Filomat 34:8 (2020), 2797–2803
https://doi.org/10.2298/FIL2008797D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On nth Roots of Normal Operators

B.P. Duggal[a], **I.H. Kim**[b]

[a] *8 Redwood Grove, London W5 4SZ, England (U.K.)*
[b] *Department of Mathematics, Incheon National University, Incheon, 22012, Korea*

Abstract. For n-normal operators A [2, 4, 5], equivalently n-th roots A of normal Hilbert space operators, both A and A^* satisfy the Bishop–Eschmeier–Putinar property $(\beta)_\epsilon$, A is decomposable and the quasi-nilpotent part $H_0(A-\lambda)$ of A satisfies $H_0(A-\lambda)^{-1}(0) = (A-\lambda)^{-1}(0)$ for every non-zero complex λ. A satisfies every Weyl and Browder type theorem, and a sufficient condition for A to be normal is that either A is dominant or A is a class $\mathcal{A}(1,1)$ operator.

1. Introduction

Let $B(\mathcal{H})$ denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Hilbert space $\mathcal{H}$ into itself. Every *normal operator* $A \in B(\mathcal{H})$, i.e., $A \in B(\mathcal{H})$ such that $[A^*, A] = A^*A - AA^* = 0$, has an nth root for every positive integer $n > 1$. Thus given a normal $A \in B(\mathcal{H})$, there exists $B \in B(\mathcal{H})$ such that $B^n = A$ (and then $\sigma(B^n) = \sigma(B)^n = \sigma(A)$). A straight forward application of the Putnam-Fuglede commutativity theorem ([14, Page 103]) applied to $[B, B^n] = 0$ then implies $[B^*, B^n] = 0$. (Conversely, $[B^*, B^n] = 0$ implies B^n is normal). Operators $B \in B(\mathcal{H})$ satisfying $[B^*, B^n] = 0$ have been called *n-normal*, and a study of the spectral structure of n-normal operators, with emphasis on the properties which B inherits from its normal avatar B^n, has been carried out in ([2], [4], [5]).

Given $A \in B(\mathcal{H})$, let $\sigma(A) \subseteq \angle < \frac{2\pi}{n}$ denote that $\sigma(A)$ is contained in an angle $\angle$, with vertex at the origin, of width less than $\frac{2\pi}{n}$. Assuming $\sigma(B) \subseteq \angle < \frac{2\pi}{n}$ for an n-normal operator $B \in B(\mathcal{H})$, the authors of ([2], [4], [5]) prove that B inherits a number of properties from B^n, amongst them that B satisfies Bishop-Eschmeier-Putinar property $(\beta)_\epsilon$, B is polaroid (hence also isoloid) and $\lim_{m\to\infty}\langle x_m, y_m\rangle = 0$ for sequences $\{x_m\}, \{y_m\} \subset \mathcal{H}$ of unit vectors such that $\lim_{m\to\infty} \|(B-\lambda)x_m\| = 0 = \lim_{m\to\infty} \left\|(B-\mu)y_m\right\|$ for distinct scalars $\lambda, \mu \in \sigma_a(B)$. (All our notation is explained in the following section.) That B inherits a property from B^n in many a case has little to do with the normality of B^n, but is instead a consequence of the fact that B^n has the property. Thus, if the approximate point spectrum $\sigma_a(B^n) = \sigma_a(B)^n$ of B^n is normal (recall: $\lambda \in \sigma_a(B^n)$) is normal if $\lim_{m\to\infty} \|(B^n-\lambda)x_m\| = 0$ for a sequence $\{x_m\} \subseteq \mathcal{H}$ of unit vectors implies $\lim_{m\to\infty} \|(B^n-\lambda)^*x_m\| = 0$; hyponormal operators, indeed dominant operators, satisfy this property), $\sigma(B) \subseteq \angle < \frac{2\pi}{n}$, and $\{x_m\}$, $\{y_m\}$

2010 *Mathematics Subject Classification*. Primary 47A05, 47A55; Secondary 47A80, 47A10
Keywords. Normal operator, n-th root, property $(\beta)_\epsilon$, decomposable, quasi-nilpotent part, pole, dominant operator, Weyl and Browder theorems
Received: 04 September 2019; Accepted: 13 February 2020
Communicated by Dragan S. Djordjević
The second named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2019R1F1A1057574)
Email addresses: bpduggal@yahoo.co.uk (B.P. Duggal), ihkim@inu.ac.kr (I.H. Kim)

are sequences of unit vectors in $\mathcal{H}$ such that $\lim_{m\to\infty} \|(B^n - \lambda^n)x_m\| = 0 = \lim_{m\to\infty} \left\|(B^n - \mu^n)y_m\right\|$ for some distinct $\lambda, \mu \in \sigma_a(B)$, then

$$\lim_{m\to\infty} \lambda^n \langle x_m, y_m\rangle = \lim_{m\to\infty} \langle B^n x_m, y_m\rangle = \lim_{m\to\infty} \langle x_m, B^{*n} y_m\rangle = \mu^n \lim_{m\to\infty} \langle x_m, y_m\rangle$$

implies

$$(\lambda - \mu) \lim_{m\to\infty} \langle x_m, y_m\rangle = 0 \Longleftrightarrow \lim_{m\to\infty} \langle x_m, y_m\rangle = 0$$

(*cf.* [4, Theorem 2.4]). It is well known that w-hyponormal operators satisfy property $(\beta)_\epsilon$ ([3]). If $B^n \in (\beta)_\epsilon$ (i.e., B^n satisfies property $(\beta)_\epsilon$) and $\sigma(B) \subseteq \angle < \frac{2\pi}{n}$, then [7, Theorem 2.9 and Corollary 2.10] imply that $B + N \in (\beta)_\epsilon$ for every nilpotent operator N which commutes with B (*cf.* [5, Theorem 3.1]). Again, if B^n is polaroid and $\sigma(B) \subseteq \angle < \frac{2\pi}{n}$, then B is polaroid (hence also, isoloid) ([9, Theorem 4.1]). Observe that paranormal operators are polaroid. Nth roots of normal operators have been studied by a large number of authors (see [18], [17], [6], [11], [13]) and there is a rich body of text available in the literature. Our starting point in this note is that an n-normal operator B considered as an nth root of a normal operator has a well defined structure ([13, Theorem 3.1]). The problem then is that of determining the "normal like" properties which B inherits. We prove in the following that the condition $\sigma(B) \subseteq \angle < \frac{2\pi}{n}$ may be dispensed with in many a case (though not always). Just like normal operators, nth roots B have SVEP (the single-valued extension property) everywhere, $\sigma(B) = \sigma_a(B)$, B is polaroid (hence also, isoloid). $B \in (\beta)_\epsilon$ (as also does B^*) and (the quasinilpotent part) $H_0(B - \lambda) = (B - \lambda)^{-1}(0)$ at every $\lambda \in \sigma_p(B)$ except for $\lambda = 0$ when we have $H_0(B) = B^{-n}(0)$. Again, just as for normal operators, B satisfies various variants of the classical Weyl's theorem $\sigma(B) \setminus \sigma_w(B) = E_0(B)$ (resp., Browder's theorem $\sigma(B) \setminus \sigma_w(B) = \Pi_0(B)$). It is proved that dominant and class $\mathcal{A}(1, 1)$ operators B are normal.

2. Notation and terminology

Given an operator $S \in B(\mathcal{H})$, *the point spectrum, the approximate point spectrum, the surjectivity spectrum* and *the spectrum* of S will be denoted by $\sigma_p(S), \sigma_a(S), \sigma_{su}(S)$ and $\sigma(S)$, respectively. The isolated points of a subset K of $\mathbb{C}$, the set of complex numbers, will be denoted by iso(K). An operator $X \in B(\mathcal{H})$ is a *quasi-affinity* if it is injective and has a dense range, and operators $S, T \in B(\mathcal{H})$ are *quasi-similar* if there exist quasi-affinities $X, Y \in B(\mathcal{H})$ such that $SX = XT$ and $YS = TY$.

$S \in B(\mathcal{H})$ has *SVEP, the single-valued extension property*, at a point $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathfrak{D}$ centered at λ_0 the only analytic function $f : \mathfrak{D} \to \mathcal{H}$ satisfying $(S - \lambda)f(\lambda) = 0$ is the function $f \equiv 0$; S has *SVEP* if it has SVEP everywhere in $\mathbb{C}$. (Here and in the sequel, we write $S - \lambda$ for $S - \lambda I$.) Let, for an open subset $\mathcal{U}$ of $\mathbb{C}$, $\mathcal{E}(\mathcal{U}, \mathcal{H})$ (resp., $O(\mathcal{U}, \mathcal{H})$) denote the Fréchet space of all infinitely differentiable (resp., analytic) H-valued functions on $\mathcal{U}$ endowed with the topology of uniform convergence of all derivatives (resp., topology of uniform convergence) on compact subsets of $\mathcal{U}$. $S \in B(\mathcal{H})$ satisfies *property* $(\beta)_\epsilon$, $S \in (\beta)_\epsilon$, at $\lambda \in \mathbb{C}$ if there exists a neighborhood $\mathcal{N}$ of λ such that for each subset $\mathcal{U}$ of $\mathcal{N}$ and sequence $\{f_n\}$ of H-valued functions in $\mathcal{E}(\mathcal{U}, \mathcal{H})$,

$$(S - z)f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H}) \Longrightarrow f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H})$$

(resp., S satisfies *property* (β), $S \in (\beta)$, at $\lambda \in \mathbb{C}$ if there exists an $r > 0$ such that, for every open subset $\mathcal{U}$ of the open disc $\mathfrak{D}(\lambda; r)$ of radius r centered at λ and sequence $\{f_n\}$ of $\mathcal{H}$-valued functions in $O(\mathcal{U}, \mathcal{H})$,

$$(S - z)f_n(z) \to 0 \text{ in } O(\mathcal{U}, \mathcal{H}) \Longrightarrow f_n(z) \to 0 \text{ in } O(\mathcal{U}, \mathcal{H})).$$

The following implications are well known ([12], [16]):

$$S \in (\beta)_\epsilon \Longrightarrow S \in (\beta) \Longrightarrow S \text{ has SVEP; } S, S^* \in (\beta) \Longrightarrow S \text{ decomposable.}$$

The *ascent* asc$(S - \lambda)$ (resp., *descent* dsc$(S - \lambda)$) of S at $\lambda \in \mathbb{C}$ is the least non-negative integer p such that $(S - \lambda)^{-p}(0) = (S - \lambda)^{-(p+1)}(0)$ (resp., $(S - \lambda)^p(\mathcal{H}) = (S - \lambda)^{(p+1)}(\mathcal{H})$). A point $\lambda \in$ iso$\sigma(S)$ (resp., $\lambda \in$ iso$\sigma_a(S)$)

is a *pole* (resp., *left pole*) of the resolvent of S if $0 < \mathrm{asc}(S-\lambda) = \mathrm{dsc}(S-\lambda) < \infty$ (resp., there exists a positive integer p such that $\mathrm{asc}(S-\lambda) = p$ and $(S-\lambda)^{p+1}(\mathcal{H})$ is closed) ([1]). Let

$$\Pi(S) = \{\lambda \in \mathrm{iso}\sigma(S) : \lambda \text{ is a pole (of the resolvent) of } S\};$$
$$\Pi^a(S) = \{\lambda \in \mathrm{iso}\sigma_a(S) : \lambda \text{ is a left pole (of the resolvent) of } S\}.$$

Then $\Pi(S) \subseteq \Pi^a(S)$, and $\Pi^a(S) = \Pi(S)$ if (and only if) S^* has SVEP at points $\lambda \in \Pi^a(S)$. We say in the following that the operator S is *polaroid* if $\{\lambda \in \mathbb{C} : \lambda \in \mathrm{iso}\sigma(S)\} \subseteq \Pi(S)$. Polaroid operators are isoloid (where S is *isoloid* if $\{\lambda \in \mathbb{C} : \lambda \in \mathrm{iso}\sigma(S)\} \subseteq \sigma_p(S)$). Let $\sigma_x = \sigma$ or σ_a. The sets $E^x(S) = E(S)$ or $E^a(S)$ and $E_0^x(S) = E_0(S)$ or $E_0^a(S)$ are then defined by

$$E^x(S) = \{\lambda \in \mathrm{iso}\sigma_x(S) : \lambda \in \sigma_p(S)\}, \text{ and}$$
$$E_0^x(S) = \{\lambda \in \mathrm{iso}\sigma_x(S) : \lambda \in \sigma_p(S),\ \dim(S-\lambda)^{-1}(0) < \infty\}.$$

It is clear that

$$\Pi^x(S) \subseteq E^x(S) \text{ and } \Pi_0^x(S) \subseteq E_0^x(S)$$

(where $\Pi_0^x(S) = \{\lambda \in \Pi^x(S) : \dim(S-\lambda)^{-p}(0) < \infty\}$).

The *quasi-nilpotent part* $H_0(S)$ and the *analytic core* $K(S)$ of $S \in B(\mathcal{H})$ are the sets

$$\begin{aligned} H_0(S) &= \left\{x \in \mathcal{H} : \lim_{n\to\infty} \|S^n x\|^{\frac{1}{n}} = 0\right\}, \text{ and} \\ K(S) &= \{x \in \mathcal{H} : \text{there exists a sequence } \{x_n\} \subset \mathcal{H} \text{ and } \delta > 0 \text{ for} \\ &\qquad \text{which } x = x_0, Sx_{n+1} = x_n \text{ and } \|x_n\| \le \delta^n \|x\| \text{ for all } n = 1, 2, \cdots\} \end{aligned}$$

([1]). If $\lambda \in \mathrm{iso}\sigma(S)$, then $\mathcal{H}$ has a direct sum decomposition $\mathcal{H} = H_0(S-\lambda) \oplus K(S-\lambda)$, $S-\lambda|_{H_0(S-\lambda)}$ is quasinilpotent and $S-\lambda|_{K(S-\lambda)}$ is invertible. A necessary and sufficient condition for a point $\lambda \in \mathrm{iso}\sigma(S)$ to be a pole of S is that there exist a positive integer p such that $H_0(S-\lambda) = (S-\lambda)^{-p}(0)$.

In the following we shall denote the upper semi-Fredholm, the lower semi-Fredholm and the Fredholm spectrum of S by $\sigma_{usf}(S), \sigma_{lsf}(S)$ and $\sigma_f(S)$; $\sigma_{uw}(S), \sigma_{lw}(S)$ and $\sigma_w(S)$ (resp., $\sigma_{ub}(S), \sigma_{lb}(S)$ and $\sigma_b(S)$) shall denote the upper Weyl, the lower Weyl and the Weyl (resp., the upper Browder, the lower Browder and the Browder) spectrum of S. Additionally, we shall denote the upper B-Weyl, the lower B-Weyl and the B-Weyl (resp., the upper B-Browder, the lower B-Browder and the B-Browder) spectrum of S by $\sigma_{ubw}(S), \sigma_{lbw}(S)$ and $\sigma_{bw}(S)$ (resp., $\sigma_{ubb}(S), \sigma_{lbb}(S)$ and $\sigma_{bb}(S)$). We refer the interested reader to the monograph ([1]) for definition, and other relevant information, on these distinguished parts of the spectrum; our interest here in these spectra is at best peripheral.

3. Results

Throughout the following, $A \in B(\mathcal{H})$ shall denote an n-normal operator. Considered as an nth root of the normal operator A^n, A has a direct sum representation

$$A = \bigoplus_{i=0}^{\infty} A\,|_{\mathcal{H}_i} = \bigoplus_{i=0}^{\infty} A_i,\ \mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}_i,$$

where A_0 is n-nilpotent and A_i, for all $i = 1, 2, \cdots$, is similar to a normal operator $N_i \in B(\mathcal{H}_i)$. Equivalently,

$$A = B_1 \oplus B_0,\ B_0 = A_0 \text{ and } B_1 = \bigoplus_{i=1}^{\infty} A_i,$$

where $B_0^n = 0$ and B_1 is quasi-similar to a normal operator $N = \bigoplus_{i=1}^{\infty} N_i \in B\left(\bigoplus_{i=1}^{n} \mathcal{H}_i\right)$. Quasi-similar operators preserve SVEP; hence, since the direct sum of operators has SVEP at a point if and only if the summands have SVEP at the point, A and A^* have SVEP (everywhere). Consequently ([1]):

$$\sigma(A) = \sigma(B_1) \cup \{0\} = \sigma(N) \cup \{0\} = \sigma_a(A) = \sigma_{su}(A),$$

$$E^a(A) = E(A),\ E_0^a(A) = E_0(A),\ \Pi^a(A) = \Pi(A),\ \Pi_0^a(A) = \Pi_0(A);$$

furthermore:

$$\sigma_f(A) = \sigma_{usf}(A) = \sigma_{lsf}(A) = \sigma_w(A) = \sigma_{uw}(A) = \sigma_{lw}(A) = \sigma_b(A) = \sigma_{ub}(A) = \sigma_{lb}(A),$$

$$\sigma_{bf}(A) = \sigma_{bw}(A) = \sigma_{ubw}(A) = \sigma_{lbw}(A) = \sigma_{bb}(A) = \sigma_{ubb}(A) = \sigma_{lbb}(A).$$

The point spectrum of a normal operator consists of normal eigenvalues (i.e., the corresponding eigenspaces are reducing): This fails for the operator A ([4, Remark 2.17]), and a sufficient condition is that $\sigma(A) \subseteq \angle < \frac{2\pi}{n}$ (for then $(A-\lambda)x = 0 \Longrightarrow (A^n - \lambda^n)x = 0 \Longrightarrow (A^{*n} - \bar{\lambda}^n)x = 0 \Longleftrightarrow (A^* - \bar{\lambda})x = 0$).

The polaroid property travels from A^n to A, no restriction on $\sigma(A)$. (This would then imply that $E^a(A) = E(A) = \Pi(A) = \Pi^a(A)$ and $E_0^a(A) = E_0(A) = \Pi_0(A) = \Pi_0^a(A)$.) We start by proving that the quasi-similarity of B_1 and N transfers to the Riesz projections $P_{B_1}(\lambda)$ and $P_N(\lambda)$ corresponding to points $\lambda \in \text{iso}\sigma(B_1) = \text{iso}\sigma(N)$. Let Γ be a positively oriented path separating λ from $\sigma(B_1)$ and let X, Y be quasi-affinities such that $B_1X = XN$ and $YB_1 = NY$. Then, for all $\mu \notin \sigma(B_1)$,

$$P_{B_1}(\lambda) = \frac{1}{2\pi i}\int_\Gamma (\mu - B_1)^{-1} d\mu \Longleftrightarrow YP_{B_1}(\lambda) = Y\left\{\frac{1}{2\pi i}\int_\Gamma (\mu - B_1)^{-1} d\mu\right\}$$
$$\Longleftrightarrow YP_{B_1}(\lambda) = \left\{\frac{1}{2\pi i}\int_\Gamma (\mu - N)^{-1} d\mu\right\} Y = P_N(\lambda)Y.$$

A similar argument proves

$$P_{B_1}(\lambda)X = XP_N(\lambda).$$

Theorem 3.1. *A is polaroid.*

Proof. Continuing with the argument above, the normality of N implies that the range $H_0(N-\lambda)$ of $P_N(\lambda)$ coincides with $(N-\lambda)^{-1}(0)$. Hence $(N-\lambda)P_N(\lambda) = 0$, and

$$Y(B_1 - \lambda)P_{B_1}(\lambda) = (N-\lambda)YP_{B_1}(\lambda) = (N-\lambda)P_N(\lambda)Y = 0$$
$$\Longrightarrow \quad (B_1 - \lambda)P_{B_1}(\lambda) = 0 \Longleftrightarrow H_0(B_1 - \lambda) = (B_1 - \lambda)^{-1}(0).$$

Since $\lambda \in \text{iso}\sigma(B_1)$,

$$\bigoplus_{i=1}^{\infty} \mathcal{H}_i = H_0(B_1 - \lambda) \oplus K(B_1 - \lambda) = (B_1 - \lambda)^{-1}(0) \oplus K(B_1 - \lambda)$$
$$\Longrightarrow \quad \bigoplus_{i=1}^{\infty} \mathcal{H}_i = (B_1 - \lambda)^{-1}(0) \oplus (B_1 - \lambda)\bigoplus_{i=1}^{\infty} \mathcal{H}_i,$$

i.e., λ is a (simple) pole. The n-nilpotent operator B_0 being polaroid, the direct sum $B_0 \oplus B_1$ is polaroid (since $\text{asc}(A-\lambda) \le \text{asc}(B_0 - \lambda) \oplus \text{asc}(B_1 - \lambda)$ and $\text{dsc}(A-\lambda) \le \text{dsc}(B_0 - \lambda) \oplus \text{dsc}(B_1 - \lambda)$ for all λ ([20, Exercise 7, Page 293])). □

Theorem 3.1 implies:

Corollary 3.2. *A is isoloid (i.e., points $\lambda \in \text{iso}\sigma(A)$ are eigenvalues of A).*

More is true and, indeed, Theorem 3.1 is a consequence of the following result which shows that $H_0(A-\lambda) = (A-\lambda)^{-1}(0)$ for all non-zero $\lambda \in \sigma(A)$.

Theorem 3.3. *$H_0(A-\lambda) = (A-\lambda)^{-1}(0)$ for all non-zero $\lambda \in \sigma(A)$ and $H_0(A) = A^{-n}(0)$. In particular, A is polaroid.*

Proof. Following the same notation as above, the normality of N implies $H_0(N-\lambda) = (N-\lambda)^{-1}(0)$ for all $\lambda \in \sigma(N)$ $(= \sigma(B_1))$. Since

$$NY = YB_1 \iff (N-\lambda)Y = Y(B_1-\lambda), \text{ all } \lambda,$$

it follows that

$$\|(N-\lambda)^n Yx\|^{\frac{1}{n}} = \|Y(B_1-\lambda)^n x\|^{\frac{1}{n}} \le \|Y\|^{\frac{1}{n}} \|(B_1-\lambda)^n x\|^{\frac{1}{n}} \to 0 \text{ as } n \to \infty$$

for all $x \in H_0(B_1-\lambda)$. Consequently,

$$Yx \in H_0(N-\lambda) = (N-\lambda)^{-1}(0) \Longrightarrow Y(B_1-\lambda)x = (N-\lambda)Yx = 0 \iff x \in (B_1-\lambda)^{-1}(0),$$

and hence

$$H_0(B_1-\lambda) = (B_1-\lambda)^{-1}(0)$$

for all $\lambda \in \sigma(B_1)$. Evidently,

$$H_0(A) = H_0(B_1 \oplus B_0) = B_1^{-1}(0) \oplus B_0^{-n}(0) \subseteq A^{-n}(0).$$

Argue now as in the proof of Theorem 3.1 to prove that A is polaroid. □

The Riesz projection $P_A(\lambda)$ corresponding to points $(0 \neq)$ $\lambda \in \text{iso}\sigma(A)$ are, in general, not self-adjoint. Since $\sigma(A) \subseteq \angle < \frac{2\pi}{n}$ ensures $(A-\lambda)^{-1}(0) \subseteq (A^*-\bar{\lambda})^{-1}(0)$ for all $0 \neq \lambda \in \sigma_p(A)$, $\sigma(A) \subseteq \angle < \frac{2\pi}{n}$ forces $P_A(\lambda) = P_A(\lambda)^*$ for all $\lambda \neq 0$.

Corollary 3.4. *If $\sigma(A) \subseteq \angle < \frac{2\pi}{n}$, then the Riesz projection corresponding to non-zero $\lambda \in \text{iso}\sigma(A)$ is self-adjoint.*

Remark 3.5. *Theorems 3.1 and 3.3 generalize corresponding results from [2], [4], [5] by removing the hypothesis that $\sigma(A) \subseteq \angle < \frac{2\pi}{n}$, and, in the case of Theorem 3.3, the hypothesis on the points λ being isolated in $\sigma(A)$. Recall from [1, Page 336] that an operator $S \in B(\mathcal{H})$ is said to have property Q if $H_0(S_\lambda)$ is closed for all λ:* Theorem 3.3 says that the nth roots A have property Q. *Another proof of Theorem 3.3, hence also of the fact that the operators A satisfy property Q, follows from the argument below proving the subscalarity of A.*

Property $(\beta)_\epsilon$ (similarly (β)) does not travel well under quasi-affinities. Thus $CX = XB$ and $B \in (\beta)_\epsilon$ does not imply $C \in (\beta)_\epsilon$ (see [7, Remark 2.7] for an example). However, $C \in (\beta)_\epsilon$ implies $B \in (\beta)_\epsilon$ holds, as the following argument proves. If $\{f_n\}$ is a sequence in $\mathcal{E}(\mathcal{U}, \mathcal{H})$ such that

$$(B-z)f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H}),$$

then

$$X(B-z)f_n(z) = (C-z)Xf_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H}).$$

Since $C \in (\beta)_\epsilon$ and X is a quasi-affinity,

$$Xf_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H}) \Longrightarrow f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U}, \mathcal{H}).$$

Thus $B \in (\beta)_\epsilon$.

Theorem 3.6. *A and A^* satisfy property $(\beta)_\epsilon$.*

Proof. Recall from [7, Lemma 2.2] that a direct sum of operators satisfies $(\beta)_\epsilon$ if and only if the individual operators satisfy $(\beta)_\epsilon$. The operator A being the direct sum $B_1 \oplus B_0$, where B_0, B_0^* being nilpotent satisfy $(\beta)_\epsilon$, to prove the theorem it will suffice to prove $B_1, B_1^* \in (\beta)_\epsilon$. But this is immediate from the argument above, since normal operators N satisfy $N, N^* \in (\beta)_\epsilon$ and since there exist quasi-affinities X and Y in $B\left(\bigoplus_{i=1}^{\infty} \mathcal{H}_i\right)$ such that $N^*X^* = X^*B_1^*$ and $NY = YB_1$. □

$A \in (\beta)_\epsilon$ implies $A \in (\beta)$, and $A, A^* \in (\beta)$ implies A is decomposable ([16]). Hence:

Corollary 3.7. *A is decomposable.*

We consider next a sufficient condition for the operator A to be normal. However, before that we point out that the operator A satisfies almost all Weyl and Browder type theorems ([1]) satisfied by normal operators.

Weyl's theorem An operator $S \in B(\mathcal{H})$ satisfies

$$\text{generalized Weyl's theorem, } S \in \text{gWt, if } \sigma(S) \setminus \sigma_{Bw}(S) = E(S);$$
$$a-\text{generalized Weyl's theorem, } S \in a-\text{gWt, if } \sigma_a(S) \setminus \sigma_{uBw}(S) = E^a(S)$$

(see [1, Definitions 6.59, 6.81]). Let $S \in \text{Wt}, S \in a-\text{Wt}, S \in \text{gBt}, S \in a-\text{gBt}, S \in \text{Bt}$ and $S \in a-\text{Bt}$ denote, respectively, that

$$S \text{ satisfies Weyl's theorem} : \sigma(S) \setminus \sigma_w(S) = E_0(S),$$
$$S \text{ satisfies a} - \text{Weyl's theorem} : \sigma_a(S) \setminus \sigma_{aw}(S) = E_0^a(S),$$
$$S \text{ satisfies generalized Browder's theorem} : \sigma(S) \setminus \sigma_{Bw}(S) = \Pi(S),$$
$$S \text{ satisfies generalized a} - \text{Browder's theorem} : \sigma_a(S) \setminus \sigma_{uBw}(S) = \Pi^a(S),$$
$$S \text{ satisfies Browder's theorem} : \sigma(S) \setminus \sigma_w(S) = \Pi_0(S),$$
$$S \text{ satisfies a} - \text{Browder's theorem} : \sigma_a(S) \setminus \sigma_{aw}(S) = \Pi_0^a(S),$$

(see [1, Chapter 6]). The following implications are well known ([1, Chapters 5, 6]):

$$S \in a-\text{gWt} \Longrightarrow \left\{ \begin{array}{l} S \in a-\text{Wt} \\ S \in \text{gWt} \end{array} \right. \Longrightarrow S \in \text{Wt} \Longrightarrow S \in \text{Bt},$$

$$S \in a-\text{gWt} \Longrightarrow \left\{ \begin{array}{l} S \in a-\text{Wt} \\ S \in a-\text{gBt} \end{array} \right. \Longrightarrow S \in a-\text{Bt} \Longrightarrow S \in \text{Bt},$$

$$S \in a-\text{gBt} \Longleftrightarrow S \in a-\text{Bt},\ S \in \text{gBt} \Longleftrightarrow S \in \text{Bt}.$$

A has SVEP (guarantees $A \in a-\text{gBt}$ ([1, Therem 5.37])) and $\sigma(A) = \sigma_a(A)$ guarantee the equivalence of a-gBt and gBt (hence also of a-gBt with a-Bt and Bt) for A. The fact that A is polaroid and $\sigma(A) = \sigma_a(A)$ guarantees also that $E(A) = E^a(A) = \Pi^a(A) = \Pi(a)$ (and $E_0(A) = E_0^a(A) = \Pi_0^a(A) = \Pi_0(a)$). Hence all Weyl's theorems (listed above) are equivalent for A and :

Theorem 3.8. $A \in a-\text{gWt}$

Normal A**.** For the operator $A = B_1 \oplus B_0$ to have any chance of being a normal operator, it is necessary that (either B_0 is missing, or) $B_0 = 0$. The hypothesis (B_0 is missing, or) $B_0 = 0$ is, however, in no way sufficient to ensure the normality of A. Additional hypotheses are required. An operator $S \in B(\mathcal{H})$ is said to be *dominant* (resp., *class* $\mathcal{A}(1,1)$) if to every complex λ there corresponds a real number $M_\lambda > 0$ such that $\|(S-\lambda)^*x\| \leq M_\lambda \|(S-\lambda)x\|$ for all $x \in \mathcal{H}$ (resp., $|S|^2 \leq \left|S^2\right|$) ([19], [15]). Recall from [10, Lemma 2.1] that if a dominant or class $\mathcal{A}(1,1)$ operator $A \in B(\mathcal{H})$ is a square root of a normal operator, then A is normal. The following theorem, which uses an argument different from that used in [10], proves that this result extends to nth roots A.

Theorem 3.9. *Dominant or* $\mathcal{A}(1,1)$ *nth roots of a normal operator in* $B(\mathcal{H})$ *are normal.*

Proof. Recall that the eigenvalues of a dominant operator are normal (i.e., they are simple and the corresponding eigenspace is reducing). Hence if our nth root of $A = B_1 \oplus B_0$ is dominant, then $A = B_1 \oplus 0$ is a dominant operator which satisfies

$$A\left(Y \oplus I|_{\mathcal{H}_o}\right) = \left(Y \oplus I|_{\mathcal{H}_o}\right)(N \oplus 0).$$

The operator $N \oplus 0$ being normal and the operator $Y \oplus I\,|_{\mathcal{H}_o}$ being a quasi-affinity it follows from [19], [8] that A is normal (and unitarily equivalent to $N \oplus 0$). We consider next $A \in \mathcal{A}(1,1)$.

It is well known that $\mathcal{A}(1,1)$ operators have ascent less than or equal to one. (Indeed, operators $S \in \mathcal{A}(1,1)$ are *paranormal*: $\|Sx\|^2 \leq \left\|S^2x\right\| \|x\|$ for all $x \in \mathcal{H}$, hence $\mathrm{asc}(S) \leq 1$.) Hence if $A = B_1 \oplus B_0 \in \mathcal{A}(1,1)$, then $B_0 = 0$ and $A \in B\left(A^{-1}(0) \oplus A^{-1}(0)^{\perp}\right)$ has an upper triangular matrix representation

$$A = \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

Let $N_1 = N \oplus 0\,|_{\mathcal{H}_0}$ have the represenation

$$N_1 = 0 \oplus N_{22} \in B\left(N_1^{-1}(0) \oplus N_1^{-1}(0)^{\perp}\right),$$

and let $Y_1 = Y \oplus I\,|_{\mathcal{H}_0} \in B\left(N_1^{-1}(0) \oplus N_1^{-1}(0)^{\perp}, A^{-1}(0) \oplus A^{-1}(0)^{\perp}\right)$ have the corresponding matrix representation

$$Y_1 = \left[Y_{ij}\right]_{i,j=1}^{2}.$$

Then, given that Y is a quasi-affinity satisfying $B_1Y = YN$, Y_1 is a quasi-affinity such that $AY_1 = Y_1N_1$. Consequently, $A_{22}Y_{21} = 0$. The operator A_{22} being injective, we must have $Y_{21} = 0$ (and then Y_{11} is injective and Y_{22} has a dense range). The operator A being an nth root of a normal operator, A^n is normal. Applying the Putnam-Fuglede commutativity theorem to ($AY_1 = Y_1N_1 \Longrightarrow$) $A^nY_1 = Y_1N_1^n$, it follows that $A^{*n}Y_1 = Y_1N_1^{*n}$, and hence $Y_{12}N_{22}^{*}{}^{n} = 0$. Since the normal operator $N_{22}^{*}{}^{n}$ has a dense range, $Y_{12} = 0$ (which than implies that Y_{11} and Y_{22} are quasi-affinities). But then $A_{22}^*Y_{22} = Y_{22}N_{22}^*$ and $A_{22}Y_{22} = Y_{22}N_{22}$ imply that A_{22} is quasi-affinity. Hence, since ($A^nY_1 = Y_1N_1^n$ implies also that) $A_{12}A_{22}^{n-1}Y_{11} = 0$, $A_{12} = 0$. Thus $A = 0 \oplus A_{22}$, where $A_{22} \in \mathcal{A}(1,1)$, $A_{22}^{-1}(0) = \{0\}$ and $A_{22}Y_{22} = Y_{22}N_{22}$. Applying Proposition 2.5 and Lemma 2.2 of [10], it follows that A_{22} and N_{22} are (unitarily equivalent) normal operators. Conclusion: $A = 0 \oplus A_{22}$ is a normal nth root. □

References

[1] P. Aiena, Fredholm and Local Spectral Theory II with Applications to Weyl-type Theorems, Lecture Notes in Mathematics 2235, Springer (2018).

[2] S.A. Alzuraiqi and A.B. Patel, *On n-normal operators*, General Math. Notes **1**(2010), 61-73.

[3] C. Benhida and E.H. Zerouali, *Local spectral theory of linear operators RS and SR*, Integr. Equat. Oper. Theory **54**(2006), 1-8.

[4] M. Chō and Načevska Nastovska, *Spectral properties of n-normal operators*, Filomat **32**(2018), 5063-5069.

[5] M. Chō, J.E. Lee, K. Tanahashi and A. Uchiyama, *Remarks on n-normal operators*, Filomat **32**(2018), 5441-5451.

[6] I. Colojoara and C. Foias, Theory of Generalized Spectral Operators, Gordan and Breach (1968), New York.

[7] B.P. Duggal, *Finite intertwinings and subscalarity*, Operators and Matrices **4**(2010), 257-271.

[8] B.P. Duggal, *On dominant operators*, Archiv der Math. **46**(1986), 353-359.

[9] B.P. Duggal, D.S. Djordjević, R.E. Harte and S.C. Živković-Zlatanović, *Polynomially meromorphic operators*, Math. Proc. Royal Irish Acad. **116 A(1)**(2016), 83-98.

[10] B.P. Duggal, S.V. Djordjević and I.H. Jeon, *A Putnam-Fuglede commutativity theorem for class $\mathcal{A}$ operators*, Rendiconti del Circolo Matematico di Palermo **63**(2014), 355-362.

[11] M.R. Embry, *nth roots of normal operators*, Proc. Amer. Math. Soc. **19**(1968), 63-68.

[12] J. Eschmeier and M. Putiner, *Bishop's property (β) and rich extensions of linear operators*,, Indiana Univ. Math. J. **37**(1988), 325-348.

[13] F. Gilfeather, *Operator valued roots of abelian analytic functions*, Pac. J. Math. **55**(1974), 127-148.

[14] P.R. Halmos, A Hilbert Space Problem Book. Second Edition (1982), Springer-Verlag, New York - Heidelbery - Berlin.

[15] M. Ito and T. Yamazaki, *Relations between inequalities $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq \left(A^{\frac{p}{2}}B^rA^{\frac{p}{2}}\right)^{\frac{p}{p+r}}$ and their applications*, Integr. Equat. Oper. Theory **44**(2002), 442-450.

[16] K.B. Laursen and M.N. Neumann, Introduction to Local Spectral Theory, Clarendon, Oxford 2000.

[17] M. Radjavi and P. Rosenthal, *On roots of normal operators*, J. Math. Anal. Appl. **34**(2)(2013), 653-665.

[18] J.G. Stampfli, *Roots of scalar operators*, Proc. Amer. Math. Soc. **13**(1962), 796-798.

[19] J.G. Stampfli and B.L. Wadhwa, *An asymmetric Putnam-Fuglede theorem for dominant operators*, Indiana Univ. Math. J. **25**(1976), 359-365.

[20] A.E. Taylor and D.C. Lay, Introduction to Functional Analysis, Wily, New York, 1980