}{A}\right)x = Bx.$$

From the above equality we get

$$x - Cx = BxAx.$$

Finally, we have

$$x = AxBx + Cx.$$

This completes the proof of our theorem. □

As an immediate consequence of the above theorem we derive the following corollary.

Corollary 3.6. *Let S be a nonempty, bounded, closed and convex subset of a WC-Banach algebra X and let $A, C : X \rightarrow X, B : S \rightarrow X$ be three operators satisfying the below listed conditions:*

- (i) C is D -Lipschitzian with the D -function ϕ_C . Moreover, C satisfies the condition (H_2) .
- (ii) A is nonexpansive, regular on X and satisfies the condition (H_2) .
- (iii) The operator B is continuous, satisfies the condition (H_1) and the set BS is relatively weakly compact.
- (iv) For each $y \in S$ the following implication holds

$$x = AxBy + Cx \Rightarrow x \in S.$$

- (v) For any $r > 0$ the following inequality is satisfied

$$Lr + \phi_C(r) < r,$$

where $L = \|BS\|$.

Then the operator equation $x = AxBy + Cx$ has at least one solution in the set S .

Indeed, let us notice that the above corollary is a particular case of Theorem 3.5 if we put $\phi_A(r) = r$.

Corollary 3.7. Let S be a nonempty, bounded, closed and convex subset of a WC–Banach algebra X and let $C : X \rightarrow X$, $B : S \rightarrow X$ be operators satisfying the following conditions:

- (i) C is a nonlinear contraction and satisfies the condition (H_2) .
- (ii) B is continuous, satisfies the condition (H_1) and the set BS is relatively weakly compact.
- (iii) For any $y \in S$ the following implication holds

$$x = By + Cx \Rightarrow x \in S.$$

Then, the operator equation $x = Bx + Cx$ has at least one solution in S .

In fact, let A be the mapping defined by $Ax = 1_X$, where 1_X is the unit element of the Banach algebra X , we see that the operator A satisfies condition (i) of Theorem 3.5 with D-function $\phi_A = 0$. Obviously A satisfies the condition (H_2) . Apart from this we have that the D-function ϕ_C of the operator C satisfies the inequality $\phi_C(r) < r$. Thus, our corollary follows easily from Theorem 3.5.

Our next result will be not an immediate corollary of Theorem 3.5.

Theorem 3.8. Let S be a nonempty, bounded, closed and convex subset of a Banach algebra X and let $A, C : X \rightarrow X$, $B : S \rightarrow X$ be operators satisfying the following conditions:

- (i) B is continuous and ω -contraction (with a constant $k \in [0, 1)$). Moreover, B satisfies the condition (H_1) .
- (ii) The operator A is regular and the set AS is relatively weakly compact.
- (iii) The operators A and C are D-Lipschitzian with the D-functions ϕ_A, ϕ_C , respectively. Moreover, C satisfies the condition (H_2) and $\phi_C(r) < (1 - kQ)r$ for each $r > 0$, where $Q = \|AS\|$ and $kQ < 1$.
- (iv) For every $y \in S$ the following implication holds

$$x = AxBy + Cx \Rightarrow x \in S.$$

- (v) For any $r > 0$ the following inequality is satisfied

$$L\phi_A(r) + \phi_C(r) < r,$$

where $L = \|BS\|$.

Then the operator equation $x = AxBx + Cx$ has at least one solution $x \in S$.

Proof. In the similar way as in the proof of Theorem 3.5 we can show that the operator

$$T = \left(\frac{I - C}{A} \right)^{-1} B$$

transforms the set S into itself, is continuous and satisfies the condition (H_1) . Thus, in view of Theorem 3.2 it is sufficient to show that the operator T is ω -condensing. In order to prove this fact let us take a subset M of S with $\omega(M) > 0$. Then, we get

$$TM \subset A(TM) \cdot BM + C(TM).$$

Further, applying Lemmas 2.6 and 3.3 and keeping in mind assumptions concerning the operators A, B and C , we obtain:

$$\begin{aligned} \omega(TM) &\leq \omega(A(TM)BM) + \omega(C(TM)) \\ &\leq kQ\omega(M) + \phi_C(\omega(TM)). \end{aligned} \tag{3.10}$$

If $k = 0$ then from (3.10) we obtain that

$$\omega(TM) \leq \phi_C(\omega(TM))$$

which implies that $\omega(TM) = 0$.

In other case, applying the inequality

$$\phi_C(r) < (1 - kQ)r$$

for $r > 0$, we derive the estimate

$$\omega(TM) < \omega(M).$$

Observe that in both cases it is shown that the operator T is ω -condensing. Thus, the use of Theorem 3.2 ends the proof. \square

The above theorem yields the following corollary.

Corollary 3.9. *Assume that S is a nonempty, bounded, closed and convex subset of a WC–Banach algebra X . Let $A, C : X \rightarrow X$ and $B : S \rightarrow X$ be operators satisfying the following conditions:*

- (i) C is nonlinear contraction with a contraction function ϕ_C . Moreover, C satisfies the condition (H_2) .
- (ii) A is nonexpansive, regular on X and satisfies the condition (H_2) .
- (iii) The operator B is continuous and satisfies the condition (H_1) .
- (iv) The sets AS and BS are relatively weakly compact.
- (v) For each $y \in S$ the following implication holds

$$x = AxBy + Cx \Rightarrow x \in S.$$

- (vi) For any $r > 0$ the following inequality is satisfied

$$Lr + \phi_C(r) < r,$$

where $L = \|BS\|$.

Then the operator equation $x = AxBx + Cx$ has at least one solution $x \in S$.

Let us observe that if we put $A = 1_X$ in the above theorem (1_X denotes the unit element in a WC–Banach algebra X) then we obtain Krasnosel’skii type fixed point theorem.

Now, we formulate the final result of this section.

Theorem 3.10. *Let S be a nonempty, bounded, closed and convex subset of a Banach algebra X and let $A, C : X \rightarrow X$ and $B : S \rightarrow X$ be operators satisfying the following conditions:*

- (i) A and C are D -Lipschitzian with the D -functions ϕ_A and ϕ_C , respectively.
- (ii) A is regular on X .
- (iii) The operator B is continuous and satisfies the condition (H_1) . Moreover, the set BS is bounded.
- (iv) There exists a constant $k \in [0, 1)$ such that

$$\omega(AM \cdot BM + CM) \leq k\omega(M),$$

for an arbitrary nonempty subset M of the set S .

- (v) For every $y \in S$ the following implication holds

$$x = AxBy + Cx \Rightarrow x \in S.$$

- (vi) For any $r > 0$ the following inequality is satisfied

$$L\phi_A(r) + \phi_C(r) < r,$$

where $L = \|BS\|$.

Then the operator equation $x = Ax + Bx + Cx$ has at least one solution in the set S .

Proof. In view of assumptions (i)-(iii), (vi) and Lemma 3.4 the operator $T : S \rightarrow X, T = \left(\frac{I-C}{A}\right)^{-1} B$, is well-defined.

We show that $TS \subset S$. To this end fix $y \in TS$. Then, there exists $x \in S$ such that $y = Tx$. This equality can be written in the form

$$y = \left(\frac{I-C}{A}\right)^{-1} Bx.$$

Hence we obtain

$$\frac{I-C}{A}y = Bx,$$

which yields that $y - Cy = AyBx$. Thus we have

$$y = AyBx + Cy.$$

Since $x \in S$ this implies, in view of assumption (v), that $y \in S$ and our conclusion follows.

Further on, let us consider the sequence (S_n) consisting of subsets of the set S such that $S_1 = S$ and $S_{n+1} = \text{Conv}S_n$ for $n = 1, 2, \dots$. Obviously all sets of this sequence are nonempty, closed and convex subsets of S . Keeping in mind that $TS \subset S$ and the definition of the sequence (S_n) , in view of the equality $T = AT \cdot B + CT$, we obtain:

$$TS_n \subset A(TS_n) \cdot BS_n + C(TS_n) \subset AS_n \cdot BS_n + CS_n.$$

Hence we get

$$\omega(S_{n+1}) = \omega(\text{Conv}TS_n) = \omega(TS_n) \leq \omega(AS_n \cdot BS_n + CS_n).$$

Further, taking into account assumption (iv), we derive the estimate

$$\omega(S_{n+1}) \leq k\omega(S_n).$$

Using the induction we infer that

$$\omega(S_n) \leq k^{n-1}\omega(S).$$

Thus $\lim_{n \rightarrow \infty} \omega(S_n) = 0$. Hence we deduce that the set $S_\infty = \bigcap_{n=1}^{\infty} S_n$ is nonempty, bounded, closed, convex and weakly compact subset of S . Obviously $TS_\infty \subset S_\infty$ which implies that the set TS_∞ is relatively weakly compact. Similarly as in the proof of Theorem 3.5 we can show that the operator $T : S_\infty \rightarrow S_\infty$ is continuous and satisfies the condition (H_1) on the set S_∞ . Applying Theorem 3.1 we complete the proof. \square

Let us point out two immediate corollaries of the result contained in Theorem 3.10.

Corollary 3.11. Assume that S is a nonempty, bounded, closed and convex subset of a WC–Banach algebra X . Let $A, C : X \rightarrow X$ and $B : S \rightarrow X$ be operators satisfying the following conditions:

- (i) A is regular and D -Lipschitzian with the D -function ϕ_A .
- (ii) C is contraction with a constant k and satisfies the condition (H_2) .
- (iii) The sets AS and BS are relatively weakly compact.
- (iv) The operator B is continuous and satisfies the condition (H_1) .
- (v) For each $y \in S$ the following implication is satisfied

$$x = Ax + By + Cx \Rightarrow x \in S.$$

- (vi) For any $r > 0$ the following inequality is satisfied

$$L\phi_A(r) + kr < r,$$

where $L = \|BS\|$.

Then the operator equation $x = Ax + Bx + Cx$ has at least one solution in S .

Corollary 3.12. Let S be a nonempty, bounded, closed and convex subset of a WC–Banach algebra X and let $A, C : X \rightarrow X, B : S \rightarrow X$ be operators which satisfy the following conditions:

- (i) A and C are D -Lipschitzian with the D -functions ϕ_A and ϕ_C , respectively. Moreover, the operator A is regular.
- (ii) The sets AS, BS and CS are relatively weakly compact.
- (iii) B is continuous and satisfies the condition (H_1) .
- (iv) For each $y \in S$ the following implication holds

$$x = AxBy + Cx \Rightarrow x \in S.$$

- (v) For any $r > 0$ the following inequality is satisfied

$$L\phi_A(r) + \phi_C(r) < r,$$

where $L = \|BS\|$.

Then the operator equation $x = Ax + Bx + Cx$ has at least one solution in the set S .

4. Applications to infinite systems of integral equations

This section is dedicated to show the applicability of the theory developed in the previous section to prove a result on the existence of solutions of the following infinite system of nonlinear integral equations

$$x_n(t) = c_n(x_n(t)) + a_n(x_n(t)) \int_0^1 b(t, s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds, \tag{4.1}$$

where $n = 1, 2, \dots$ and $t \in I = [0, 1]$. Moreover, the integral in (4.1) is understood as the Lebesgue integral. The infinite system of integral equations (4.1) will be investigated in the Banach space $C_0 = C(I, c_0)$ (cf. Examples 2.8 and 2.9) consisting of all functions acting from the interval I into the sequence space c_0 , which are continuous on I . Obviously, the norm in the space C_0 has the form

$$\|x\|_{C_0} = \|(x_1, x_2, \dots)\|_{C_0} = \sup_{t \in I} \{\sup_n |x_n(t)| : n = 1, 2, \dots\}.$$

In what follows we will usually write $\|\cdot\|$ instead of $\|\cdot\|_{C_0}$. Moreover, let us recall that by $C(I)$ we will denote the space $C(I, \mathbb{R})$ equipped with the norm $\|x\|_{C(I)} = \sup\{|x(t)| : t \in I\}$ (cf. Example 2.8).

Let us mention that C_0 forms a WC–Banach algebra (cf. Examples 2.8 and 2.9).

Now, we formulate assumptions under which we will investigate infinite system (4.1).

- (i) For any $n \in \mathbb{N}$ the function $a_n : \mathbb{R} \rightarrow (0, \infty)$ and there exists a function $\phi_A : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ being nondecreasing, $\phi_A(0) = 0$, continuous at 0 and such that

$$|a_n(x) - a_n(y)| \leq \phi_A(|x - y|)$$

for $x, y \in \mathbb{R}$ and $n = 1, 2, \dots$. Moreover, there exists a constant $N > 0$ such that $\phi_A(r) \leq N$ for $r \geq 0$.

- (ii) For any sequence (x_n) with $x_n \rightarrow 0$ as $n \rightarrow \infty$ we have that $a_n(x_n) \rightarrow 0$ as $n \rightarrow \infty$.
- (iii) $c_n : \mathbb{R} \rightarrow \mathbb{R}, c_n(0) = 0$ for $n = 1, 2, \dots$ and there exists a function $\phi_c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is nondecreasing, continuous at the point 0 and such that

$$|c_n(x) - c_n(y)| \leq \phi_c(|x - y|)$$

for $x, y \in \mathbb{R}$ and $n = 1, 2, \dots$

- (iv) The function ϕ_c is bounded on \mathbb{R}_+ i.e., there exists a constant $M_2 > 0$ such that $\phi_c(r) \leq M_2$ for $r \geq 0$.

Remark 4.1. Observe that from assumptions (i) and (iii) it follows that the functions a_n, c_n ($n = 1, 2, \dots$) are continuous on \mathbb{R} .

Remark 4.2. Notice that in view of assumption (i), for each $x \in \mathbb{R}$ we get:

$$a_n(x) \leq |a_n(x) - a_n(0)| + a_n(0) \leq \phi_A(|x|) + a_n(0).$$

On the other hand, taking in assumption (ii) the sequence $x_n = 0$ ($n = 1, 2, \dots$) we deduce that the sequence $(a_n(0))$ is bounded. Thus, the constant A_0 defined by putting $A_0 = \sup\{a_n(0) : n = 1, 2, \dots\}$ is finite. Linking this fact with the above obtained estimate, we conclude that

$$a_n(x) \leq N + A_0$$

for any $x \in \mathbb{R}$ and $n = 1, 2, \dots$

Further on, we will denote $M_1 = N + A_0$.

Remark 4.3. Let us pay attention to the fact that from assumption (iii) we have

$$|c_n(x)| = |c_n(x) - c_n(0)| \leq \phi_C(|x|)$$

for $n = 1, 2, \dots$ and for every $x \in \mathbb{R}$. Joining this inequality with assumption (iv) we infer that $|c_n(x)| \leq M_2$ for any $n = 1, 2, \dots$ and $x \in \mathbb{R}$.

Now we formulate other assumptions needed in our considerations.

- (v) The function f_n acts from the set $I \times \mathbb{R}^\infty$ into \mathbb{R} for any $n = 1, 2, \dots$. Moreover, we assume that there exist two sequences $(k_n), (l_n)$ with positive terms such that $k_n \rightarrow 0$ as $n \rightarrow \infty$, (l_n) is bounded and the following inequality is satisfied

$$|f_n(t, x_n, x_{n+1}, \dots)| \leq k_n + l_n \sup\{|x_i| : i \geq n\}$$

for any $t \in I, x = (x_i) \in c_0$ and for $n = 1, 2, \dots$

Remark 4.4. From the above formulated assumption we deduce that $K < \infty$ and $L < \infty$, where the constants K and L are defined by the equalities:

$$K = \sup\{k_n : n = 1, 2, \dots\}, \quad L = \sup\{l_n : n = 1, 2, \dots\}.$$

Our further assumptions are as follows.

- (vi) The family of function $\{f_n\}_{n \in \mathbb{N}}$ is uniformly equicontinuous on the set $I \times c_0$. This means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathbb{N}$ and $t \in I$, and for all $x = (x_n), y = (y_n) \in c_0$ with $\|x - y\|_{c_0} \leq \delta$ we have that

$$|f_n(t, x_n, x_{n+1}, \dots) - f_n(t, y_n, y_{n+1}, \dots)| \leq \varepsilon.$$

- (vii) The function $b(t, s) = b : I \times I \rightarrow \mathbb{R}$ is continuous in t uniformly with respect to the variable $s \in I$ and is integrable with respect to s for any $t \in I$.

Remark 4.5. In view of assumption (vii) the function

$$\bar{b}(t) = \int_0^1 b(t, s) ds$$

is well defined on I . Observe that this function is continuous on the interval I . In fact, for a fixed $\varepsilon > 0$ and for arbitrary $t_1, t_2 \in I$ such that $|t_2 - t_1| \leq \varepsilon$, we have

$$\begin{aligned} |\bar{b}(t_2) - \bar{b}(t_1)| &\leq \int_0^1 |b(t_2, s) - b(t_1, s)| ds \\ &\leq \int_0^1 \mu_1(b, \varepsilon) ds = \mu_1(b, \varepsilon), \end{aligned}$$

where the function $\mu_1(b, \varepsilon)$ denotes the modulus of continuity of the function $t \rightarrow b(t, s)$ defined by the formula

$$\mu_1(b, \varepsilon) = \sup\{|b(t_2, s) - b(t_1, s)| : t_1, t_2, s \in I, |t_2 - t_1| \leq \varepsilon\}.$$

Obviously, in view of assumption (vii) we have that $\mu_1(b, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. This shows that the function $\bar{b} = \bar{b}(t)$ defined above is continuous on I .

Taking into account the above statement we can define the finite constant \bar{B} by putting

$$\bar{B} = \sup \left\{ \int_0^1 |b(t, s)| ds : t \in I \right\}.$$

Now, we formulate our further assumptions. To this end, let us put $M = \max\{M_1, M_2\}$.

(viii) The following inequality holds

$$M\bar{B}L < 1,$$

where the constants \bar{B} and L were defined earlier.

For further purposes we define the number r_0 by putting

$$r_0 = M \frac{\bar{B}K + 1}{1 - M\bar{B}L}. \tag{4.2}$$

(ix) For any $r > 0$ the following inequality is satisfied

$$\bar{B} \frac{K + LM}{1 - M\bar{B}L} \phi_A(r) + \phi_C(r) < r.$$

Before formulating our main result we provide an auxiliary lemma which will be useful in our investigations.

Lemma 4.6. *Let the function $x(t) = (x_1(t), x_2(t), \dots) = (x_n(t))$ be an element of the space $C_0 = C(I, c_0)$. Then*

$$\lim_{n \rightarrow \infty} \|x_n\|_{C(I)} = 0.$$

Proof. Observe that for an arbitrary $t \in I$ we have that $\lim_{n \rightarrow \infty} x_n(t) = 0$. Thus the function sequence (x_n) converges pointwise to the function $x = 0$ on the interval I .

Since $x \in C_0$, the sequence $(x_n(t))$ is equibounded on the interval I . Further, let us pay attention to the fact that the function $x = x(t) = (x_n(t))$, as an element of the space C_0 , acts continuously from the interval I into the space c_0 . This implies that the function x is uniformly continuous on I . But this means that the following condition is satisfied

$$\forall \varepsilon > 0 \exists \delta > 0 \forall t_1, t_2 \in I [|t_2 - t_1| \leq \delta \Rightarrow \|x(t_2) - x(t_1)\|_{c_0} \leq \varepsilon].$$

Equivalently, we can write

$$\forall \varepsilon > 0 \exists \delta > 0 \forall t_1, t_2 \in I [|t_2 - t_1| < \varepsilon \Rightarrow \sup\{|x_1(t_2) - x_1(t_1)|, |x_2(t_2) - x_2(t_1)|, \dots, |x_n(t_2) - x_n(t_1)|, \dots\} \leq \varepsilon].$$

The above condition simply means that the function sequence (x_n) is equicontinuous on the interval I . This assertion in conjunction with the above established equiboundedness of the sequence (x_n) , in view of the Ascoli-Arzelá criterion [11] implies that the sequence (x_n) is relatively compact in the space $C(I)$. Thus there exists a subsequence (x_{k_n}) of the sequence (x_n) which converges uniformly (i.e., in the norm of the space $C(I)$) to a function $x \in C(I)$. Since (x_{k_n}) converges pointwise to the function vanishing identically on I , this yields that $x = 0$. Thus we have that $\lim_{n \rightarrow \infty} \|x_{k_n}\|_{C(I)} = 0$. Since we can repeat the same reasoning starting from an arbitrary subsequence of the sequence (x_n) , in view of the well-known facts from mathematical analysis we infer that $\lim_{n \rightarrow \infty} \|x_n\|_{C(I)} = 0$.

The proof is complete. □

Now, we are prepared to formulate our main result of this section.

Theorem 4.7. *Under assumptions (i)-(ix) the infinite system of integral equations (4.1) has at least one solution $x(t) = (x_n(t))$ in the space $C_0 = C(I, c_0)$.*

Proof. In order to prove our theorem we will apply the result contained in Theorem 3.5. To this end let us define on the space C_0 three operators A, B, C by putting:

$$(Ax)(t) = (a_n(x_n(t))) = (a_1(x_1(t)), a_2(x_2(t)), \dots),$$

$$\begin{aligned} (Bx)(t) &= ((B_n x)(t)) \\ &= \left(\int_0^1 b(t, s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds \right), \end{aligned}$$

$$(Cx)(t) = (c_n(x_n(t))) = (c_1(x_1(t)), c_2(x_2(t)), \dots),$$

for $t \in I = [0, 1]$. We show that these operators satisfy the assumptions of Theorem 3.5.

We start with investigations concerning the operator A . At first, let us observe that assumption (i) guarantees that A is regular. This means that there is satisfied assumption (ii) of Theorem 3.5.

Next, for fixed elements $x = (x_n), y = (y_n) \in c_0$ we get:

$$\begin{aligned} \|A(x) - A(y)\|_{c_0} &= \|(a_1(x_1) - a_1(y_1), a_2(x_2) - a_2(y_2), \dots)\|_{c_0} \\ &= \sup\{|a_n(x_n) - a_n(y_n)| : n = 1, 2, \dots\} \\ &\leq \sup\{\phi_A(|x_n - y_n|) : n = 1, 2, \dots\} \\ &\leq \phi_A(\|x - y\|_{c_0}). \end{aligned}$$

This shows that the operator A is D-Lipschitzian with the D-function ϕ_A . Further, let us observe that on the basis of assumption (ii) we deduce that the operator A transforms the space C_0 into itself. Indeed, let us take a function $x(t) = (x_n(t)) \in C_0$. This means, that for each fixed $t \in I$ we have that $x_n(t) \rightarrow 0$ when $n \rightarrow \infty$. Hence, in view of assumption (ii) we get that $a_n(x_n(t)) \rightarrow 0$ as $n \rightarrow \infty$. This proves our claim.

Apart from this, taking into account Remark 4.2, for each fixed $t \in I$ we obtain

$$\begin{aligned} \|(Ax)(t)\|_{c_0} &= \|(a_1(x_1(t)), a_2(x_2(t)), \dots)\|_{c_0} \\ &= \sup\{a_n(x_n(t)) : n = 1, 2, \dots\} \leq M_1. \end{aligned}$$

This implies that

$$\|Ax\|_0 \leq M_1. \tag{4.3}$$

In order to check that the operator A satisfies the condition (H_2) let us take a sequence $(x_n) \subset C_0 = C(I, c_0)$ which is weakly convergent to a function $x \in C_0$. This means (cf. [10]) that if we denote

$$x_n(t) = (x_1^n(t), x_2^n(t), x_3^n(t), \dots)$$

for $n = 1, 2, \dots$ and for an arbitrary $t \in I$ and if we denote $x(t) = (x_1(t), x_2(t), x_3(t), \dots)$, then we have that $x_1^n(t) \rightarrow x_1(t)$, $x_2^n(t) \rightarrow x_2(t)$, ..., $x_k^n(t) \rightarrow x_k(t)$, ... for any $t \in I$, if $n \rightarrow \infty$.

Now, let us consider the sequence (Ax_n) i.e.,

$$\begin{aligned} (Ax_n) &= (A(x_1^n, x_2^n, x_3^n, \dots)) = (a_1(x_1^n), a_2(x_2^n), a_3(x_3^n), \dots) \\ &= (a_k(x_k^n)). \end{aligned}$$

Then, for an arbitrarily fixed $t \in I$ we obtain:

$$\begin{aligned} ((Ax_n)(t)) &= (a_1(x_1^n(t)), a_2(x_2^n(t)), a_3(x_3^n(t)), \dots) \\ &= (a_k(x_k^n(t))). \end{aligned}$$

Since, according to our assumptions we have that $x_k^n(t) \rightarrow x_k(t)$ as $n \rightarrow \infty$ ($k = 1, 2, \dots$), this implies that $a_k(x_k^n(t)) \rightarrow a_k(x_k(t))$ ($k = 1, 2, \dots$), which is a simple consequence of the continuity of each function a_k ($k = 1, 2, \dots$) on the set \mathbb{R} (cf. Remark 4.1). But this means that the sequence $((Ax_n))$ is weakly convergent in the space $C_0 = C(I, c_0)$. Thus the operator A satisfies the condition (H_2) (it satisfies even a stronger condition called the ww-compactness (cf. [6])).

In a similar way we can show that the operator C is D-Lipschitzian with the D-function ϕ_C (cf. Remark 4.3). Moreover, the operator C transforms the space C_0 into itself and satisfies the condition (H_2) .

Summing up we deduce that there are satisfied assumptions (i) and (ii) of Theorem 3.5.

In what follows we will consider the operator B . To this end let us take the set $S = B(\theta, r_0)$, where r_0 is a number described by equality (4.2).

At the beginning we show that B transforms the set S into the space C_0 . Thus, take an arbitrary function $x(t) = (x_i(t)) \in S$. Fix arbitrarily $n \in \mathbb{N}$ and a number $t \in I$. Then, keeping in mind assumptions (v) and (vii), we obtain:

$$\begin{aligned} |(B_n x)(t)| &\leq \int_0^1 |b(t, s)| |f_n(s, x_n(s), x_{n+1}(s), \dots)| ds \\ &\leq \int_0^1 |b(t, s)| \{k_n + l_n \sup\{|x_i(s)| : i \geq n\}\} ds \\ &\leq k_n \int_0^1 |b(t, s)| ds + l_n \int_0^1 |b(t, s)| \sup\{|x_i(s)| : i \geq n\} ds \\ &\leq k_n \bar{B} + l_n \sup_{i \geq n} \{\sup\{|x_i(t)| : t \in I\}\} \bar{B}, \end{aligned}$$

where \bar{B} was defined in Remark 4.5.

Consequently, in view of assumption (v) and Remark 4.4, we get

$$|(B_n x)(t)| \leq k_n \bar{B} + L \bar{B} \sup_{i \geq n} \{\sup\{|x_i(t)| : t \in I\}\},$$

for any $n \in \mathbb{N}$ and for each $t \in I$. The above estimate implies the following one

$$|(B_n x)(t)| \leq k_n \bar{B} + L \bar{B} \sup_{i \geq n} \|x_i\|_{C(I)}. \tag{4.4}$$

Now, taking into account that $k_n \rightarrow 0$ as $n \rightarrow \infty$ and keeping in mind Lemma 4.6 we conclude from estimate (4.4) that the operator B transforms the set S (even the space C_0) into itself. Moreover, from estimate (4.4) we infer the following inequality

$$\|Bx\|_{C_0} \leq K\bar{B} + L\bar{B}\|x\|_{C_0}.$$

Consequently (since $x \in S$), we obtain

$$\|Bx\|_{C_0} \leq K\bar{B} + L\bar{B}r_0. \tag{4.5}$$

Further on, we show that the operator B is continuous on the set S . To this end fix $\varepsilon > 0$ and choose a number $\delta > 0$ according to assumption (vi). Next, take $y, x \in S$ such that $\|x - y\|_{C_0} \leq \delta$. This means that for any $t \in I$ we have:

$$\sup_{t \in I} \|x(t) - y(t)\|_{C_0} \leq \delta.$$

Equivalently, we can write:

$$\sup_{t \in I} \{\max [|x_n(t) - y_n(t)| : n = 1, 2, \dots]\} \leq \delta.$$

Thus, linking the above estimates, we derive the following inequalities:

$$\begin{aligned} \|Bx - By\|_{C_0} &= \sup_{t \in I} \|(Bx)(t) - (By)(t)\|_{C_0} \\ &= \sup_{t \in I} \{\max [|(B_n x)(t) - (B_n y)(t)| : n = 1, 2, \dots]\} \\ &= \sup_{t \in I} \left\{ \max_{n \in \mathbb{N}} \left[\int_0^1 b(t, s) (f_n(s, x_n(s), x_{n+1}(s), \dots) - f_n(s, y_n(s), y_{n+1}(s), \dots)) ds \right] \right\} \\ &\leq \sup_{t \in I} \left\{ \max_{n \in \mathbb{N}} \left[\int_0^1 |b(t, s)| |f_n(s, x_n(s), x_{n+1}(s), \dots) - f_n(s, y_n(s), y_{n+1}(s), \dots)| ds \right] \right\}. \end{aligned}$$

Hence, in view of assumption (vi), we obtain

$$\|Bx - By\|_{C_0} \leq \sup_{t \in I} \int_0^1 |b(t, s)| \varepsilon ds = \bar{B} \cdot \varepsilon.$$

This shows that the operator B is continuous (even uniformly continuous) on the set S .

Now, we are going to prove that the operator B satisfies the condition (H_1) on the set S . At first let us observe that from estimate (4.5) we have that functions belonging to the set BS are equibounded on the interval I .

Further, for arbitrarily fixed $t_1, t_2 \in I$ and for a function $x \in S, n \in \mathbb{N}$, on the basis of the estimate from Remark 4.5, we have:

$$\begin{aligned} |(B_n x)(t_2) - (B_n x)(t_1)| &\leq \int_0^1 |b(t_2, s) - b(t_1, s)| |f_n(s, x_n(s), x_{n+1}(s), \dots)| ds \\ &\leq \int_0^1 \mu_1(b, |t_2 - t_1|) (KB + LBr_0) ds \\ &= (KB + LBr_0) \mu_1(b, |t_2 - t_1|), \end{aligned}$$

where we utilized estimate (4.5). From the above established facts and Ascoli-Arzelá criterion for the relative compactness, we deduce that the set BS is relatively compact. Obviously, this automatically implies, that

the operator B satisfies the condition (H_1) . This means that there is satisfied assumption (iii) of Theorem 3.5.

In our next step we show that there is satisfied assumption (iv) of Theorem 3.5. To this end let us fix arbitrarily $y \in S = B(\theta, r_0)$. Next, assume that an element $x \in C_0$ satisfies the equality

$$x = AxBy + Cx.$$

This yields

$$\|x\|_{C_0} \leq \|Ax\|_{C_0} \|By\|_{C_0} + \|Cx\|_{C_0}.$$

Further, in view of Remarks 4.2, 4.3 and estimate (4.5) we obtain

$$\begin{aligned} \|x\|_{C_0} &\leq M_1 \|By\|_{C_0} + M_2 \\ &\leq M(K\bar{B} + L\bar{B}r_0 + 1) = M[\bar{B}(K + Lr_0) + 1]. \end{aligned}$$

Hence, keeping in mind assumption (viii) we obtain that $\|x\|_{C_0} \leq r_0$. This implies that $x \in S$ and shows that there is satisfied assumption (iv) of Theorem 3.5.

Finally, let us notice that in view of equality (4.2) and estimate (4.5) we have:

$$L = \|BS\|_{C_0} = K\bar{B} + L\bar{B}r_0 = \bar{B} \frac{K + ML}{1 - \bar{B}ML}.$$

Linking the above equality with assumption (ix) we see that there is satisfied assumption (v) of Theorem 3.5.

The proof is complete. □

In what follows we are going to illustrate our result contained in Theorem 4.7 by a few examples.

At the beginning we provide examples related to assumptions (i)-(iv) of the mentioned theorem. To this end we recollect some auxiliary facts which will be needed in our considerations (cf. [5]).

At first, let us assume that the function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, nondecreasing on \mathbb{R}_+ and such that $\varphi(r) \leq r$ for $r > 0$. Moreover, we assume that φ is subadditive i.e., $\varphi(\alpha + \beta) \leq \varphi(\alpha) + \varphi(\beta)$ for $\alpha, \beta \in \mathbb{R}_+$. Next, let us take the function $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ defined by the formula

$$\phi(x) = \frac{P\varphi(|x|)}{Q + \varphi(|x|)}, \tag{4.6}$$

where P, Q are positive constant and $P < Q$.

Then, we can show that

$$|\phi(x) - \phi(y)| \leq \phi(|x - y|) \tag{4.7}$$

and

$$\phi(r) \leq \frac{P}{Q}r < r$$

for $r > 0$.

Example 4.8. As concrete examples of the functions φ appearing above we can give the functions defined by the following formulas:

$$\begin{aligned} \varphi(r) &= r, \\ \varphi(r) &= \ln(1 + r), \\ \varphi(r) &= \arctan r, \\ \varphi(r) &= 2(\sqrt{1 + r} - 1). \end{aligned}$$

Example 4.9. Let us take into account the sequence (a_n) of functions defined on the set \mathbb{R} by the formula

$$a_n(x) = \gamma_n + \frac{P_n \varphi(|x|)}{Q_n + \varphi(|x|)},$$

where γ_n is a real sequence of positive numbers such that $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover, (P_n) and (Q_n) are sequences of positive numbers such that (P_n) is nondecreasing and $P_n \rightarrow P$ as $n \rightarrow \infty$ and (Q_n) is nonincreasing and $Q_n \rightarrow Q$ as $n \rightarrow \infty$. Apart from this $Q > 0$ and $P < Q$. Additionally we assume that $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, nondecreasing, $\varphi(r) \leq r$ for $r \geq 0$ and φ is subadditive.

Observe that $a_n(x) \geq \gamma_n > 0$ for $n = 1, 2, \dots$ which implies that $a_n : \mathbb{R} \rightarrow (0, \infty)$. Next, let us denote by ϕ_A the function defined by formula (4.6) i.e.,

$$\phi_A(r) = \frac{P\varphi(r)}{Q + \varphi(r)}.$$

Obviously, the function ϕ_A is nondecreasing and continuous. Moreover, for arbitrary $x, y \in \mathbb{R}$ and for a fixed natural number n we have:

$$|a_n(x) - a_n(y)| \leq P_n \left| \frac{\varphi(|x|)}{Q_n + \varphi(|x|)} - \frac{\varphi(|y|)}{Q_n + \varphi(|y|)} \right|.$$

Hence, in view of (4.7) we get:

$$\begin{aligned} |a_n(x) - a_n(y)| &\leq P_n \frac{\varphi(|x - y|)}{Q_n + \varphi(|x - y|)} \leq \frac{P\varphi(|x - y|)}{Q + \varphi(|x - y|)} \\ &= \phi_A(|x - y|). \end{aligned}$$

Further, for $r \geq 0$ we obtain

$$\phi_A(r) = P \frac{\varphi(r)}{Q + \varphi(r)} \leq P.$$

Moreover, let us pay attention to the fact that $a_n(x_n) \rightarrow 0$ as $x_n \rightarrow 0$ which is a consequence of the continuity of the function $\varphi(r)$ at $r = 0$ and the equality $\varphi(0) = 0$.

Hence we see that the functions $a_n = a_n(x)$ ($n = 1, 2, \dots$) and $\phi_A = \phi_A(r)$ satisfy assumptions (i), (ii) of Theorem 4.7.

Example 4.10. Now, let us take a sequence (c_n) of function $c_n = c_n(x)$ ($n = 1, 2, \dots$) defined on \mathbb{R} by the equality

$$c_n(x) = \frac{P_n \varphi(|x|)}{Q_n + \varphi(|x|)},$$

where the constants P_n, Q_n ($n = 1, 2, \dots$) and the function φ satisfy the same conditions as in the previous example. In the same way as in Example 4.9 we can show that there are satisfied assumptions (iii), (iv) of Theorem 4.7 with the function

$$\phi_C(r) = \frac{P\varphi(r)}{Q + \varphi(r)}$$

for $r \geq 0$.

Example 4.11. For a fixed $n \in \mathbb{N}$ let us consider the function $f_n : I \times \mathbb{R}^\infty \rightarrow \mathbb{R}$ defined in the following way

$$f_n(t, x_n, x_{n+1}, \dots) = \frac{t}{3nt + 1} + \frac{n}{n + 1} \cdot \frac{x_n}{x_n^2 + 1} + \frac{2n}{2n + 1} \cdot \frac{x_{n+1}}{x_{n+1}^2 + 2}.$$

Then, we have the following estimate:

$$\begin{aligned}
 |f_n(t, x_n, x_{n+1}, \dots)| &\leq \frac{1}{3n} + \frac{n}{n+1} \cdot \frac{|x_n|}{x_n^2 + 1} + \frac{2n}{2n+1} \cdot \frac{|x_{n+1}|}{x_{n+1}^2 + 2} \\
 &\leq \frac{1}{3n} + \frac{2n}{2n+1} [|x_n| + |x_{n+1}|] \\
 &\leq \frac{1}{3n} + \frac{2n}{2n+1} \cdot 2 \max\{|x_n|, |x_{n+1}|\} \\
 &\leq \frac{1}{3n} + \frac{2n}{2n+1} \cdot 2 \sup\{|x_n|, |x_{n+1}|, |x_{n+2}|, \dots\} \\
 &= \frac{1}{3n} + \frac{4n}{2n+1} \sup\{|x_i| : i \geq n\}.
 \end{aligned}$$

Hence we see that the function f_n satisfies the inequality from assumption (v) with the sequences (k_n) and (l_n) defined as follows:

$$k_n = \frac{1}{3n}, \quad l_n = \frac{4n}{2n+1}$$

for $n = 1, 2, \dots$. Hence (cf. Remark 4.4) we have that $K = \frac{1}{3}$ and $L = 2$. Thus the functions f_n ($n = 1, 2, \dots$) satisfy assumption (v) of Theorem 4.7.

Further, observe that the functions $x_n \rightarrow \frac{x_n}{x_n^2+1}, x_{n+1} \rightarrow \frac{x_{n+1}}{x_{n+1}^2+2}$ are Lipschitzian on the set \mathbb{R} with the constant 1. Thus, the function $f_n = f_n(t, x_n, x_{n+1}, \dots)$ satisfies on the set $I \times c_0$ the Lipschitz condition with the constant 1. This implies that the family of functions $\{f_n\}_{n \in \mathbb{N}}$ is uniformly equicontinuous on the set $I \times c_0$. This allows us to conclude that there is satisfied assumption (vi) of Theorem 4.7.

Example 4.12. Consider the infinite system of integral equations of the form

$$\begin{aligned}
 x_n(t) &= \frac{P_n \arctan |x_n(t)|}{Q_n + \arctan |x_n(t)|} + \left(\frac{1}{n^2 + n} \right. \\
 &+ \left. \frac{P'_n \ln(1 + |x_n(t)|)}{Q'_n + \ln(1 + |x_n(t)|)} \right) \int_0^1 [t + sD(s)] \left\{ \frac{s}{3ns + 1} + \frac{n}{n+1} \cdot \frac{x_n(s)}{x_n^2(s) + 1} \right. \\
 &\quad \left. + \frac{2n}{2n+1} \frac{x_{n+1}(s)}{x_{n+1}^2(s) + 2} \right\} ds,
 \end{aligned} \tag{4.8}$$

where $D = D(s)$ denotes the so-called Dirichlet function defined on the interval $I = [0, 1]$ as

$$D(t) = \begin{cases} 0 & \text{for } t \text{ rational} \\ 1 & \text{for } t \text{ irrational.} \end{cases}$$

Moreover, $(P_n), (P'_n)$ are nondecreasing sequences of positive numbers converging to P and P' , respectively, while $(Q_n), (Q'_n)$ are nonincreasing sequences of positive numbers converging to positive limits Q and Q' , respectively. Apart from this we assume that $P < Q$ and $P' < Q'$.

Obviously $t \in I$ and $n = 1, 2, \dots$

Let us observe that infinite system (4.8) is a special case of infinite system (4.1) if we put

$$\begin{aligned}
 a_n(x_n) &= \frac{1}{n^2 + n} + \frac{P'_n \ln(1 + |x_n|)}{Q'_n + \ln(1 + |x_n|)}, \\
 c_n(x_n) &= \frac{P_n \arctan |x_n|}{Q_n + \arctan |x_n|}
 \end{aligned}$$

$$f_n(t, x_n, x_{n+1}, \dots) = \frac{t}{3nt + 1} + \frac{n}{n + 1} \cdot \frac{x_n}{x_n^2 + 1} + \frac{2n}{2n + 1} \cdot \frac{x_{n+1}}{x_{n+1}^2 + 2}$$

$$b(t, s) = t + sD(s)$$

for $n = 1, 2, \dots$ and for $t, s \in I = [0, 1]$.

Further, let us observe that $a_n(0) = \gamma_n$ for $n = 1, 2, \dots$. Thus we have that $A_0 = \sup\{\gamma_n : n \in \mathbb{N}\}$. Moreover, we have that $M_1 = N + A_0 = P + A_0, M_2 = P'$ and $M = \max\{M_1, M_2\} = \max\{P + A_0, P'\}$.

In the sequel of this chapter we will assume that $M < \frac{1}{12}$ and

$$\frac{\max\{P, P'\}}{\min\{Q, Q'\}} \leq \frac{1}{2}. \tag{4.9}$$

Apart from this let us notice that in our considerations we can accept that

$$\phi_A(r) = \frac{P' \ln(1 + r)}{Q' + \ln(1 + r)},$$

$$\phi_C(r) = \frac{P \arctan r}{Q + \arctan r}.$$

Now, we check that there are satisfied assumptions imposed in Theorem 4.7.

Indeed, in view of Examples 4.8 - 4.11 we see that there are satisfied assumptions (i)-(vii), where $K = \frac{1}{3}, L = 2$. Moreover, we have:

$$\bar{B} = \sup \left\{ \int_0^1 |b(t, s)| ds : t \in I \right\} = \sup \left\{ t + \int_0^1 sD(s) ds : t \in [0, 1] \right\} = \frac{3}{2}.$$

In view of the above accepted assumptions, we get:

$$M\bar{B}L = \frac{3}{2}M \cdot 2 = 3M < \frac{1}{4} < 1.$$

Thus we see that there is satisfied assumption (viii).

Finally, let us observe that for an arbitrarily fixed $r > 0$ we have:

$$\begin{aligned} \bar{B} \frac{K + LM}{1 - M\bar{B}L} \phi_A(r) + \phi_C(r) &= \frac{1 + 6M}{2(1 - 3M)} \phi_A(r) + \phi_C(r) \\ &< \phi_A(r) + \phi_C(r) = \frac{P' \ln(1 + r)}{Q' + \ln(1 + r)} + \frac{P \arctan r}{Q + \arctan r} \\ &\leq \max\{P, P'\} \left[\frac{\psi(r)}{Q' + \psi(r)} + \frac{\psi(r)}{Q + \psi(r)} \right], \end{aligned}$$

where we denoted $\psi(r) = \max\{\ln(1 + r), \arctan r\}$ for $r \geq 0$.

From the above estimate we derive the following one:

$$\begin{aligned} &\bar{B} \frac{K + LM}{1 - M\bar{B}L} \phi_A(r) + \phi_C(r) \\ &\leq \max\{P, P'\} \frac{2\psi(r)}{\min\{Q, Q'\} + r} \leq \frac{2 \max\{P, P'\}}{\min\{Q, Q'\}} \psi(r) < r, \end{aligned}$$

where the last inequality is a consequence of the above assumed inequality (4.9) and the fact that $\psi(r) < r$ for $r > 0$.

This shows that assumption (ix) is satisfied.

Finally, let us notice that all assumptions of Theorem 4.7 are satisfied. In view of that theorem the infinite system of integral equations (4.8) has at least one solution in the space $C_0 = C(I, c_0)$.

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