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The g-Drazin Inverse Involving Power Commutativity

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Abstract. Let $\mathcal{A}$ be a complex Banach algebra. An element $a \in \mathcal{A}$ has g-Drazin inverse if there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a - a^2b \in A^{qnil}.$$

Let $a, b \in \mathcal{A}^d$. If $a^3b = ba, b^3a = ab,$ and $a^2a^db = aa^dba$, we prove that $a + b \in \mathcal{A}^d$ if and only if $1 + a^db \in \mathcal{A}^d$. We present explicit formula for $(a + b)^d$ under certain perturbations. These extend the main results of Wang, Zhou and Chen (Filomat, **30**(2016), 1185–1193) and Liu, Xu and Yu (Applied Math. Comput., **216**(2010), 3652–3661).

1. Introduction

Throughout the paper, $\mathcal{A}$ denotes a complex Banach algebra with identity. An element a in $\mathcal{A}$ has a g-Drazin inverse provided that there exists some $b \in \mathcal{A}$ such that

$$ab = ba, b = bab, a - a^2b \in \mathcal{A}^{qnil}.$$

Here, $\mathcal{A}^{qnil}$ is the set of all quasinilpotents in $\mathcal{A}$, i.e.,

$$\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid \lim_{n\to\infty} \parallel a^n \parallel^{\frac{1}{n}} = 0\}.$$

As is well known, we have

$$a \in \mathcal{A}^{qnil} \Leftrightarrow 1 + \lambda a \in \mathcal{A}^{-1} \text{ for any } \lambda \in \mathbb{C}.$$

Here, $\mathcal{A}^{-1}$ stands for the set of invertible elements in $\mathcal{A}$. The preceding b is unique, if exists, and is called the g-Drazin inverse of a. We denote it by a^d. We use $\mathcal{A}^d$ to stand for the set of all g-Drazin invertible elements in $\mathcal{A}$. In [10, Theorem 4.2], it was proved that $a \in \mathcal{A}^d$ if and only if there exists an idempotent $p \in \mathcal{A}$ such that $ap = pa, a + p \in \mathcal{A}^{-1}$ and $ap \in \mathcal{A}^{qnil}$.

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The g-Drazin invertibility of the sum of two elements in a Banach algebra is very attractive. It plays an important role in matrix and operator theory, e.g., [3–6, 12, 15]. The Drazin inverse a^D of $a \in \mathcal{A}$ is defined as the g-Drazin inverse by replacing $\mathcal{A}^{qnil}$ by the set of all nilpotents in $\mathcal{A}$. In [13], Liu et al. investigated Drazin inverse $(A+B)^D$ of two complex matrices A and B which satisfying $A^3B = BA$ and $B^3A = AB$. Wang et al. gave representations of $(a+b)^D$ as a function of a, b, a^D and b^D whenever $a^3b = ba$ and $b^3a = ab$ in a ring R in which 2 has Drazin inverse (see [16, Theorem 3.7]). The motivation of this paper is to explore such conditions involving power commutativity under which the sum of two g-Drazin invertible elements in a Banach algebra has g-Drazin inverse.

Let $a, b \in \mathcal{A}^d$. If $ab = ba$, then $a + b \in \mathcal{A}^d$ if and only if $1 + a^d b \in \mathcal{A}^d$ (see [6, Theorem 1]). Zou et al. extended this result to weaker conditions $a^2b = aba$ and $b^2a = bab$ (see [18, Theorem 3.3]). In Section 2, we investigate the relations of $a+b, aa^d(a+b), (a+b)bb^d$ and $aa^d(a+b)bb^d$ for $a, b \in \mathcal{A}^d$. Let $a, b \in \mathcal{A}^d$. If $a^3b = ba, b^3a = ab,$ and $a^d ab = a^d ba$, we prove that $a + b \in \mathcal{A}^d$ if and only if $1 + a^d b \in \mathcal{A}^d$.

Let $x \in \mathcal{A}^d$. The element $x^\pi = 1 - xx^d$ is called the spectral idempotent of x. Let $A, B \in M_n(\mathbb{C})$. Hartwig et al. gave the formula of $(A+B)^D$ under condition $AB = 0$ (see [9, Theorem 2.1]). In [1, Theorem 2.5], Castro-González gave a formula for the Drazin inverse of a sum of two complex matrices less restrictive conditions:

$$A^D B = 0, AB^D = 0 \text{ and } B^\pi ABA^\pi = 0.$$

Guo et al. extended the preceding results and considered representations for the Drazin inverse of the sum of two complex matrices of

$$A^D B = 0, AB^D = 0, B^\pi ABAA^\pi = 0, B^\pi AB^2 A^\pi = 0$$

(see [7, Theorem 2]). In [8, Theorem 2], Guo et al. deduced the expressions for the g-Drazin inverse $(a+b)^d$ under the conditions

$$a^d b = 0, ab^d = 0\ b^\pi abaa^\pi = 0 \text{ and } b^\pi ab^2 a^\pi = 0.$$

In Section 3, we obtain the explicit formula for the g-Drazin inverse of $a+b$ under the perturbation conditions involving power commutativity. We shall derive explicit representation for $(a+b)^d$ under a new condition:

$$ab^d = 0, a^d b = 0, b^\pi a^3 ba^\pi = b^\pi baa^\pi, b^\pi b^3 aa^\pi = b^\pi aba^\pi.$$

Let $p \in \mathcal{A}$ be an idempotent, and let $x \in \mathcal{A}$. Then we write

$$x = pxp + px(1-p) + (1-p)xp + (1-p)x(1-p),$$

and induce a Pierce representation given by the matrix

$$x = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{pmatrix}_p.$$

2. Additive results

The main purpose of this section is to prove the equivalence of the g-Drazin invertibility for $a + b$ and $1 + a^d b$ under certain power communicative condition. We begin with

Lemma 2.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$ and $a^3b = ba, b^3a = ab$. Then*

(1) $aa^d b = baa^d$.
(2) $bb^d a = abb^d$.

Proof. (1) Since $a \in \mathcal{A}^{qnil}$, we have $\| (a - a^2a^d)^n \|^{\frac{1}{n}} \to 0$ $(n \to \infty)$. Let $p = aa^d$. Then

$$\begin{array}{rcl} pb - pbp & = & (a^d)^{3n}a^{3n}b(1 - aa^d) \\ & = & (a^d)^{3n}a^{3(n-1)}ba(1 - aa^d) \\ & \vdots & \\ & = & (a^d)^{3n}ba^n(1 - aa^d) \\ & = & (a^d)^{3n}b(a - a^2a^d)^n. \end{array}$$

This shows that

$$\| pb - pbp \|^{\frac{1}{n}} \leq \| a^d \|^3 \| b \|^{\frac{1}{n}} \| (a - a^2a^d)^n \|^{\frac{1}{n}},$$

and then

$$\| pb - pbp \|^{\frac{1}{n}} \to 0 \ (n \to \infty).$$

This implies that $pb = pbp$. Similarly, we have $bp = pbp$. Accordingly, $aa^db = pb = bp = baa^d$.

(2) This is proved as in (1). □

Lemma 2.2. *Let*

$$x = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}_p \ or \ \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}_p$$

Then

$$x^d = \begin{pmatrix} a^d & 0 \\ z & b^d \end{pmatrix}_p, \ or \ \begin{pmatrix} b^d & z \\ 0 & a^d \end{pmatrix}_p,$$

where

$$z = (b^d)^2\Big(\sum_{i=0}^{\infty}(b^d)^ica^i\Big)a^{\pi} + b^{\pi}\Big(\sum_{i=0}^{\infty}b^ic(a^d)^i\Big)(a^d)^2 - b^dca^d.$$

Proof. See [14, Lemma 1.2]. □

Lemma 2.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^{qnil}$. Suppose $a^3b = ba, b^3a = ab$. Then $a + b \in \mathcal{A}^{qnil}$.*

Proof. By induction, we have

$$ab = b^3a = a^{26}(ab)b^2 = \cdots = a^{26n}(ab)b^{2n},$$

and so

$$\| ab \|^{\frac{1}{n}} \leq \| a^{26n} \|^{\frac{1}{n}} \| ab \|^{\frac{1}{n}} \| b^{2n} \|\|^{\frac{1}{n}} .$$

Hence, $ab = 0$. In view of [18, Lemma 2.10], $a + b \in \mathcal{A}^{qnil}$. □

Lemma 2.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}^d, b \in \mathcal{A}^{qnil}$. If $a^3b = ba, b^3a = ab$, then $a + b \in \mathcal{A}^d$ if and only if $aa^d(a + b)bb^d \in \mathcal{A}^d$.*

Proof. Let $p = aa^d$. Then we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{pmatrix}_p.$$

In view of Lemma 2.1, $aa^db = baa^d$, and so $b_{12} = aa^db(1 - aa^d) = baa^d(1 - aa^d) = 0$. Likewise, we have $b_{21} = 0$. Thus

$$a + b = \begin{pmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{pmatrix}_p.$$

Clearly, $a_2 = (1 - aa^d)a \in \mathcal{A}^{qnil}$. Since $b_2 = (1 - aa^d)b(1 - aa^d) = (1 - aa^d)b$, we have $b_2 \in \mathcal{A}^{qnil}$ by Lemma 2.1 and [18, Lemma 2.11]. One easily checks that $a_2b_2 = b_2^3a_2$ and $b_2a_2 = a_2^3b_2$, it follows by Lemma 2.3 that $a_2 + b_2 \in \mathcal{A}^{qnil}$.

Therefore $a + b \in \mathcal{A}^d$ if and only if $aa^d(a + b)bb^d = a_1 + b_1 \in \mathcal{A}^d$. □

We are now ready to prove the following.

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $a^3b = ba, b^3a = ab$, then the following are equivalent:*

(1) $a + b \in \mathcal{A}^d$.
(2) $aa^d(a + b) \in \mathcal{A}^d$.
(3) $(a + b)bb^d \in \mathcal{A}^d$.
(4) $aa^d(a + b)bb^d \in \mathcal{A}^d$.

Proof. (1) ⇔ (4) Let $p = aa^d$. Then we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{pmatrix}_p.$$

As in the proof of Lemma 2.4, we show that $b_{12} = 0$ and $b_{21} = 0$. Thus

$$a + b = \begin{pmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{pmatrix}_p.$$

We claim that $a_2 + b_2 \in \mathcal{A}^d$. We have $a_2 = a - a^2a^d \in \mathcal{A}^{qnil}$. We will suffice to prove $b_2 = (1 - aa^d)b(1 - aa^d) \in \mathcal{A}^d$. In light of Lemma 2.1, $(aa^d)b = b(aa^d)$, and so $(1 - aa^d)b = b(1 - aa^d)$. Clearly, $1 - aa^d \in \mathcal{A}^d$. Therefore $b_2 = (1 - aa^d)b \in \mathcal{A}^d$ by [18, Theorem 3.1]. Accordingly, $a_2 + b_2 \in \mathcal{A}^d$ by using Lemma 2.4.

Thus, $a + b \in \mathcal{A}^d$ if and only if $a_1 + b_1 \in \mathcal{A}^d$. In view of [18, Theorem 3.1], $(aa^d)b \in \mathcal{A}^d$. By Cline's formula (see [11, Theorem 2.1]), we have $b_1 = aa^dbaa^d \in \mathcal{A}^d$. By using [18, Theorem 3.1] again, $a_1 = aa^da \in \mathcal{A}^d$. Therefore $a_1 + b_1 = aa^d(a + b)aa^d$, as desired.

(2) ⇔ (3) ⇔ (4) These are obvious by Cline's formula (see [11, Theorem 2.1]). □

As an immediate consequence, we now derive

Corollary 2.6. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $a^3b = ba, b^3a = ab$, then the following are equivalent:*

(1) $a + b \in \mathcal{A}^d$.
(2) $a(1 + a^db) \in \mathcal{A}^d$.

Proof. (1) ⇒ (2) In view of [18, Theorem 3.1], $aa^d(a + b) = aa^da(1 + a^db) \in \mathcal{A}^d$. It is easy to check that $(1 - aa^d)a(1 + a^db) = a - a^2a^d \in \mathcal{A}^{qnil}$. In view of Lemma 2.1,

$$aa^da(1 + a^db)(1 - aa^d)a(1 + a^db) = 0,$$

and so

$$a(1 + a^db) = aa^da(1 + a^db) + (1 - aa^d)a(1 + a^db) \in \mathcal{A}^d$$

by [6, Theorem 2.3].

(2) ⇒ (1) Clearly, $aa^d(a + b) = a^2a^d + aa^db = a^2a^d(1 + a^db) = aa^da(1 + a^db)$. By virtue of Lemma 2.1, $aa^da(1+a^db) = a(1+a^db)aa^d$. Thus, it follows by [18, Theorem 3.1] that $aa^da(1+a^db) \in \mathcal{A}^d$. Hence, $aa^d(a+b) \in \mathcal{A}^d$. Therefore we complete the proof by Theorem 2.5. □

We have accumulated all the information necessary to prove the following.

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $a^3b = ba, b^3a = ab$, and $a^dab = a^dba$, then the following are equivalent:*

(1) $a + b \in \mathcal{A}^d$.
(2) $1 + a^db \in \mathcal{A}^d$.

Proof. (1) ⇒ (2) In view of Lemma 2.1, we see that $aa^d(a+b) = a^2a^d + aa^db \in \mathcal{A}^d$. Since $a^dab = a^dba$, we have

$$(a^2a^d)(aa^db) = (aa^db)(a^2a^d).$$

By virtue of [18, Theorem 3.3], $1 + (a^2a^d)^d(aa^db) = 1 + a^db \in \mathcal{A}^d$, as desired.

(2) ⇒ (1) In view of [18, Theorem 3.1], $a^2a^d = a(aa^d) \in \mathcal{A}^d$. By hypothesis and Lemma 2.1, we easily check that

$$a^2a^d(1 + a^db) = a^2a^d + a^dba = a^2a^d + a(a^d)^2ba = (1 + a^db)a^2a^d.$$

Thus, $aa^d(a+b) = a^2a^d(1 + a^db) \in \mathcal{A}^d$ by [18, Theorem 3.1]. Therefore we complete the proof by Theorem 2.5. □

Corollary 2.8. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $a^3b = ba, b^3a = ab$, and $a^dab = a^dba$ then the following are equivalent:*

(1) $a - b \in \mathcal{A}^d$.
(2) $1 - a^db \in \mathcal{A}^d$.

Proof. In view of [2, Theorem 2.2], $-b \in \mathcal{A}^d$. Applying Theorem 2.7 to a and $-b$, we complete the proof. □

3. Perturbations

The aim of this section is to provide conditions on a and b in $\mathcal{A}^d$ with multiplicative perturbations so that the sum $a + b$ will have g-Drazin inverse. For further use, we now derive the following.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $ab^d = 0, a^db = 0, a^3b = ba, b^3a = ab$, then $a + b \in \mathcal{A}^d$ and $(a+b)^d = a^d + b^d$.*

Proof. In view of Lemma 2.1, we easily check that $ba^d = ba(a^d)^2 = aa^dba^d = 0$ and $b^da = (b^d)^2ba = (b^d)ab^db = 0$. Thus,

$$(a+b)(a^d + b^d) = aa^d + ba^d + bb^d = a^da + b^da + b^db = (a^d + b^d)(a+b).$$

Also we have

$$(a^d + b^d)(a+b)(a^d + b^d) = a^d + b^d.$$

Moreover, we have $(a+b) - (a+b)^2(a^d + b^d) = x + y$, where $x = a - a^2a^d, y = b - b^2d^d \in \mathcal{A}^{qnil}$. We easily check that $x^3y = yx$ and $y^3x = xy$. According to Lemma 2.3, $x + y \in \mathcal{A}^{qnil}$. Therefore $(a+b)^d = a^d + b^d$, as asserted. □

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $ab^d = 0, a^db = 0, a^3ba^\pi = baa^\pi, b^3aa^\pi = aba^\pi$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = b^\pi a^d + b^da^\pi + b^\pi a^\pi \sum_{i=0}^{\infty}(a+b)^ib(a^d)^{i+2}.$$

Proof. Let $p = aa^d$. Then we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_{11} & b_{12} \\ b_1 & b_2 \end{pmatrix}_p.$$

Since $a^db = 0$, we see that $b_{11} = b_{12} = 0$. Hence we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}_p.$$

Moreover, $a_2 = (1-p)a(1-p) = a - a^2a^d \in \mathcal{A}^{qnil}$. Since $b \in \mathcal{A}^d$ and $a^db = 0$, we have $a^\pi b = b \in \mathcal{A}^d$. In light of Cline's formula, $b_2 = a^\pi ba^\pi \in \mathcal{A}^d$, and so $b_2 \in ((1-p)\mathcal{A}(1-p))^d$. One easily checks that

$$a_2b_2^d = 0, a_2^db_2 = 0, a_2^3b_2 = b_2a_2, b_2^3a_2 = a_2b_2.$$

In view of Lemma 3.1, $(a_2+b_2)^d = a_2^d + b_2^d = b^da^\pi$. In light of Lemma 2.2, we have

$$(a+b)^d = \begin{pmatrix} a_1^d & 0 \\ z & (a_2+b_2)^d \end{pmatrix} = \begin{pmatrix} a^d & 0 \\ z & b^da^\pi \end{pmatrix},$$

where

$$z = b^\pi a^\pi \sum_{i=0}^{\infty} (a_2+b_2)^i b_1 (a^d)^{i+2} - b^db_1a^d.$$

Moreover, we have

$$\begin{aligned} & \begin{pmatrix} 0 & 0 \\ (a_2+b_2)^i b_1 (a^d)^{i+2} & 0 \end{pmatrix} \\ = & \begin{pmatrix} 0 & 0 \\ (a_2+b_2)^i b_1 & (a_2+b_2)^i b_2 \end{pmatrix} \begin{pmatrix} (a^d)^{i+2} & 0 \\ 0 & 0 \end{pmatrix} \\ = & (a+b)^i b (a^d)^{i+2}, \end{aligned}$$

and

$$\begin{pmatrix} 0 & 0 \\ b^db_1a^d & 0 \end{pmatrix} = bb^da^d.$$

Therefore

$$(a+b)^d = b^\pi a^d + b^da^\pi + b^\pi a^\pi \sum_{i=0}^{\infty} (a+b)^i b (a^d)^{i+2},$$

as asserted. □

Corollary 3.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If $ab^d = 0, a^db = 0, b^\pi a^3ba^\pi = b^\pi baa^\pi, b^\pi b^3aa^\pi = b^\pi aba^\pi$, then $a+b \in \mathcal{A}^d$ and*

$$\begin{aligned} (a+b)^d = \; & b^da^\pi + b^\pi a^d + b^\pi a^\pi \sum_{i=0}^{\infty} (a+b)^i b (a^d)^{i+2} \\ & + \sum_{i=0}^{\infty} \Big[(b^d)^{i+2} a (a+b)^i - (b^d)^{i+2} a (a+b)^{i+1} a^d \\ & - \sum_{j=0}^{\infty} (b^d)^{i+2} a (a+b)^{i+j+1} b (a^d)^{j+2} \Big] \\ & - \sum_{i=0}^{\infty} b^d a (a+b)^i b (a^d)^{i+2}. \end{aligned}$$

Proof. Let $p = bb^d$. Then we have

$$a = \begin{pmatrix} a_{11} & a_1 \\ a_{21} & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p.$$

Clearly, $a_{11} = a_{21} = 0$, and so

$$a = \begin{pmatrix} 0 & a_1 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p.$$

Moreover, $b_2 = (1-p)b(1-p) = b - b^2b^d \in \mathcal{A}^{qnil}$. Since $a \in \mathcal{A}^d$ and $ab^d = 0$, we see that $ab^\pi = a \in \mathcal{A}^d$. By using Cline's formula, $a_2 = b^\pi ab^\pi \in \mathcal{A}^d$, and so $a_2 \in ((1-p)\mathcal{A}(1-p))^d$. It is easy to verify that

$$a_2^db_2 = 0, a_2b_2^d = 0, a_2^3b_2a_2^\pi = b_2a_2a_2^\pi, b_2^3a_2a_2^\pi = a_2b_2a_2^\pi.$$

Since $b_2^d = 0$, it follows by Theorem 3.2 that

$$(a_2+b_2)^d = a_2^d + a_2^\pi \sum_{i=0}^{\infty} (a_2+b_2)^i b_2 (a_2^d)^{i+2}.$$

Therefore

$$(a+b)^d = \begin{pmatrix} b^d & z \\ 0 & (a_2+b_2)^d \end{pmatrix},$$

where

$$z = \sum_{i=0}^{\infty} (b^d)^{i+2} a_1 (a_2+b_2)^i (a_2+b_2)^\pi - b^d a_1 (a_2+b_2)^d.$$

Therefore we have

$$\begin{aligned} z &= \sum_{i=0}^{\infty} (b^d)^{i+2} a_1 (a_2+b_2)^i \\ &\quad \Big(1 - (a_2+b_2)(a_2^d + a_2^\pi \sum_{j=0}^{\infty} (a_2+b_2)^j b_2 (a_2^d)^{j+2})\Big) \\ &- b^d a_1 (a_2+b_2)^d. \end{aligned}$$

We see that

$$\begin{pmatrix} 0 & (b^d)^{i+2} a_1 (a_2+b_2)^i \\ 0 & 0 \end{pmatrix}_p = (b^d)^{i+2} a (a+b)^i;$$

$$\begin{aligned} &\begin{pmatrix} 0 & (b^d)^{i+2} a_1 (a_2+b_2)^{i+1} a_2^d \\ 0 & 0 \end{pmatrix}_p \\ = &\begin{pmatrix} 0 & (b^d)^{i+2} a_1 (a_2+b_2)^{i+1} \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} 0 & bb^d a^d \\ 0 & a_2^d \end{pmatrix}_p \\ = &(b^d)^{i+2} a (a+b)^{i+1} a^d; \end{aligned}$$

$$\begin{aligned} &\begin{pmatrix} 0 & (b^d)^{i+2} a_1 (a_2+b_2)^{i+1} a_2^\pi (a_2+b_2)^j b_2 (a_2^d)^{j+2} \\ 0 & 0 \end{pmatrix}_p \\ = &\begin{pmatrix} 0 & (b^d)^{i+2} a_1 (a_2+b_2)^{i+1} a_2^\pi (a_2+b_2)^j b_2 \\ 0 & 0 \end{pmatrix}_p \begin{pmatrix} 0 & bb^d (a^d)^{j+2} \\ 0 & (a_2^d)^{j+2} \end{pmatrix}_p \\ = &(b^d)^{i+2} a (a+b)^{i+1} a^\pi (a+b)^j b (a^d)^{j+2} \\ = &(b^d)^{i+2} a (a+b)^{i+j+1} b (a^d)^{j+2}. \end{aligned}$$

Since $b^d a_1 (a_2+b_2)^d = b^d a_1 a_2^d + b^d a_1 a_2^\pi \sum\limits_{j=0}^{\infty} (a_2+b_2)^j b_2 (a_2^d)^{j+2}$, we have

$$\begin{pmatrix} 0 & b^d a_1 (a_2+b_2)^d \\ 0 & 0 \end{pmatrix}_p = b^d a a^d + \sum_{i=0}^{\infty} b^d a (a+b)^i b (a^d)^{i+2}.$$

Therefore we complete the proof. □

Theorem 3.4. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$*, a and d have g-Drazin inverses. If* $ab = 0, cb = 0, bd^2 = 0$ *and* $d^3 c a^\pi = 0$*, then* $M \in M_2(\mathcal{A})^d$ *and*

$$M^d = Q^\pi P^d + Q^d P^\pi + Q^\pi P^\pi \sum_{i=0}^{\infty} M^i Q (P^d)^{i+2}.$$

where

$$P=\begin{pmatrix} a & 0\\ c & 0\end{pmatrix}, Q=\begin{pmatrix} 0 & b\\ 0 & d\end{pmatrix};$$
$$P^d=\begin{pmatrix} a^d & 0\\ c(a^d)^2 & 0\end{pmatrix}, Q^d=\begin{pmatrix} 0 & 0\\ 0 & d^d\end{pmatrix}.$$

Proof. Let $M=P+Q$, where $P=\begin{pmatrix} a & 0\\ c & 0\end{pmatrix}$, and $Q=\begin{pmatrix} 0 & b\\ 0 & d\end{pmatrix}$. Then P,Q have g-Drazin inverses. Moreover, we have

$$P^d=\begin{pmatrix} a^d & 0\\ c(a^d)^2 & 0\end{pmatrix}, Q^d=\begin{pmatrix} 0 & b(d^d)^2\\ 0 & d^d\end{pmatrix}.$$

Since $a^db=0$ and $bd^d=0$, we see that $P^dQ=0$ and $PQ^d=0$. Moreover, we have

$$P^\pi=\begin{pmatrix} a^\pi & 0\\ -ca^d & 1\end{pmatrix}, Q^\pi=\begin{pmatrix} 1 & 0\\ 0 & d^\pi\end{pmatrix}.$$

By hypothesis, we directly check

$$P^3QP^\pi=QPP^\pi, Q^3PP^\pi=PQP^\pi.$$

In light of Theorem 3.2, we complete the proof. □

Corollary 3.5. *Let $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathcal{A})$, a and d have g-Drazin inverses. If $bc=0, dc=0, ca^2=0$ and $a^3bd^\pi=0$, then $M\in M_2(\mathcal{A})^d$ and*

$$M^d=P^\pi Q^d+P^dQ^\pi+P^\pi Q^\pi\sum_{i=0}^{\infty}M^iP(Q^d)^{i+2},$$

where

$$P=\begin{pmatrix} a & 0\\ c & 0\end{pmatrix}, Q=\begin{pmatrix} 0 & b\\ 0 & d\end{pmatrix};$$
$$P^d=\begin{pmatrix} a^d & 0\\ 0 & 0\end{pmatrix}, Q^d=\begin{pmatrix} 0 & b(d^d)^2\\ 0 & d^d\end{pmatrix}.$$

Proof. It is easy to verify that

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} d & c\\ b & a\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

Applying Theorem 3.4 to the matrix $\begin{pmatrix} d & c\\ b & a\end{pmatrix}$, we complete the proof. □

We note that the Drazin and g-Drazin inverse are the same for a complex matrix, and so we have

Example 3.6. *Let $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in M_4(\mathbb{C})$, where*

$$A=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}, B=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, C=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, D=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}.$$

Then

$$AB=0, CB=0, BD^2=0 \text{ and } D^3CA^\pi=0$$

and

$$M^D=\begin{pmatrix} A & 0\\ -C & D\end{pmatrix}.$$

Proof. Clearly, $A^D = A, D^D = D, A^\pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, D^\pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We easily check that

$$AB = 0, CB = 0, BD^2 = 0 \text{ and } D^3CA^\pi = 0.$$

Then M has g-Drazin inverse by Theorem 3.4. In this case,

$$M^D = Q^\pi P^D + Q^D P^\pi + Q^\pi P^\pi \sum_{i=0}^{\infty} M^i Q(P^D)^{i+2},$$

where

$$P = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix};$$
$$P^d = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, Q^D = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

By computing, we deduce thet

$$M^i Q(P^D)^{i+2} = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix},$$

and so $Q^\pi P^\pi M^i Q(P^D)^{i+2} = 0$ for all $i \geq 0$. Therefore

$$\begin{aligned} M^D &= Q^\pi P^D + Q^D P^\pi \\ &= \begin{pmatrix} I_2 & 0 \\ 0 & I_2 - D \end{pmatrix}\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I_2 - A & 0 \\ -C & I_2 \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ -C & D \end{pmatrix}, \end{aligned}$$

as desired. □

References

[1] N. Castro-González, Additive perturbation results for the Drazin inverse, *Linear Algebra Appl.*, **397**(2005), 279–297.
[2] H. Chen and M. Sheibani, Generalized Drazin inverses in a ring, *Filomat*, **32**(2018), 5289–5295.
[3] D.S. Cvetković-Ilić, The generalized Drazin inverse with commutativity up to a factor in a Banach algebra, *Linear Algebra Appl.*, **431**(2009), 783–791.
[4] D.S. Cvetković-Ilić; X. Liu and Y. Wei, Some additive results for the generalized Drazin inverse in a Banach algebra, *Electronic J. Linear Algebra*, **22**(2011), 1049–1058.
[5] C. Deng, D. S. Cvetcović-Ilić and Y. Wei, Some results on the genrealized Derazin inverse of operator matrices, *Linear and Multilinear Algebra* **58**(2010), 503-521.
[6] D.S. Djordjevic and Y. Wei, Additive results for the generalized Drazin inverse, *J. Austral. Math. Soc.*, **73**(2002), 115–125.
[7] L. Guo; H. Zou and W. Du, Representations for Drazin inverses of sum of two matrices, *J. Jilin Univ.*, **54**(2016), 1033–1035.
[8] L. Guo; J. Chen and H. Zou, Some additive results for the generalized Drazin inverse in a Banach algebra, *J. Southeast Univ. (English Edition)*, **33**(2017), 382–386.
[9] R. Hartwig; G. Wang and Y. Wei, Some additive results on Drazin inverse, *Linear Algebra Appl.*, **322**(2001), 207–217.
[10] J.J. Koliha, A generalized Drazin Inverse, *Glasgow Math. J.*, **38**(1996), 367–381.
[11] Y. Liao, J. Chen and J. Cui, Cline's formula for the generalized Drazin inverse, *Bull. Malays. Math. Sci. Soc.*, **37**(2014), 37–42.
[12] X. Liu; X. Qin and J. Benítez, New additive results for the generalized Drazin inverse in a Banach Algebra, *Filomat*, **30**(2016), 2289-2294.
[13] X. Liu; L. Xu and Y. Yu, The representations of the Drazin inverse of differences of two matrices, *Applied Math. Comput.*, **216**(2010), 3652–3661.
[14] J. Ljubisavljevic and D.S. Cvetkovic-Ilic, Additive results for the Drazin inverse of block matrices and applications, *J. Comput. Appl. Math.* **235**(2011), 3683-3690.
[15] D. Mosić; H. Zou and J. Chen, The generalized Drazin inverse of the sum in a Banach algebra, *Ann. Funct. Anal.*, **8**(2017), 90–105.
[16] L. Wang; Xia Zhu and J. Chen, Additive property of Drazin invertibility of elements in a ring, *Filomat*, **30**(2016), 1185–1193.
[17] H. Zhu and J. Chen, Additive and product properties of Drazin inverses of elements in a ring, *Bull. Malays. Math. Sci. Soc.*, **40**(2017), 259–278.
[18] H. Zou; D. Mosić and J. Chen, Generalized Drazin invertiblity of the product and sum of two elements in a Banach algebra and its applications, *Turkish. J. Math.*, **41**(2017), 548–563.