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On Measures of Parameterized Fuzzy Compactness

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Abstract. In the present study, the parameterized degree of compactness of a lattice valued fuzzy soft set is described in a fuzzy soft topological space. The extended versions of the basic compactness properties known in general topology are investigated for the given notion and some further characterizations of parameterized degree of fuzzy compactness are specified. In addition, a generalized version of Tychonoff Theorem is proved in the product fuzzy soft topological space.

1. Introduction

Since Molodtsov [18] proposed soft set theory to overcome some of the difficulties involving the parametrization process in handling uncertainties, many researchers have applied soft set theory in different directions. Especially, the definition of fuzzy soft set given by Maji et al. [17], has gained acceleration to the improvement of the investigations of proposed theories in several aspects [1, 3, 4, 8, 16, 19, 20, 22, 27-30].

Compactness is one of the most significant concepts in general topology since it is sort of a topological generalization of finiteness. If you have some object, then compactness allows you to extend results that you know are true for all finite subobjects to the object itself. A very closely related example is the compactness theorem in propositional logic: an infinite collection of sentences is consistent if every finite subcollection is consistent. The notion of compactness has been generalized to L-topological space by many authors [5, 14, 15, 23, 24]. In these approaches of compactness, fuzzy set does not have a degree of compactness except the empty set and the whole space. Lowen and Lowen [13] considered compactness of I-topological spaces in a matter of degree. Then Šostak [25] introduced the compactness degree of an L-fuzzy set in the case $L = I$. Later Li et al. [12] observed the fuzzy compactness degree in L-fuzzy topological spaces and Çetkin et al. [7] introduced the parameterized compactness of a fuzzy soft set in the parameterized fuzzy topological spaces.

The goal of this study is to measure the parameterized degree of compactness of a lattice valued fuzzy soft set in a fuzzy soft universe and study its fundamental characteristics by enlarging the well-known properties to the parameterized fuzzy case. In this manner the paper is arranged as follows. In section 2, we recall some lattice theoretical properties, the notion of a fuzzy soft set and some operations on fuzzy soft sets, fuzzy soft topology. Moreover, we propose the definitions of a base and a subbase for a given fuzzy soft topology by considering the parametrization tool. In section 3, we describe the degree of compactness

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of a lattice valued fuzzy soft set according to a parameter by means of the implication operation of the lattice, in such spaces. We also give a characterization and extend the elementary compactness properties. For instance, we gain the results that the union of two compact L-fuzzy soft sets is compact, the intersection of a compact L-fuzzy soft set and a closed L-fuzzy soft set is also compact, and the compactness is preserved under a continuous L-fuzzy soft mapping, in the described meaning. In section 4, we observe some other characterizations of parameterized degree of fuzzy compactness in terms of different coverings. In the last section, we give a generalized version of the Tychonoff theorem for the product fuzzy soft topology. Hence, we proved that the degree of compactness of a product space is computed by the meet of the degrees of compactness of the production spaces.

2. Preliminaries

Throughout this paper, X refers to a nonempty initial universe, E, K denotes the arbitrary nonempty sets viewed on the sets of parameters and $L = (L, \vee, \wedge, ')$ denotes a complete DeMorgan algebra with the smallest element 0_L and the largest element 1_L. With the underlying lattice L, a mapping $A : X \to L$ is said to be an L-fuzzy set on X and by L^X, we denote the family of all L-fuzzy sets on X.

Let α, β and γ be elements in L. An element α in L is said to be coprime (respectively, prime) if $\alpha \leq \beta \vee \gamma$ implies that $\alpha \leq \beta$ or $\alpha \leq \gamma$ (respectively, if $\alpha \geq \beta \wedge \gamma$ implies $\alpha \geq \beta$ or $\alpha \geq \gamma$) The set of all prime and coprime elements of L is denoted by $p(L)$ and $c(L)$, respectively. We say α is wedge below β, in symbols, $\alpha \lhd \beta$ or $\beta \rhd \alpha$, if for every arbitrary subset $D \subseteq L$, $\vee D \geq \beta$ implies $\alpha \leq d$ for some $d \in D$. As shown by Raney [21] a complete lattice L is completely distributive if and only if $\beta = \bigvee\{\alpha \in L \mid \alpha \lhd \beta\}$ for each $\beta \in L$. For any $\beta \in L$, define $\Theta(\beta) = \bigvee\{\alpha \in L \mid \alpha \lhd \beta\}$ as the greatest minimal family and denote the greatest maximal family by $\Omega(\beta)$ through the paper. The wedge below operation in a completely distributive lattice has an interpolation property, this means $\alpha \lhd \beta$ implies there exists $\gamma \in L$ such that $\alpha \lhd \gamma \lhd \beta$. For the details of lattices and the wedge below relation, see [10].

The binary operation $\mapsto$ in the complete DeMorgan algebra L is given by $\alpha \mapsto \beta = \bigvee\{\gamma \in L \mid \alpha \wedge \gamma \leq \beta\}$. For all $\alpha, \beta, \gamma, \delta \in L$ and $\{\alpha_i\}, \{\beta_i\} \subseteq L$, the following properties are satisfied:

(1) $(\alpha \mapsto \beta) \geq \gamma$ iff $\alpha \wedge \gamma \leq \beta$.

(2) $\alpha \mapsto \beta = 1_L$ iff $\alpha \leq \beta$.

(3) $\alpha \mapsto (\bigwedge_i \beta_i) = \bigwedge_i (\alpha \mapsto \beta_i)$.

(4) $(\bigvee_i \alpha_i) \mapsto \beta = \bigwedge_i (\alpha_i \mapsto \beta)$.

The parameterized version of an L-fuzzy set is called an L-fuzzy soft set and it is defined as follows.

Definition 2.1. ([17]) f is called an L-fuzzy soft set on X, where f is a mapping from E into L^X. This means that $f_e := f(e) : X \to L$, is an L-fuzzy set on X, for each parameter $e \in E$.
The family of all L-fuzzy soft sets on X is denoted by $(L^X)^E$.

The set-theoretical operations for lattice valued fuzzy soft sets are described as follows.

Definition 2.2. ([2, 17, 22]) Let f and g be two L-fuzzy soft sets on X. Then:

(1) we say that f is an L-fuzzy soft subset of g and write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. f and g are called equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.

(2) the union of f and g is an L-fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(3) the intersection of f and g on X is an L-fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(4) the complement of an L-fuzzy soft set f is denoted by f', where $f' : E \to L^X$ is a mapping given by $f'_e = (f_e)'$, for each $e \in E$. Clearly $(f')' = f$.

Definition 2.3. ([22]) We say that

(1) An L-fuzzy soft set f on X is called a null L-fuzzy soft set and denoted by $\widetilde{0}$, if $f_e(x) = 0$, for each $e \in E$ and $x \in X$.

(2) An L-fuzzy soft set f on X is called an absolute L-fuzzy soft set and denoted by $\widetilde{1}$, if $f_e(x) = 1$, for each $e \in E, x \in X$. Clearly $(\widetilde{1})' = \widetilde{0}$ and $\widetilde{0}' = \widetilde{1}$.

Notation 2.4. ([9]) The fuzzy soft inclusion $[\,\widetilde{\sqsubseteq}\,] : (L^X)^E \times (L^X)^E \to L$ is defined by the following equality

$$[f\widetilde{\sqsubseteq}g] = \bigwedge_{x\in X}\bigwedge_{e\in E}(f'_e(x) \vee g_e(x)).$$

Definition 2.5. ([11]) Let $\varphi : X_1 \to X_2$ and $\psi : E_1 \to E_2$ be two functions, where E_1 and E_2 are parameter sets for the crisp sets X_1 and X_2, respectively. Then the pair φ_ψ is called an L-fuzzy soft mapping from X_1 to X_2. Let f and g be two L-fuzzy soft sets on X_1 and X_2, respectively.

(1) The image of f under the L-fuzzy soft mapping φ_ψ, denoted by $\varphi_\psi(f)$, is an L-fuzzy soft set on X_2 defined by for all $k \in E_2$, $y \in X_2$,

$$\varphi_\psi(f)_k(y) = \begin{cases} \bigvee_{\varphi(x)=y}\bigvee_{\psi(a)=k} f_a(x), & \text{if } x \in \varphi^{-1}(y), a \in \psi^{-1}(k), \\ 0, & \text{otherwise.} \end{cases}$$

(2) The pre-image of g under the L-fuzzy soft mapping φ_ψ, denoted by $\varphi_\psi^{-1}(g)$, is an L-fuzzy soft set on X_1 defined by

$$\varphi_\psi^{-1}(g)_e(x) = g_{\psi(e)}(\varphi(x)), \qquad \text{for all } e \in E_1, x \in X_1.$$

If φ and ψ is injective (surjective), then φ_ψ is said to be injective (surjective).

Definition 2.6. ([4]) A mapping $\tau : K \to L^{(L^X)^E}$ is called an L-fuzzy (E, K)-soft topology on X if it satisfies the following conditions for each $k \in K$.

(O1) $\tau_k(\widetilde{0}) = \tau_k(\widetilde{1}) = 1_L$.
(O2) $\tau_k(f \sqcap g) \geq \tau_k(f) \wedge \tau_k(g)$, for all $f, g \in (L^X)^E$.
(O3) $\tau_k(\bigsqcup_{i\in\Delta} f_i) \geq \bigwedge_{i\in\Delta} \tau_k(f_i)$, for all $f_i \in (L^X)^E, i \in \Delta$.

Then the pair (X, τ) is called an L-fuzzy (E, K)-soft topological space. The value $\tau_k(f)$ is interpreted as the degree of openness of an L-fuzzy soft set f with respect to the parameter $k \in K$. The parameterized gradation of closedness of f is computed as $\tau^*_k(f) = \tau_k(f')$, where f' denotes the complement of the L-fuzzy soft set f. Let $\mathcal{U}$ be a subfamily of $(L^X)^E$, then the value $\tau_k(\mathcal{U}) = \bigwedge_{f\in\mathcal{U}} \tau_k(f)$ will be called the parameterized degree of openness of the subfamily $\mathcal{U} \subseteq (L^X)^E$ with respect to the parameter $k \in K$.

Example 2.7. ([7]) Let $L = \{(0, 0), (1, 1)\} \cup \{(a, 0), (0, b), (a, a) \mid a, b \in (0, 1)\}$ and a relation $'' \leq ''$ on L be defined as follows: $(m, b) \leq (n, d)$ if and only if $m \leq n$ and $b \leq d$. Define an order reversing involution $' : L \to L$ is as follows: for each $x, y \in (0, 1)$, $(x, 0)' = (1 - x, 0), (0, y)' = (0, 1 - y), (x, x)' = (1 - x, 1 - x)$ and $(1, 1)' = (0, 0)$. Then $(L, \leq, ')$ is a complete DeMorgan algebra. Let $X = \{x, y\}, E = (0, 0.5]$ and $f_e(x) = f_e(y) = (e, 0)$, $g_e(x) = g_e(y) = (0, e)$ and $h_e(x) = h_e(y) = (e, e)$ for each $e \in E$. If a mapping $\tau : E \to L^{(L^X)^E}$ is defined as follows:

$$\tau_e(u) = \begin{cases} (1, 1), & \text{if } u \in \{\widetilde{0}, \widetilde{1}, h\}; \\ (e, 0), & \text{if } u = f; \\ (0, e), & \text{if } u = g; \\ (0, 0), & \text{otherwise,} \end{cases}$$

then τ is an L-fuzzy (E, E)-soft topology on X.

Definition 2.8. ([4]) Let (X_1, τ^1) be an L-fuzzy (E_1, K_1)-soft topological space and (X_2, τ^2) be an L-fuzzy (E_2, K_2)-soft topological space. Let $\varphi : X_1 \to X_2, \psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be functions. Then $\varphi_{\psi,\eta} : (X_1, \tau^1) \to (X_2, \tau^2)$ is said to be continuous if $\tau^1_k(\varphi^{-1}_{\psi}(g)) \geq \tau^2_{\eta(k)}(g)$ for all $g \in (L^{X_2})^{E_2}, k \in K_1$.

The parameterized fuzzy base and the subbase of a fuzzy soft topology are described in the following manner.

Definition 2.9. Let τ be an L-fuzzy (E, K)-soft topology on X. Then the mapping

(1) $\mathcal{B} : K \to L^{(L^X)^E}$ is called a base of τ if it satisfies the following condition, for each $f \in (L^X)^E$ and $k \in K$:
$\tau_k(f) = \bigvee \{ \bigwedge_{i\in\Gamma} \mathcal{B}_k(g_i) \mid \sqcup_{i\in\Gamma} g_i = f \}$
where the expression on the right-hand side of the equality will be denoted by $\mathcal{B}^{\sqcup}_k(f)$.

(2) $\mathcal{S} : K \to L^{(L^X)^E}$ is called a subbase of τ if $\mathcal{S}^{\sqcap} : K \to L^{(L^X)^E}$ is a base of τ, where for all $k \in K$ and $f \in (L^X)^E$, $\mathcal{S}^{\sqcap}_k(f) = \bigvee \{ \bigwedge_{\lambda\in J} \mathcal{S}_k(g_\lambda) \mid (\sqcap)_{\lambda\in J}\ g_\lambda = f \}$. Here $(\sqcap)$ standing for the finite intersection.

The product of the family of given fuzzy soft topologies are described as follows.

Definition 2.10. Let $\{(X_i, \tau^i)\}_{i\in\Gamma}$ be a family of L-fuzzy (E_i, K_i)-soft topological spaces and $p_i : \Pi X_i \to X_i, q_i : \Pi E_i \to E_i, r_i : \Pi K_i \to K_i$ be the crisp projections. And let $X = \Pi X_i$, $K = \Pi K_i$ and $E = \Pi E_i$ be the cartesian product sets. Then the L-fuzzy (E, K)-soft topology τ on X whose subbase is defined as follows, for each $k \in K$ and $f \in (L^X)^E$,

$$\mathcal{S}_k(f) = \bigvee_{i\in\Gamma} \bigwedge_{(p_q)^{-1}_i(g)=f} \tau^i_{r_i(k)}(g)$$

is called the product L-fuzzy (E, K)-soft topology of the family $\{\tau^i\}_{i\in\Gamma}$ and the pair (X, τ) is called the product space of the family $\{(X_i, \tau^i)\}_{i\in\Gamma}$.

Lemma 2.11. ([7]) *Let $\varphi : X_1 \to X_2, \psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be three crisp functions. Then for each subfamily $\mathcal{U} \subseteq (L^{X_2})^{E_2}$, the following equality is satisfied.*

$$\bigvee_{k\in E_2} \bigvee_{y\in X_2} \left(\varphi_\psi(g)'_k(y) \wedge \bigwedge_{f\in\mathcal{U}} f_k(y) \right) = \bigvee_{e\in E_1} \bigvee_{x\in X_1} \left(g'_e(x) \wedge \bigwedge_{f\in\mathcal{U}} \varphi^{-1}_\psi(f)_e(x) \right).$$

3. Measures of Parameterized Fuzzy Compactness

In order to generalize the notion of compactness to the fuzzy soft universe, let us consider the following definition which reflects us the parameterized extension version of the fuzzy-crisp case.

Definition 3.1. ([9]) Let $\mathcal{T} = \{\mathcal{T}_k\}_{k\in K}$ be an (E, K)-soft L-topology on X (see [6]) and $h \in (L^X)^E$. The L-fuzzy soft set h is said to be compact in $(X, \mathcal{T})$, if for each k and each cover $\mathcal{U} \subseteq \mathcal{T}_k$ of h there exists a finite subfamily $\mathcal{V} \subseteq \mathcal{U}$ which covers h, i.e.,

$$[h \widetilde{\sqsubseteq} \bigvee \mathcal{U}] \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [h \widetilde{\sqsubseteq} \bigvee \mathcal{V}]$$

For each $k \in K, \mathcal{U} \subseteq \mathcal{T}_k$ we define $\chi_{\mathcal{T}_k}(\mathcal{U}) = 1$, where $\chi_{\mathcal{T}_k}(\mathcal{U}) = \bigwedge_{f\in\mathcal{U}} \chi_{\mathcal{T}_k}(f)$ and $\chi_{\mathcal{T}_k}(f) = 1$ when $f \in \mathcal{T}_k$, $\chi_{\mathcal{T}_k}(f) = 0$ when $f \notin \mathcal{T}_k$. Hence we conclude that

$$[h \widetilde{\sqsubseteq} \bigvee \mathcal{U}] \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [h \widetilde{\sqsubseteq} \bigvee \mathcal{V}] \ \Leftrightarrow\ [[h \widetilde{\sqsubseteq} \bigvee \mathcal{U}] \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [h \widetilde{\sqsubseteq} \bigvee \mathcal{V}]] = 1.$$

Here the symbol $2^{(\mathcal{U})}$ demonstrates the finite subfamily of $\mathcal{U} \subseteq (L^X)^E$.

As a result of the above discussion, we may say that $h \in (L^X)^E$ is compact according to the parameter k if and only if for every family $\mathcal{U} \subseteq (L^X)^E$, it follows that

$$\mathcal{T}_k(\mathcal{U}) \leq [[h\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [h\widetilde{\sqsubseteq} \bigvee \mathcal{V}]]$$

So we now naturally generalize the notion of compactness degree to the fuzzy soft universe by using the underlying lattice implication in the following way.

Definition 3.2. Let $\tau : K \to L^{(L^X)^E}$ be a map and $g \in (L^X)^E$. Define such a map $com_\tau : K \to L^{(L^X)^E}$ as follows.

$$\begin{aligned} com_\tau(k,g) &= \bigwedge_{\mathcal{U}\subseteq (L^X)^E} [\tau_k(\mathcal{U}) \leq [[g\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [g\widetilde{\sqsubseteq} \bigvee \mathcal{V}]]] \\ &= \bigwedge_{\mathcal{U}\subseteq (L^X)^E} (\tau_k(\mathcal{U}) \mapsto ([g\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [g\widetilde{\sqsubseteq} \bigvee \mathcal{V}])) \\ &= \bigwedge_{\mathcal{U}\subseteq (L^X)^E} (\bigwedge_{h\in\mathcal{U}} \tau_k(h) \mapsto (\bigwedge_{x\in X} \bigwedge_{e\in E} (g'_e(x) \vee \bigvee_{h\in\mathcal{U}} h_e(x)) \mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \bigwedge_{e\in E} (g'_e(x) \vee \bigvee_{h\in\mathcal{V}} h_e(x)))) \end{aligned}$$

If (X, τ) is an L-fuzzy (E, K)-soft topological space, then the value $com_\tau(k, g)$ is called the compactness degree of g with respect to the parameter k. So g is said to be compact L-fuzzy soft set with respect to k if $com_\tau(k, g) = 1_L$. In this manner, the compactness degree of g in the whole space (X, τ) is computed by the value $com_\tau(g) = \bigwedge_{k\in K} com_\tau(k, g)$. So the L-fuzzy soft set g is said to be compact in the fuzzy soft space (X, τ) if $com_\tau(g) = 1_L$. Hence if $com_\tau(\widetilde{1}) = 1_L$, then the whole space (X, τ) is said to be compact.

According to the properties of implication operation $\mapsto$, the following lemma can be proved.

Lemma 3.3. *Let (X, τ) be an L-fuzzy (E, K)-soft topological space, $g \in (L^X)^E$ and $k \in K$. Then $com_\tau(k, g) \geq a$ if and only if for each $\mathcal{U} \in (L^X)^E$,*

$$\tau_k(\mathcal{U}) \wedge [\, g\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [\, g\widetilde{\sqsubseteq} \bigvee \mathcal{V}]$$

Theorem 3.4. *Let (X, τ) be an L-fuzzy (E, K)-soft topological space, $k \in K$ and $g \in (L^X)^E$. Then we can characterize the parameterized compactness degree by the following equality.*

$$com_\tau(k,g) = \bigvee \{a \in L \mid \tau_k(\mathcal{U}) \wedge [\, g\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [\, g\widetilde{\sqsubseteq} \bigvee \mathcal{V}], \forall\, \mathcal{U} \subseteq (L^X)^E\}.$$

Proof. The claim of the theorem is proved via Lemma 3.3. □

The following result shows that the union of two compact L-fuzzy soft sets is compact, too.

Theorem 3.5. *Let (X, τ) be an L-fuzzy (E, K)-soft topological space and $g, h \in (L^X)^E$. Then the following inequality is satisfied for each $k \in K$,*

$$com_\tau(k, g \sqcup h) \geq com_\tau(k, g) \wedge com_\tau(k, h).$$

Proof. Let $g, h \in (L^X)^E$ and $k \in K$ be given, then the following is true.

$$com_\tau(k, g \sqcup h) = \bigvee \{a \in L \mid \tau_k(\mathcal{U}) \wedge [(g \sqcup h)\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [(g \sqcup h)\widetilde{\sqsubseteq} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq (L^X)^E\}$$

$$= \bigvee \{a \in L \mid \tau_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge [h\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [g\widetilde{\sqsubseteq} \bigvee \mathcal{V}] \wedge [h\widetilde{\sqsubseteq} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq (L^X)^E\}$$

$$\geq \bigvee\{a \in L \mid \tau_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [g\widetilde{\sqsubseteq} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq (L^X)^E\}$$
$$\wedge \bigvee\{a \in L \mid \tau_k(\mathcal{U}) \wedge [h\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [h\widetilde{\sqsubseteq} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq (L^X)^E\}$$
$$= com_\tau(k, g) \wedge com_\tau(k, h). \quad \square$$

The following result gives that the intersection of a compact L-fuzzy soft set and a closed L-fuzzy soft set, is compact.

Theorem 3.6. *Let (X, τ) be an L-fuzzy (E, K)-soft topological space and $g, h \in (L^X)^E$. Then the following inequality is valid for each $k \in K$,*

$$com_\tau(k, g \sqcap h) \geq com_\tau(k, g) \wedge \tau_k^*(h).$$

Proof. Let $g, h \in (L^X)^E$ and $k \in K$ be given, then we have

$$com_tau(k, g \sqcap h) = \bigvee\{a \in L \mid \tau_k(\mathcal{U}) \wedge [(g \sqcap h)\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [(g \sqcap h)\widetilde{\sqsubseteq} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq (L^X)^E\}$$
$$= \bigvee\{a \in L \mid \tau_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} h' \sqcup \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [g\widetilde{\sqsubseteq} h' \sqcup \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq (L^X)^E\}$$
$$\geq \{a \wedge \tau_k^*(h) \mid \tau_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge q \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [g\widetilde{\sqsubseteq} \bigvee \mathcal{V}]\}$$
$$\geq com_\tau(k, g) \wedge \tau_k^*(h)$$

as desired. □

Corollary 3.7. *Let (X, τ) be an L-fuzzy (E, K)-soft topological space. Then for each $g \in (L^X)^E$ the relation between the k-parameterized degree of compactness of g and the degree of closedness is as follows:*

$$com_\tau(k, g) \geq com_\tau(k, \widetilde{1}) \wedge \tau_k^*(g).$$

Theorem 3.8. *Let $\tau^1, \tau^2 : K \to L^{(L^X)^E}$ be two maps which satisfy $\tau^2 \leq \tau^1$. Then $com_{\tau^1}(k, g) \leq com_{\tau^2}(k, g)$ for every $g \in (L^X)^E$ and $k \in K$.*

Proof. It is straightforward and therefore omitted. □

Corollary 3.9. *Let τ^1 and τ^2 be two L-fuzzy (E, K)-soft topologies on X which satisfy $\tau^2 \leq \tau^1$, i.e., $\tau_k^2(f) \leq \tau_k^1(f)$ for any $k \in K$ and $f \in (L^X)^E$. Then $com_{\tau^1}(k, g) \leq com_{\tau^2}(k, g)$ for each $k \in K$ and $g \in (L^X)^E$.*

Corollary 3.10. *Let (X, τ) be an L-fuzzy (E, K)-soft topological space and $\mathcal{B}$ be a base or subbase of τ. Then $com_\tau(k, g) \leq com_{\mathcal{B}}(k, g)$ for any $g \in (L^X)^E$ and $k \in K$.*

The following result shows that the compactness of an L-fuzzy soft set is preserved under continuous L-fuzzy soft mapping.

Theorem 3.11. *Let $\varphi_{\psi,\eta} : (X_1, \tau^1) \to (X_2, \tau^2)$ be a continuous L-fuzzy soft mapping between L-fuzzy (E_1, K_1)-soft and L-fuzzy (E_2, K_2)-soft topological spaces. Then for each $k \in K_1$ and $g \in (L^{X_1})^{E_1}$, we have*

$$com_{\tau^1}(k, g) \leq com_{\tau^2}(\eta(k), \varphi_\psi(g)).$$

Proof. Let $k \in K_1$ and g be a fuzzy soft set on X_1 with the parameter set E_1. Then one gets

$$com_{\tau^2}(\eta(k), \varphi_\psi(g)) = \bigvee\{a \in L \mid \tau^2_{\eta(k)}(\mathcal{U}) \wedge [\varphi_\psi(g)\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [\varphi_\psi(g)\widetilde{\sqsubseteq} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq (L^{X_2})^{E_2}\}$$
$$\geq \bigvee\{a \mid (\tau^1)_k(\varphi_\psi^{-1}(\mathcal{U})) \wedge [g\widetilde{\sqsubseteq} \bigvee \varphi_\psi^{-1}(\mathcal{U})] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{\mathcal{U}}} [g\widetilde{\sqsubseteq} \bigvee \varphi_\psi^{-1}(\mathcal{V})], \forall \mathcal{U} \subseteq (L^{X_2})^{E_2}\}$$
$$\geq \bigvee\{a \in L \mid \tau^1_k(\mathcal{P}) \wedge [g\widetilde{\sqsubseteq} \bigvee \mathcal{P}] \wedge a \leq \bigvee_{\mathcal{R}\in 2^{\mathcal{P}}} [g\widetilde{\sqsubseteq} \bigvee \mathcal{R}], \forall \mathcal{P} \in (L^{X_1})^{E_1}\}$$
$$= com_{\tau^1}(k, g).$$

Hence the proof is completed as desired. □

4. Some Further Characterizations

In this section we give some equivalent conditions in order to characterize the parameterized degree of fuzzy compactness of an L-fuzzy soft set in terms of different coverings. Let us first propose the following coverings.

Definition 4.1. Let (X, τ) be an L-fuzzy (E, K)-soft topological space, $\alpha \in L \setminus \{0_L\}$ and $f \in (L^X)^E$. Then the subfamily $\mathcal{U} \subseteq (L^X)^E$ is said to be

(1) an "α-shading of f" if for any $x \in X$ and $e \in E$, it follows that $f'_e(x) \vee \bigvee_{g\in\mathcal{U}} g_e(x) \nleq \alpha$.

(2) a "strong α-shading of f" if $[f\widetilde{\sqsubseteq} \bigvee \mathcal{U}] \nleq \alpha$.

(3) an "α-remote family of f" if for any $x \in X$ and $e \in E$, it follows that $f_e(x) \wedge \bigvee_{h\in\mathcal{U}} h_e(x) \ngeq \alpha$.

(4) a "strong α-remote family of f" if $\bigvee_{e\in E}\bigvee_{x\in X}(f_e(x) \wedge \bigvee_{h\in\mathcal{U}} h_e(x)) \ngeq \alpha$.

(5) a "Q_α-cover of f" if for any $x \in X$ and $e \in E$ with $f_e(x) \nleq \alpha'$, it follows that $\bigvee_{g\in\mathcal{U}} g_e(x) \geq \alpha$.

(6) a "Θ_α-cover of f" if for any $x \in X$ and $e \in E$, it follows that $\alpha \in \Theta(f'_e(x) \vee \bigvee_{g\in\mathcal{U}} g_e(x))$.

(7) a "strong Θ_α-cover of f" if $\alpha \in \Theta([f\widetilde{\sqsubseteq} \bigvee \mathcal{U}])$.

It is noted that if for $\alpha \in c(L)$, $\mathcal{U}$ is a Q_α-cover of f iff $\alpha \leq [f\widetilde{\sqsubseteq} \bigvee \mathcal{U}]$. It is obvious that a strong Θ_α-cover of f is a Θ_α-cover of f.

Let $\tau : K \to L^{(L^X)^E}$ be an L-fuzzy (E, K)-soft topology on X and $a \in L$, define $\tau^a_k = \{f \in (L^X)^E \mid \tau_k(f) \geq a\}$. Then the family $\mathcal{T} = \{\tau^a_k\}_{k\in K}$ constitutes an (E, K)-soft L-topology on X.

Theorem 4.2. *Let (X, τ) be an L-fuzzy (E, K)-soft topological space, $f \in (L^X)^E$ and $\alpha \in L \setminus \{0_L\}$. Then the following statements are equivalent:*

(1) $com_\tau(k, f) \geq \alpha$.

(2) *For each $\beta \leq \alpha, \gamma \in \Theta(\beta), \beta, \gamma \neq 0$, every Q_β-cover $\mathcal{U} \subseteq \tau^\beta_k$ of f has a finite subfamily $\mathcal{V}$ which is a Q_γ-cover of f.*

Proof. (1) $\Rightarrow$ (2): Suppose that $com_\tau(k,g) \geq a$. Then by Lemma 3.3, $\tau_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} \vee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [g\widetilde{\sqsubseteq} \vee \mathcal{V}]$ is satisfied for any $\mathcal{U} \subseteq (L^X)^E$. $\forall b \leq a, \forall r \in \Theta(b)$, if $\mathcal{U} \subseteq \tau_k^b$ is a Q_b-cover of g with respect to k, then $b \leq \tau_k(\mathcal{U})$ and $b \leq [g\widetilde{\sqsubseteq} \vee \mathcal{U}]$. Hence $b \leq \tau_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} \vee \mathcal{U}] \wedge a$. So we have that $r \in \Theta(b) \subseteq \Theta(\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [g\widetilde{\sqsubseteq} \vee \mathcal{V}])$. Therefore $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ which is a Q_r-cover of g.

(2) $\Rightarrow$ (1): Suppose $\forall b \leq a, \forall r \in \Theta(b)$, each Q_b-cover $\mathcal{U} \subseteq \tau_k^b$ of g has a finite subfamily $\mathcal{V}$ which is a Q_r-cover of g. $\forall \mathcal{U} \subseteq (L^X)^E, \forall r \in c(L)$, if $r \in \Theta(\tau_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} \vee \mathcal{U}] \wedge a)$, then there exists $b \in \Theta(\tau_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} \vee \mathcal{U}] \wedge a)$ such that $r \in \Theta(b)$. We have that $b \leq a, b \leq \tau_k(\mathcal{U})$ and $b \leq [g\widetilde{\sqsubseteq} \vee \mathcal{U}]$, this means that $\mathcal{U} \subseteq \tau_k^b$ is a Q_b-cover of g. By the hypothesis, $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ such that a Q_r-cover of g. This implies that $r \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [g\widetilde{\sqsubseteq} \vee \mathcal{V}]$. Therefore, $com_\tau(k,g) \geq a$. □

We may also obtain some other characterizations of the parameterized degree of fuzzy compactness as follows.

Remark 4.3. Let (X,τ) be an L-fuzzy (E,K)-soft topological space, $f \in (L^X)^E$ and $\alpha \in L \setminus \{0_L\}$. Then the following statements are equivalent each other.

(1) $com_\tau(k,f) \geq \alpha$.

(2) For each $\beta \in p(L), \beta \ngeq \alpha$, every strong β-shading $\mathcal{U}$ of f with $\bigwedge_{h\in\mathcal{U}} \tau_k(h) \nleq \beta$ has a finite subcollection $\mathcal{V}$ which is a strong β-shading of f.

(3) For each $\beta \in p(L), \beta \ngeq \alpha$, every strong β-shading $\mathcal{U}$ of f with $\bigwedge_{h\in\mathcal{U}} \tau_k(h) \nleq \beta$ has a finite subcollection $\mathcal{V}$ and $\gamma \in \Omega^*(\beta)$ where $\mathcal{V}$ is a γ-shading of f.

(4) For each $\beta \in p(L), \beta \ngeq \alpha$, every strong β-shading $\mathcal{U}$ of f with $\bigwedge_{h\in\mathcal{U}} \tau_k(h) \nleq \beta$ has a finite subcollection $\mathcal{V}$ and $\gamma \in \Omega^*(\beta)$ where $\mathcal{V}$ is a strong γ-shading of f.

(5) For each $\beta \in c(L), \beta \nleq \alpha'$, every strong β-remote family $\mathcal{P}$ of f with $\bigvee_{h\in\mathcal{P}} \tau_k^*(h) \nleq \beta'$ has a finite subcollection $\mathcal{R}$ which is a strong β-remote family of f.

(6) For each $\beta \in c(L), \beta \nleq \alpha'$, every strong β-remote family $\mathcal{P}$ of f with $\bigvee_{h\in\mathcal{P}} \tau_k^*(h) \nleq \beta'$ has a finite subcollection $\mathcal{R}$ and $\gamma \in \Theta^*(\beta)$ where $\mathcal{R}$ is a γ-remote family of f.

(7) For each $\beta \in c(L), \beta \nleq \alpha'$, every strong β-remote family $\mathcal{P}$ of f with $\bigvee_{h\in\mathcal{P}} \tau_k^*(h) \nleq \beta'$ there is a finite subcollection $\mathcal{R}$ of $\mathcal{P}$ and $\gamma \in \Theta^*(\beta)$ where $\mathcal{R}$ is a strong γ-remote family of f.

(8) For each $\beta \leq \alpha, \gamma \in \Theta(\beta), \beta, \gamma \neq 0$, every Q_β-cover $\mathcal{U} \subseteq \tau_k^\beta$ of f has a finite subfamily $\mathcal{V}$ which is a strong Θ_β-cover of f.

(9) For each $\beta \leq \alpha, \gamma \in \Theta(\beta), \beta, \gamma \neq 0$, every Q_β-cover $\mathcal{U} \subseteq \tau_k^\beta$ of f has a finite subfamily $\mathcal{V}$ which is a Θ_γ-cover of f.

(10) For each $\beta \leq \alpha, \gamma \in \Theta(\beta), \beta, \gamma \neq 0$, every strong Θ_β-cover $\mathcal{U} \subseteq \tau_k^\beta$ of f has a finite subfamily $\mathcal{V}$ which is a Q_γ-cover of f.

(11) For each $\beta \leq \alpha, \gamma \in \Theta(\beta), \beta, \gamma \neq 0$, every strong Θ_β-cover $\mathcal{U} \subseteq \tau_k^\beta$ of f has a finite subfamily $\mathcal{V}$ which is a strong Θ_γ-cover of f.

(12) For each $\beta \leq \alpha, \gamma \in \Theta(\beta), \beta, \gamma \neq 0$, every strong Θ_β-cover $\mathcal{U} \subseteq \tau_k^\beta$ of f has a finite subfamily $\mathcal{V}$ which is a Θ_γ-cover of f.

Theorem 4.4. *Let (X, τ) be an L-fuzzy (E, K)-soft topological space, $f \in (L^X)^E$ and $\alpha \in L \setminus \{0_L\}$. If $\Theta(\gamma \wedge \delta) = \Theta(\gamma) \cap \Theta(\delta)$, for each $\gamma, \delta \in L$, then the following statements are equivalent.*

(1) $com_\tau(k, f) \geq \alpha$.

(2) *For each $\beta \in \Theta(\alpha), \beta \neq 0$, every strong Θ_β-cover $\mathcal{U}$ of f with $\beta \in \Theta(\tau_k(\mathcal{U}))$ has a finite subcollection $\mathcal{V}$ which is a Q_β-cover of f.*

(3) *For each $\beta \in \Theta(\alpha), \beta \neq 0$, every strong Θ_β-cover $\mathcal{U}$ of f with $\beta \in \Theta(\tau_k(\mathcal{U}))$ has a finite subcollection $\mathcal{V}$ which is a strong Θ_β-cover of f.*

(4) *For each $\beta \in \Theta(\alpha), \beta \neq 0$, every strong Θ_β-cover $\mathcal{U}$ of f with $\beta \in \Theta(\tau_k(\mathcal{U}))$ has a finite subcollection $\mathcal{V}$ which is a Θ_β-cover of f.*

Proof. (1) $\Rightarrow$ (3): Let $com_\tau(k, f) \geq \alpha$ be satisfied for some $k \in K$, $\beta \in \Theta(\alpha)$ and $\mathcal{U} \subseteq (L^X)^E$ be a strong Θ_β-cover of f with $\beta \in \Theta(\tau_k(\mathcal{U}))$. By the hypothesis, $\beta \in \Theta(\tau_k(\mathcal{U})) \cap \Theta([f \widetilde{\sqsubseteq} \bigvee \mathcal{U}]) \cap \Theta(\alpha) = \Theta(\tau_k(\mathcal{U}) \wedge [f \widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge \alpha)$. From Lemma 3.3, we get $\beta \in \Theta([f \widetilde{\sqsubseteq} \bigvee \mathcal{V}])$, for some finite subcollection $\mathcal{V}$ of $\mathcal{U}$. This means that $\mathcal{V}$ is a finite subcollection of $\mathcal{U}$ which is a strong Θ_β-cover of f.

(3) $\Rightarrow$ (1): Let $\beta \in c(L)$ with $\beta \lhd (\tau_k(\mathcal{U}) \wedge [f \widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge \alpha)$, for some subcollection of $\mathcal{U}$ of $(L^X)^E$. Hence we obtain that $\beta \lhd \alpha$, $\beta \lhd \tau_k(\mathcal{U})$ and $\beta \lhd ([f \widetilde{\sqsubseteq} \bigvee \mathcal{U}])$. These results imply that $\beta \in \Theta(\alpha)$ and $\mathcal{U}$ is a strong θ_β-cover of f with $\beta \in \Theta(\tau_k(\mathcal{U}))$. From the hypothesis of (3), there exists a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ which is a strong Θ_β-cover of f. That is, $\beta \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [f \widetilde{\sqsubseteq} \bigvee \mathcal{V}]$ is satisfied. In conclude, we have $\tau_k(\mathcal{U}) \wedge [f \widetilde{\sqsubseteq} \bigvee \mathcal{U}] \wedge \alpha \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [f \widetilde{\sqsubseteq} \bigvee \mathcal{V}]$. From Lemma 3.3, $com_\tau(k, f) \geq \alpha$ is obtained as desired.

The other implications are proved similarly. □

5. Tychonoff Theorem

In this section we enlarge the Tychonoff theorem for the parameterized degree of fuzzy compactness. Throughout this section we assume that L is completely distributive.

The following lemma shows that the parameterized degree of compactness of an L-fuzzy soft set can be characterized by any subbase of the topology.

Lemma 5.1. *Let (X, τ) be an L-fuzzy (E, K)-soft topological space and $\mathcal{S}$ be a subbase of τ. Then*

$$com_\tau(k, g) = \bigvee \{a \in L \mid \mathcal{S}_k(\mathcal{U}) \wedge [g \widetilde{\sqsubseteq} \vee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [g \widetilde{\sqsubseteq} \vee \mathcal{V}]\}.$$

Proof. The claim is proved by using Corollary 3.10 and Theorem 4.2. □

The following result shows that the product of compact L-fuzzy soft sets is also compact.

Theorem 5.2. *Let $\{(X_i, \tau^i)\}_{i \in I}$ be a family of L-fuzzy (E_i, K_i)-soft topological spaces and (X, τ) be the product L-fuzzy (E, K)-soft topological space. Then the relation between the parameterized degree of compactness of $g = \Pi g_i \in (L^X)^E$ in the product space and the compactness degree of the productions is as follows:*

$$com_\tau(k, g) \geq \bigwedge_{i \in I} com_{\tau^i}(k_i, g_i).$$

Proof. Let $\mathcal{S}$ denote the subbase of the product topology τ. In order to prove that $com_\tau(k,g) \geq \bigwedge_{i\in I} com_{\tau^i}(k_i, g_i)$, let $\bigwedge_{i\in I} com_{\tau^i}(k_i, g_i) = a$. Then for any $i \in I, com_{\tau^i}(k_i, g_i) \geq a$. From Lemma 5.1, it is sufficient to show that for any $\mathcal{U} \subseteq (L^X)^E, \mathcal{S}_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} \vee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [g\widetilde{\sqsubseteq} \vee \mathcal{V}]$.

$\forall \mathcal{U} \subseteq (L^X)^E, \forall b \in \Theta(\mathcal{S}_k(\mathcal{U}) \wedge [g\widetilde{\sqsubseteq} \vee \mathcal{U}] \wedge a)$, we have that $b \in \Theta(\mathcal{S}_k(\mathcal{U})), b \in \Theta([g\widetilde{\sqsubseteq} \vee \mathcal{U}])$ and $b \in \Theta(a)$. For each $f \in \mathcal{U}$, there exists $i \in I$ and $f_i \in (L^{X_i})^{E_i}$ such that $b \in \Theta(\tau^i_k(f_i))$ by $b \in \Theta(\mathcal{S}_k(\mathcal{U})) = \Theta(\bigwedge_{f\in\mathcal{U}} \bigvee_{i\in I} \bigvee_{(p_q)_i^{-1}(f_i)=f} \tau^i_k(f_i))$. Let for $j \in J \subseteq I$,

$\mathcal{U}_j = \{f_j \in (L^{X_j})^{E_j} \mid (p_q)_j^{-1}(f_j) = f,\ f \in \mathcal{U}\}$ and $\mathcal{B}_j = \{(p_q)_j^{-1}(f_j) \mid f_j \in (L^{X_j})^{E_j}, (p_q)_j^{-1}(f_j) = f, f \in \mathcal{U}\}$.

We have that $\mathcal{U} = \cup_{j\in J}\mathcal{B}_j$ and $\forall j \in J \subseteq I$, $\tau^j_k(\mathcal{U}) \wedge [g_j\widetilde{\sqsubseteq} \vee \mathcal{U}_j] \wedge a \leq \bigvee_{\mathcal{V}_j\in 2^{(\mathcal{U}_j)}} [g_j\widetilde{\sqsubseteq} \vee \mathcal{V}_j]$.

Besides for any $k \in K$ and $e \in E$, we have

$b \in \Theta(g'_e(x) \vee \bigvee_{f\in\mathcal{U}} f_e(x)) = \Theta(\bigvee_{i\in I} (g_i)'_{e_i}(x_i) \vee \bigvee_{j\in J} \bigvee_{f\in\mathcal{B}_j} f_e(x))$ by $b \in \Theta([g\widetilde{\sqsubseteq} \vee \mathcal{U}])$.

(1) If $b \in \Theta(\bigvee_{i\in I}(g_i)'_{e_i}(x_i))$ for any $x = (x_i) \in X$, then clearly $b \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [g\widetilde{\sqsubseteq} \vee \mathcal{V}]$.

(2) Suppose $b \notin \Theta(\bigvee_{i\in I}(g_i)'_{e_i}(x_i))$ for some $x = (x_i) \in X$, then $b \notin \Theta((g_i)'_{e_i}(x_i))$ for any $i \in I$. Now we prove that there exists $j_0 \in J$ such that $b \in ((g_{j_0})'_{e_{j_0}}(y_{j_0})) \vee \bigvee \mathcal{U}_{j_0}(g_{j_0}))$ for some $g_{j_0} \in X_{j_0}$, if $j \in J$ there exists $g_j \in X_j$ such that $b \notin \Theta((g_j)'_{e_j}(y_j) \vee \bigvee_{h\in\mathcal{U}_j} h_{e_j}(y_j))$. Let $z = (z_i)$ such that $z_i = y_i$ when $i \in J$, $z_i = x_i$ otherwise. By the equality $g'_e(z) = \bigvee_{i\in J}(g_i)'_{e_i}(y_i) \vee \bigvee_{i\notin J}(g_i)'_{e_i}(x_i)$, we obtain $b \notin \Theta(g'_e(z))$. In addition for any $j \in J$, by the following fact

$b \notin \Theta(\bigvee_{h\in\mathcal{U}_j} h_{e_j}(y_j)) = \Theta(\bigvee_{h\in\mathcal{U}_j} (p_q)_j^{-1}(h)_e(z)) = \Theta(\bigvee_{f\in\mathcal{B}_j} f_e(z))$, we have $b \notin \cup_{j\in J}\Theta(\bigvee_{h\in\mathcal{B}_j} h_e(z)) = \Theta(\bigvee_{j\in J} \bigvee_{f\in\mathcal{B}_j} f_e(z))$.

This implies $b \notin \Theta(g'_e(z) \vee \bigvee_{f\in\mathcal{U}} f_e(z))$. This yields a contradiction. Thus we obtain the proof that there exists $j_0 \in J$ such that $b \in \Theta((g_{j_0})'_{e_{j_0}}(x_{j_0}) \vee \bigvee \mathcal{U}_{j_0}(g_{j_0}))$ for any $g_{j_0} \in (L^{X_{j_0}})^{E_{j_0}}$. This shows that $b \leq [g_{j_0}\widetilde{\sqsubseteq} \bigvee \mathcal{U}_{j_0}]$. Thus $b \leq \tau^{j_0}_k(\mathcal{U}_{j_0}) \wedge [g_{j_0}\widetilde{\sqsubseteq} \bigvee \mathcal{U}_{j_0}] \wedge a$. We have

$$
\begin{aligned}
b &\leq \bigvee_{\mathcal{W}_{j_0}\in 2^{(\mathcal{U}_{j_0})}} [g_{j_0}\widetilde{\sqsubseteq} \bigvee \mathcal{W}_{j_0}] = \bigvee_{\mathcal{W}_{j_0}\in 2^{(\mathcal{U}_{j_0})}} \bigwedge_{y_{j_0}\in X_{j_0}} \bigwedge_{e_{j_0}\in E_{j_0}} (g'_{j_0} \vee \bigvee \mathcal{W}_{j_0})_{e_{j_0}}(y_{j_0}) \\
&= \bigvee_{\mathcal{W}_{j_0}\in 2^{(\mathcal{U}_{j_0})}} \bigwedge_{y\in X} \bigwedge_{e\in E} ((p_q)_{j_0}^{-1}(g'_{j_0}) \vee \bigvee_{h\in\mathcal{W}_{j_0}} (p_q)_{j_0}^{-1}(h))_e(y) \\
&\leq \bigvee_{\mathcal{W}_{j_0}\in 2^{(\mathcal{U}_{j_0})}} \bigwedge_{y\in X} \bigwedge_{e\in E} (g' \vee \bigvee_{h\in\mathcal{W}_{j_0}} (p_q)_{j_0}^{-1}(h))_e(y) \\
&\leq \bigvee_{\mathcal{V}_{j_0}\in 2^{(\mathcal{B}_{j_0})}} \bigwedge_{y\in X} \bigwedge_{e\in E} (g' \vee \bigvee \mathcal{V}_{j_0})_e(y) \\
&\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [g\widetilde{\sqsubseteq} \bigvee \mathcal{V}].
\end{aligned}
$$

Thus we obtain the desired result. □

Corollary 5.3. *Let (X,τ) be the product L-fuzzy (E,K)-soft topological space of the family $\{(X_i,\tau^i)\}_{i\in I}$ of L-fuzzy (E_i,K_i)-soft topological spaces, then the parameterized degree of compactness of the product space is as follows*

$com_\tau(k, \widetilde{1}_X) = \bigwedge_{i\in I} com_{\tau^i}(k_i, \widetilde{1}_{X_i})$,

where $X = \Pi X_i, K = \Pi K_i, E = \Pi E_i$ and $k = (k_i)$.

Hence we may conclude that the whole product space is compact if and only if the production spaces are all compact.

6. Conclusion

To the best of our knowledge, the tool of fuzzy soft set theory is a new efficacious technique to dispose uncertainties and it focuses on the parametrization. From this point of view to describe the compactness degree in the parameterized fuzzy universe seems meaningful to us since the compactness is sort of a topological generalization of finiteness. Also we thought that it could be interesting to carry out such an important notion to the parameterized fuzzy case. As a result, in the present paper we pictured the compactness degree of an L-fuzzy soft set in a fuzzy soft topological space as an extension of the existed ones in the literature. For further research, we aim to present parameterized degree of fuzzy countably compactness and Lindelöf property with their applications.

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