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Linear Functionals on Hypervector Spaces

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Abstract. The study of linear functionals, as an important special case of linear transformations, is one of the key topics in linear algebra and plays a significant role in analysis. In this paper we generalize the crucial results from the classical theory and study main properties of linear functionals on hypervector spaces. In this way, we obtain the dual basis of a given basis for a finite-dimensional hypervector space. Moreover, we investigate the relation between linear functionals and subhyperspaces and conclude the dimension of the vector space of all linear functionals over a hypervector space, the dimension of sum of two subhyperspaces and the dimension of the annihilator of a subhyperspace, under special conditions. Also, we show that every superhyperspace is the kernel of a linear functional. Finally, we check out whether every basis for the vector space of all linear functionals over a hypervector space V is the dual of some basis for V.

1. Introduction

In algebra the composition of two elements under an operation is an element, whereas the composition of two elements by a hyperoperation, as a generalization of operation, is a non-empty set and an algebraic structure endowed with at least one hyperoperation is known as an algebraic hyperstructure. The theory of algebraic hyperstructures was born in 1934, when Marty [11] introduced the notion of hypergroups. Afterwards this theory has been studied in various branches of mathematics such as fields, lattices, rings, quasigroups, semigroups, modules, ordered structures, combinatorics, topology, geometry, graphs, codes, etc. The reader can find the most results of algebraic hyperstructures in the books of Corsini [3] and [4], Davvaz [5–8] and Vougiuklis [19].

In 1990, M. Scafati-Tallini [13] introduced the notion of hypervector spaces and studied basic properties of them, such as norms in such spaces ([14]), geometric point of view ([15]) and characterization of remarkable hypervector spaces ([16]). Hypervector spaces have developed by some other mathematicians: Ameri [1], the author [9], [10] and Sedghi [17] from an algebraic perspective, as well as Raja [12] and Taghavi [18] from an analytic perspective.

In previous mentioned papers about hypervector spaces, the important notions of subhyperspaces, basis, dimension, linear transformations have been studied. In this paper we generalize some main properties of linear algebra into hypervector spaces. In this regards, we study the dual basis of a finite-dimensional hypervector space and the relation between linear functionals and subhyperspaces. Also, we obtain some important results about the dimensions of special hypervector spaces and conclude $\dim V^* = \dim V$, $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$ and $\dim W + \dim W^\circ = \dim V$, where V^* is the vector

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space of all linear functionals over a hypervector space V, W, W_1, W_2 are subhyperspaces of V and W° is the annihilator of W. Also, we define the notion of a superhyperspace of V and show that every superhyperspace of V is the kernel of a linear functional over V. Moreover, we investigate whether every basis for V^* is the dual of some basis for V.

2. Preliminaries

In the following we present some definitions and simple properties of hypervector spaces that we shall use in later.

Definition 2.1. *[13] Let K be a field, $(V, +)$ be an Abelian group and $P^*(V)$ be the set of all non-empty subsets of V. We define a hypervector space over K to be the quadruplet $(V, +, \circ, K)$, where " $\circ$ " is an external hyperoperation*

$$\circ : K \times V \longrightarrow P^*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

(H_1) $a \circ (x + y) \subseteq a \circ x + a \circ y$, *right distributive law,*
(H_2) $(a + b) \circ x \subseteq a \circ x + b \circ x$, *left distributive law,*
(H_3) $a \circ (b \circ x) = (ab) \circ x$,
(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
(H_5) $x \in 1 \circ x$,

where in (H_1), $a \circ x + a \circ y = \{p + q : p \in a \circ x, q \in a \circ y\}$. Similarly it is in ($H_2$). Also in ($H_3$), $a \circ (b \circ x) = \bigcup_{t \in b \circ x} a \circ t$.

V is called strongly right distributive, if we have equality in (H_1). In a similar way we define the strongly left distributive hypervector spaces. V is called strongly distributive, if it is strongly right and left distributive.

A non-empty subset W of V is called a subhyperspace of V, denoted by $W \leqslant V$, if W is itself a hypervector space with the external hyperoperation on V, i.e. for all $a \in K$ and $x, y \in W$, $x - y \in W$ and $a \circ x \subseteq W$.

Example 2.2. *[1] In $(\mathbb{R}^2, +)$ define the external hyperoperations $\circ_1, \circ_2 : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P^*(\mathbb{R}^2)$ by $a \circ_1 (x, y) = ax \times \mathbb{R}$ and $a \circ_2 (x, y) = \mathbb{R} \times ay$. Then $V_1 = (\mathbb{R}^2, +, \circ_1, \mathbb{R})$ and $V_2 = (\mathbb{R}^2, +, \circ_2, \mathbb{R})$ are hypervector spaces.*

In the sequel of this paper, V denotes a hypervector space over the field K, unless otherwise is specified. Also, the zero of V is denoted by $\underline{0}$.

Definition 2.3. *[1] A subset S of V is called linearly independent if for every vectors $v_1, \ldots, v_n$ in S, and $c_1, \ldots, c_n \in K$, $\underline{0} \in c_1 \circ v_1 + \cdots + c_n \circ v_n$, implies that $c_1 = \cdots = c_n = 0$. S is called linearly dependent if it is not linearly independent. A basis for V is a linearly independent subset of V such that spans V, i.e. $V = \langle S \rangle$, where*

$$\begin{aligned} \langle S \rangle &= \left\{ t \in V : t \in \sum_{i=1}^{n} a_i \circ s_i , a_i \in K , s_i \in S , n \in \mathbb{N} \right\} \\ &= \{ t_1 + t_2 + \cdots + t_n : t_i \in a_i \circ s_i , a_i \in K , s_i \in S , n \in \mathbb{N} \} . \end{aligned}$$

We say that V is finite-dimensional if it has a finite basis. If V is strongly left distributive, invertible (V is said to be invertible if and only if $u \in a \circ v$ implies that $v \in a^{-1} \circ u$) and finite-dimensional, then every two basis of V have the same cardinality. In this case the cardinality of any basis of V is called the dimension of V and denoted by $\dim V$.

Lemma 2.4. *[1] Let V be strongly left distributive. Then*

1. *if $\beta = \{x_1, \ldots, x_n\}$ is a basis for V, then every element of V belongs to a unique linear combination in the form $a_1 \circ x_1 + \cdots + a_n \circ x_n$, with $a_i \in K$,*
2. *if V is finite-dimensional and invertible and if U is a subhyperspace of V, then U is finite-dimensional and* $\dim U \leq \dim V$.

3. *if V is finite-dimensional and invertible, then every linearly independent subset of V is contained in a finite basis.*

Proposition 2.5. *[9] Let W_1 and W_2 be strongly left distributive and invertible subhyperspaces of V such that $W_1 \subseteq W_2$ and $\dim W_1 = \dim W_2$. Then $W_1 = W_2$.*

Definition 2.6. *[1] Let V and W be hypervector spaces over the field K. A mapping $T : V \longrightarrow W$ is called*

1. *linear transformation iff $T(x + y) = T(x) + T(y)$ and $T(a \circ x) \subseteq a \circ T(x)$,*
2. *good transformation iff $T(x + y) = T(x) + T(y)$ and $T(a \circ x) = a \circ T(x)$.*

The kernel of a linear transformation $T : V \longrightarrow W$ id defined by

$$\ker T = \{x \in V : T(x) \in 0 \circ 0_W\}.$$

$L(V, W)$ denotes the set of all good transformations from V into W.

Proposition 2.7. *[1] Let $T : V \longrightarrow W$ be a linear transformation. Then*

1. *if T is a good transformation and $U \leqslant V$, then $T(U) \leqslant W$,*
2. *if V and W are strongly left distributive, then $\ker T \leqslant V$.*

Theorem 2.8. *[9] Let V and W be strongly left distributive, invertible and finite-dimensional hypervector spaces. If $T : V \longrightarrow W$ is a linear transformation, then*

$$\dim \ker T + \dim T(V) = \dim V.$$

3. Dual Basis

In this section we introduce a basis β^* for the vector space V^* of all linear functionals over a hypervector space V, which is obtained from a basis β for V and is called the dual basis of β. Moreover, we conclude the coordinates of a linear functional based on coordinates of vectors of V relative to β.

Raja [12] introduced the hypervector space $L(V, W)$ of all good transformations from V into W over the field $\mathbb{R}$. In the following, it is generalized to a hypervector space over an arbitrary field K.

Let V and W be hypervector spaces over the field K. For every $T, S \in L(V, W)$, $a \in K$ and $x \in V$ suppose that:

1. $(T + S)(x) = T(x) + S(x)$,
2. $a \odot T = \{\acute{T} \in L(V, W) : \ \acute{T}(x) \in T(a \circ x), \ \forall x \in V\}$.

It is easy to verify that $(L(V, W), +, \odot, K)$ is a hypervector space. If V and W are strongly left distributive, then $L(V, W)$ is strongly left distributive.

Definition 3.1. *[16] Let $(V, +, \circ, K)$ be a hypervector space over the field K. Then a linear transformation $T : V \to K$ is called a linear functional on V, i.e. T is a function from V into K, where K is considered as the classical vector space over itself, such that for all $a \in K$ and $x, y \in V$ the followings hold:*

1. *$T(x + y) = T(x) + T(y)$,*
2. *$T(a \circ x) = a \cdot T(x)$,*

where the condition (2) means that:

$$\forall t \in a \circ x; \ T(t) = a \cdot T(x).$$

The set V^* of all linear functionals over V is a classical vector space with the external operation $\cdot : K \times V^* \to V^*$ defined by $(a \cdot T)(x) = a \cdot T(x)$.

Example 3.2. *Consider the hypervector spaces $V_1 = (\mathbb{R}^2, +, \circ_1, \mathbb{R})$ and $V_2 = (\mathbb{R}^2, +, \circ_2, \mathbb{R})$ in Example 2.2. Then the functions $T_1, T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $T_1(x, y) = x$ and $T_2(x, y) = y$ are linear functionals on V_1 and V_2, respectively.*

Recall that every vector space $(V, +, \cdot, K)$ is a trivial induced hypervector space by the external hyperoperation $\circ : K \times V \longrightarrow P^*(V)$ with $a \circ x = \{ax\}$. In this case every linear functional on $(V, +, \cdot, K)$ is a linear functional on $(V, +, \circ, K)$.

Proposition 3.3. *Let V and W be hypervector spaces over the field K, with basis $\beta = \{x_1, \ldots, x_n\}$ and $\acute{\beta} = \{y_1, \ldots, y_m\}$, respectively. If W is strongly left distributive, $a \circ \underline{0}_W = \{\underline{0}_W\}$, for all $a \in K$ and $0 \circ y = \{\underline{0}_W\}$, for all $y \in W$, then* $\dim L(V, W) = \dim V \times \dim W$.

Proof. It is similar to the proof of Corollary 5.20 of [9]. □

Lemma 3.4. *If $K = (K, +, \cdot)$ is a field, then $\dim(K, +, \circ, K) = 1$, where $\circ : K \times K \longrightarrow P^*(K)$ is defined by $a \circ b = \{ab\}$.*

Proof. It is clear that $\{1\}$ is a basis for K. □

Theorem 3.5. *If V is a finite-dimensional hypervector space over the field K, then*

$$\dim V^* = \dim V.$$

Proof. By Proposition 3.3 and Lemma 3.4 it follows that:

$$\begin{aligned} \dim V^* &= \dim L(V, K) \\ &= \dim V \times \dim K \\ &= \dim V. \end{aligned}$$

□

Theorem 3.6. *Let $\beta = \{x_1, \ldots, x_n\}$ be a basis for strongly left distributive hypervector space V. For each $i = 1, \ldots, n$, define*

$$\left[\begin{array}{l} T_i : V \to K \\ T_i(x) = a_i, \end{array} \right.$$

where $a_1, \ldots, a_n \in K$, such that $x \in a_1 \circ x_1 + \cdots + a_n \circ x_n$. Then the set $\beta^* = \{T_1, \ldots, T_n\}$ of distinct linear functionals on V is a basis for V^*, which is called the dual basis of β.

Proof. By Lemma 2.4(1), T_i's are well-defined. Now if $x, y \in V$ such that $x \in a_1 \circ x_1 + \cdots + a_n \circ x_n$ and $y \in b_1 \circ x_1 + \cdots + b_n \circ x_n$ for unique scalars $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$, then

$$\begin{aligned} x + y &\in a_1 \circ x_1 + \cdots + a_n \circ x_n + b_1 \circ x_1 + \cdots + b_n \circ x_n \\ &= (a_1 + b_1) \circ x_1 + \cdots + (a_n + b_n) \circ x_n, \end{aligned}$$

thus $T_i(x + y) = a_i + b_i = T_i(x) + T_i(y)$, for all $1 \leq i \leq n$.

Also for all $a \in K$, $x \in V$, $t \in a \circ x$ and $x \in a_1 \circ x_1 + \cdots + a_n \circ x_n$, it follows that $t \in a \circ (a_1 \circ x_1 + \cdots + a_n \circ x_n) = (aa_1) \circ x_1 + \cdots + (aa_n) \circ x_n$. Thus $T_i(a \circ x) = aa_i = a.T_i(x)$. Hence T_i's are linear functionals on V.
It is clear that T_i's are distinct.
Now suppose $a_1 T_1 + \cdots + a_n T_n = 0_{V^*}$, for some $a_1, \ldots, a_n \in K$. Then

$$\begin{aligned} 0 &= 0_{V^*}(x_j) \\ &= (a_1 T_1 + \cdots + a_n T_n)(x_j) \\ &= a_1 T_1(x_j) + \cdots + a_j T_j(x_j) + \cdots + a_n T_n(x_j) \\ &= a_1 \times 0 + \cdots + a_j \times 1 + \cdots + a_n \times 0 \\ &= a_j, \end{aligned}$$

and so $a_j = 0$ for all $j = 1, \dots, n$. Consequently, β^* is linearly independent.
Finally, we prove that β^* generates V^*. Let $T \in V^*$ and $x \in V$ such that $x \in a_1 \circ x_1 + \dots + a_n \circ x_n$, for some $a_1, \dots, a_n \in K$. Then

$$\begin{aligned}(T(x_1)T_1 + \dots + T(x_n)T_n)(x) &= T(x_1)T_1(x) + \dots + T(x_n)T_n(x)\\ &= T(x_1)a_1 + \dots + T(x_n)a_n\\ &= a_1T(x_1) + \dots + a_nT(x_n)\\ &= T(a_1 \circ x_1 + \dots + a_n \circ x_n)\\ &= T(x).\end{aligned}$$

Hence $T = T(x_1)T_1 + \dots + T(x_n)T_n$. □

Theorem 3.7. *Let V be a finite-dimensional strongly left distributive hypervector space and let $\beta = \{x_1, \dots, x_n\}$ be a basis for V. If $\beta^* = \{T_1, \dots, T_n\}$ is the dual basis of β, then*

$$\forall T \in V^*; \;\; T = \sum_{i=1}^{n} T(x_i)T_i,$$

and

$$\forall x \in V; \;\; x \in \sum_{i=1}^{n} T_i(x) \circ x_i.$$

Proof. The first equality has been shown in the proof of Theorem 3.6. Similarly, if $x \in V$, then $x \in a_1 \circ x_1 + \dots + a_n \circ x_n$, for some $a_1, \dots, a_n \in K$. Thus for all $j = 1, \dots, n$, it follows that:

$$\begin{aligned}T_j(x) &= T_j(a_1 \circ x_1 + \dots + a_n \circ x_n)\\ &= a_1T_j(x_1) + \dots + a_nT_j(x_n)\\ &= a_j.\end{aligned}$$

Hence x belongs to the unique linear combination of $x_1, \dots, x_n$ as the form $T_1(x) \circ x_1 + \dots + T_n(x) \circ x_n$. □

The following Corollaries are direct results of Theorem 3.7 and it's proof.

Corollary 3.8. *If $\beta = \{x_1, \dots, x_n\}$ is an ordered basis for a strongly left distributive hypervector space V, and $\beta^* = \{T_1, \dots, T_n\}$ is the dual basis of β, then T_i is precisely the function which assigns to each vector x in V, the ith coordinate of x relative to the ordered basis β.*

Corollary 3.9. *If $\beta = \{x_1, \dots, x_n\}$ is an ordered basis for a strongly left distributive hypervector space V, then every linear functional T on V has the form*

$$T(x) = a_1T(x_1) + \dots + a_nT(x_n),$$

where $(a_1, \dots, a_n)$ is the coordinates of x relative to β.

For more study on ordered basis and coordinates refer to [9].

4. Relation between Linear Functionals and Subhyperspaces

In this section we investigate the relation between linear functionals and special subhyperspaces are called superhyperspaces. We see that the kernel of a non-zero linear functional on a finite-dimensional strongly left distributive and invertible hypervector space is a superhyperspace and every superhyperspace is the kernel of a linear functional. Moreover, some important theorems about the dimension of special hypervector spaces are obtained. Let us start by definition of a superhyperspace.

Definition 4.1. *Let V be an n-dimensional hypervector space over the field K. Then any subhyperspace of dimension $n-1$ is called a superhyperspace. Such subspaces are sometimes called superhyperplanes or subhyperspaces of codimension 1.*

Example 4.2. *In $(\mathbb{R}^3,+)$ define the external hyperoperation $\circ : \mathbb{R}\times\mathbb{R}^3 \to P^*(\mathbb{R}^3)$ by $a\circ(x,y,z) = \{(r,ay,az); r\in\mathbb{R}\}$. Then $V = (\mathbb{R}^3,+,\circ,\mathbb{R})$ is a hypervector space and the xy-plane and xz-plane are superhyperspaces of V. More precisely $\{(0,1,0),(0,0,1)\}$ is a basis for V and so $\dim V = 2$, also $\{(0,1,0)\}$ and $\{(0,0,1)\}$ are basis for xy-plane and xz-plane, respectively, and so both of them are one-dimensional.*

Proposition 4.3. *Let V be a finite-dimensional strongly left distributive and invertible hypervector space over the field K and $T : V \to K$ be a non-zero linear functional on V. Then $\ker T$ is a superhyperspace in V.*

Proof. By Proposition 2.7(1), $Im\ T$ is a non-zero subhyperspace of the scalar field K and so $0 \neq \dim Im\ T \leq \dim K = 1$, by Lemma 2.4(2), ($K$ is finite-dimensional, invertible and strongly left distributive) and Lemma 3.4. Thus $\dim Im\ T = 1$. Hence by Theorem 2.8, $\dim\ker T = (\dim V) - 1$. Therefore $\ker T$ is a superhyperspace. □

Question: Is every superhyperspace the kernel of a linear functional? The answer (yes) will be seen in Corollary 4.12.

Definition 4.4. *Let V be a hypervector space over the field K and $S \subseteq V$. Then the annihilator of S is defined by*

$$S^\circ = \{T \in V^* : T(x) = 0,\ \forall x \in S\}.$$

Example 4.5. *Consider the hypervector space $V_1 = (\mathbb{R}^2,+,\circ_1,\mathbb{R})$ in Example 2.2. If S is a subset of $X = \{0\}\times\mathbb{R}$, then $S^\circ = V^*$ and if S consists of any element of $\mathbb{R}^2 \setminus X$, then $S^\circ = \{0_{V^*}\}$.*

Example 4.6. *Consider the hypervector space $V = (\mathbb{R}^3,+,\circ,\mathbb{R})$ in Example 4.2 with the basis $\{(0,1,0),(0,0,1)\}$, in fact $(x,y,z) \in y\circ(0,1,0) + z\circ(0,0,1)$ for all $(x,y,z)\in\mathbb{R}^3$. Then*

$$\begin{aligned}\{(x_0,y_0,z_0)\}^\circ &= \{T:\mathbb{R}^3\to\mathbb{R};\ T(x_0,y_0,z_0)=0\}\\ &= \{T:\mathbb{R}^3\to\mathbb{R};\ y_0T(0,1,0)+z_0T(0,0,1)=0\}\\ &= \{T:\mathbb{R}^3\to\mathbb{R};\ T(0,0,1) = -\frac{y_0}{z_0}T(0,1,0)\}\\ &= \{T:\mathbb{R}^3\to\mathbb{R};\ T(x,y,z) = (y-\frac{y_0}{z_0}z)T(0,1,0)\},\end{aligned}$$

for all $(x_0,y_0,z_0)\in\mathbb{R}^3$ such that $z_0\neq 0$. Also if W is the xy-plane, then $W^\circ = \langle T\rangle$, where $T\in V^*$ is defined by $T(x,y,z) = z$.

Proposition 4.7. *If V is a hypervector space over the field K and $S\subseteq V$, then $S^\circ \leqslant V^*$.*

Proof. Let $a\in K$ and $T_1,T_2\in S^\circ$. Then $(T_1-T_2)(x) = T_1(x)-T_2(x) = 0$, for all $x\in S$, thus $T_1-T_2\in S^\circ$. Also if $T\in S^\circ$ and $\acute{T}\in a\odot T$, then

$$\begin{aligned}&\forall x\in V,\ \acute{T}(x)\in T(a\circ x) = a\cdot_K T(x),\\ \Rightarrow\ &\forall x\in S,\ \acute{T}(x)\in\{a\cdot_K 0_K\} = \{0_k\},\\ \Rightarrow\ &\forall x\in S,\ \acute{T}(x) = 0_k,\\ \Rightarrow\ &\acute{T}\in S^\circ,\end{aligned}$$

hence $a\odot T\subseteq S^\circ$. Therefore $S^\circ\leqslant V^*$. □

Theorem 4.8. *Let V be a strongly left distributive and invertible hypervector space and let W_1, W_2 be finite-dimensional subhyperspaces of V such that $0\circ w = \{0_V\}$, for all w in any basis of W_2. Then W_1+W_2 is finite-dimensional and*

$$\dim(W_1+W_2) = \dim(W_1)+\dim(W_2)-\dim(W_1\cap W_2).$$

Proof. By Lemma 2.4(2), $W_1 \cap W_2$ is finite-dimensional and has a finite basis $\beta_0 = \{x_1, \ldots, x_k\}$, which by Lemma 2.4(3) can expand to a basis $\beta_1 = \{x_1, \ldots, x_k, y_1, \ldots, y_m\}$ for W_1 and a basis $\beta_2 = \{x_1, \ldots, x_k, z_1, \ldots, z_n\}$ for W_2. We show that

$$\beta = \{x_1, \ldots, x_k, y_1, \ldots, y_m, z_1, \ldots, z_n\}$$

is a basis for $W_1 + W_2$. For any $w_1 + w_2 \in W_1 + W_2$,

$$\begin{aligned} w_1 + w_2 &\in \sum_{i=1}^{k} a_i \circ x_i + \sum_{j=1}^{m} b_j \circ y_j + \sum_{i=1}^{k} \acute{a}_i \circ x_i + \sum_{r=1}^{n} c_r \circ z_r \\ &= \sum_{i=1}^{k} (a_i + \acute{a}_i) \circ x_i + \sum_{j=1}^{m} b_j \circ y_j + \sum_{r=1}^{n} c_r \circ z_r. \end{aligned}$$

Thus β generates $W_1 + W_2$. Now suppose

$$\underline{0} \in \sum_{i=1}^{k} a_i \circ x_i + \sum_{j=1}^{m} b_j \circ y_j + \sum_{r=1}^{n} c_r \circ z_r.$$

Then $\underline{0} = \acute{x}_1 + \cdots + \acute{x}_k + \acute{y}_1 + \cdots + \acute{y}_m + \acute{z}_1 + \cdots + \acute{z}_n$, for some $\acute{x}_i \in a_i \circ x_i$, $\acute{y}_j \in b_j \circ y_j$ and $\acute{z}_r \in c_r \circ z_r$. Hence $-\acute{z}_1 - \cdots - \acute{z}_n = \acute{x}_1 + \cdots + \acute{x}_k + \acute{y}_1 + \cdots + \acute{y}_m \in W_1 \cap W_2$ and so $-\acute{z}_1 - \cdots - \acute{z}_n \in \sum_{i=1}^{k} d_i \circ x_i$. Thus $-\acute{z}_1 - \cdots - \acute{z}_n = \bar{x}_1 + \cdots + \bar{x}_k$ for some $\bar{x}_i \in d_i \circ x_i$ and so

$$\underline{0} = \acute{z}_1 + \cdots + \acute{z}_n + \bar{x}_1 + \cdots + \bar{x}_k \in \sum_{r=1}^{n} c_r \circ z_r + \sum_{i=1}^{k} d_i \circ x_i.$$

Since the set β_2 is independent, $c_r = d_i = 0$, $1 \le r \le n$, $1 \le i \le k$. Then by assumption, $\sum_{r=1}^{n} c_r \circ z_r = \{\underline{0}\}$, which it implies that $\underline{0} \in \sum_{i=1}^{k} a_i \circ x_i + \sum_{j=1}^{m} b_j \circ y_j$. From independency of β_1, it follows that $a_i = b_j = 0$, $1 \le i \le k$, $1 \le j \le m$. Therefore β is linearly independent and so it is a basis for $W_1 + W_2$. Consequently,

$$\dim(W_1 + W_2) = k + m + n = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

□

Proposition 4.9. *Let V be a hypervector space over the field K. Then*

1. $\{\underline{0}\}^\circ = V^*$ *and* $V^\circ = \{\underline{0}\}$,
2. *if V is finite-dimensional and $W_1, W_2 \leqslant V$, then* $(W_1 + W_2)^\circ = W_1^\circ \cap W_2^\circ$,
3. *if V is strongly left distributive and invertible and if W_1, W_2 are finite-dimensional subhyperspaces of V such that $0 \circ w = \{\underline{0}\}$, for all w in any basis of W_2, then* $(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$,
4. *if $S \subseteq V$ and $W = \langle S \rangle$, then* $W^\circ = S^\circ$.

Proof. 1) Straightforward.

2) $T \in (W_1 + W_2)^\circ \Leftrightarrow T(x_1 + x_2) = 0_k, \forall x_1 + x_2 \in W_1 + W_2 \Leftrightarrow T(x_1) + T(x_2) = 0_k, \forall x_1 \in W_1, x_2 \in W_2 \Leftrightarrow T(x_1) = 0_k, \forall x_1 \in W_1$ and $T(x_2) = 0_k, \forall x_2 \in W_2 \Leftrightarrow T \in W_1^\circ \cap W_2^\circ$.

3) Suppose $T \in W_1^\circ + W_2^\circ$ and $w \in W_1 \cap W_2$. Then $T = T_1 + T_2$, for some $T_1 \in W_1^\circ$ $T_2 \in W_2^\circ$ and so $T(w) = T_1(w) + T_2(w) = 0$. Thus $T \in (W_1 \cap W_2)^\circ$ and $W_1^\circ + W_2^\circ \subseteq (W_1 \cap W_2)^\circ$.
Conversely, let $T \in (W_1 \cap W_2)^\circ$. In the proof of Theorem 4.10 it was shown that we can choose a basis $\beta = \{x_1, \ldots, x_k, y_1, \ldots, y_m, z_1, \ldots, z_n\}$ for $W_1 + W_2$, where $\{x_1, \ldots, x_k\}$ is a basis for $W_1 \cap W_2$, $\{x_1, \ldots, x_k, y_1, \ldots, y_m\}$ is a basis for W_1 and $\{x_1, \ldots, x_k, z_1, \ldots, z_n\}$ is a basis for W_2. Then β can be expanded to a basis

$$\{x_1, \ldots, x_k, y_1, \ldots, y_m, z_1, \ldots, z_n, t_1, \ldots, t_l\}$$

for V. Thus

$$x \in \sum_{i=1}^{k} a_i \circ x_i + \sum_{j=1}^{m} b_j \circ y_j + \sum_{p=1}^{n} c_p \circ z_p + \sum_{q=1}^{l} d_q \circ t_q, \tag{*}$$

for all $x \in V$. Hence $T(x) = \sum_{i=1}^{k} a_i \cdot T(x_i) + \sum_{j=1}^{m} b_j \cdot T(y_j) + \sum_{p=1}^{n} c_p \cdot T(z_p) + \sum_{q=1}^{l} d_q \cdot T(t_q)$. Since $T(x_i) = 0, 1 \leq i \leq k$, it follows that $T(x) = \sum_{j=1}^{m} b_j \cdot T(y_j) + \sum_{p=1}^{n} c_p \cdot T(z_p) + \sum_{q=1}^{l} d_q \cdot T(t_q)$. Define $T_1(x) = \sum_{p=1}^{n} c_p \cdot T(z_p) + \sum_{q=1}^{l} d_q \cdot T(t_q)$ and $T_2(x) = \sum_{j=1}^{m} b_j \cdot T(y_j)$. Then $T = T_1 + T_2$, such that if $w_1 \in W_1$ and $w_1 \in \acute{a}_1 \circ x_1 + \cdots + \acute{a}_k \circ x_k + \acute{b}_1 \circ y_1 + \cdots + \acute{b}_m \circ y_m$, then $w_1 \in \acute{a}_1 \circ x_1 + \cdots + \acute{a}_k \circ x_k + \acute{b}_1 \circ y_1 + \cdots + \acute{b}_m \circ y_m + 0 \circ z_1 + \cdots + 0 \circ z_p + 0 \circ t_1 + \cdots + 0 \circ t_l$, and so $T_1(w_1) = 0 \cdot T(z_1) + \cdots + 0 \cdot T(z_p) + 0 \cdot T(t_1) + \cdots + 0 \cdot T(t_l) = 0$. Thus $T_1 \in W_1^{\circ}$. Similarly $T_2 \in W_2^{\circ}$. Hence $T \in W_1^{\circ} + W_2^{\circ}$. Therefore $(W_1 \cap W_2)^{\circ} \subseteq W_1^{\circ} + W_2^{\circ}$.

4) Clearly $W^{\circ} \subseteq S^{\circ}$. Also if $T \in S^*$ and $w \in W$, then $w \in \sum_{i=1}^{n} a_i \circ x_i$ for some $a_i \in K$ and $x_i \in S$, so $T(w) \in T(\sum_{i=1}^{n} a_i \circ x_i) = \sum_{i=1}^{n} a_i T(x_i) = 0$. Thus $T(w) = 0$ and $T \in W^*$. □

Theorem 4.10. *Let V be a finite-dimensional strongly left distributive and invertible hypervector space and $W \leqslant V$. Then*

$$\dim W + \dim W^{\circ} = \dim V.$$

Proof. Let $\dim W = d$ (by Lemma 2.4(2) W is finite-dimensional) and $\beta_W = \{x_1, \ldots, x_d\}$ be a basis for W. Then by Lemma 2.4(3) there exists a basis $\beta = \{x_1, \ldots, x_d, x_{d+1}, \ldots, x_n\}$ for V. Suppose $\{T_1, \ldots, T_n\}$ is the basis of V^* which is the dual of β (by Theorem 3.6). We show that $\{T_{d+1}, \ldots, T_n\}$ is a basis for W°:
Firstly, $T_i \in W^{\circ}, d+1 \leq i \leq n$, because if $x \in W$ then $x \in a_1 \circ x_1 + \cdots + a_d \circ x_d$, for some $a_1, \ldots, a_d \in K$ and so $T_i(x) = a_1 \cdot T_i(x_1) + \cdots + a_d \cdot T_i(x_d) = 0$. The linear functionals $T_{d+1}, \ldots, T_n$ are independent, so we must show that they span W°. Suppose $T \in W^{\circ}$. Then

$$T = \sum_{i=1}^{n} T(x_i) \cdot T_i = \sum_{i=d+1}^{n} T(x_i) \cdot T_i,$$

since $T(x_i) = 0$, for $i \leq k$. Hence $\{T_{d+1}, \ldots, T_n\}$ is a basis for W°. Therefore

$$\dim W^{\circ} = n - d = \dim V - \dim W.$$

□

Example 4.11. *Suppose $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ is the 2-dimensional hypervector space in Example 4.2. Then $\{(0,1,0)\}$ is a basis for the subhyperspace $W = \mathbb{R} \times \mathbb{R} \times \{0\}$ and so $\dim W = 1$. Moreover the singleton $\{T : \mathbb{R}^3 \to \mathbb{R};\ T(x,y,z) = z\}$ is a basis for W° and so $\dim W^{\circ} = 1$. Hence as we expected from Theorem 4.10, $\dim W + \dim W^{\circ} = \dim V$.*

Corollary 4.12. *Let V be an n-dimensional strongly left distributive and invertible hypervector space and $W \leqslant V$ such that $\dim W = d$. Then W is the intersection of $(n-d)$ superhyperspaces in V. Therefore every superhyperspace is the kernel of a linear functional.*

Proof. By using the notations of the proof of Theorem 4.10,

$$W = \{x \in V : T_i(x) = 0, \forall d+1 \leq i \leq n\}.$$

Thus $W = \bigcap_{i=d+1}^{n} \ker T_i$, where $\ker T_i$, $d+1 \leq i \leq n$, is a superhyperspace of V, by Proposition 4.3. □

Corollary 4.13. *Let V be a finite-dimensional strongly left distributive and invertible hypervector space and $W_1, W_2 \leqslant V$. Then $W_1 = W_2$ if and only if $W_1^{\circ} = W_2^{\circ}$.*

Proof. If $W_1 = W_2$, then it is clear that $W_1^{\circ} = W_2^{\circ}$. If $W_1 \neq W_2$, without loss of generality, suppose there exists $\acute{x} \in W_2 \setminus W_1$. Then by using the notations of the proof of Theorem 4.10, there exists a linear functional T such that $T(x) = 0$, for all $x \in W_1$, but $T(\acute{x}) \neq 0$. Thus $T \in W_1^{\circ} \setminus W_2^{\circ}$ and hence $W_1^{\circ} \neq W_2^{\circ}$. □

5. The Double Dual

In this section we check out whether every basis for V^* is the dual of some basis for V.

Lemma 5.1. *Every $x \in V$ induces a linear functional $L_x : V^* \longrightarrow K$ defined by $L_x(T) = T(x)$, for all $T \in V^*$.*

Proof. Let $x \in V$, $a \in K$ and $T_1, T_2 \in V^*$. Then $L_x(T_1 + T_2) = (T_1 + T_2)(x) = T_1(x) + T_2(x) = L_x(T_1) + L_x(T_2)$. Also for all $T \in a \odot T_1$, $L_x(T) = T(x) = a \cdot T_1(x) = a \cdot L_x(T_1)$. Thus $L_x(a \odot T_1) = a \cdot L_x(T_1)$. $\square$

Proposition 5.2. *If V is finite-dimensional and $x \in V \setminus \{\underline{0}\}$, then $L_x \neq 0$.*

Proof. Let $\beta = \{x_1, \ldots, x_n\}$ be an ordered basis for V such that $x_1 = x$. Suppose T is the linear functional which assigns to each vector in V its first coordinate in the ordered basis β. Then $T(x) = 1 \neq 0$. Thus $L_x(T) = T(x) \neq 0$. Hence $L_x \neq 0$. $\square$

Lemma 5.3. *Let $T : V \longrightarrow K$ be a linear functional on V. Then*

1. $\bigcup_{x \in V} 0 \circ x \subseteq \ker T$,
2. *T is injective if and only if* $\ker T = \{\underline{0}\}$,
3. *if V is strongly left distributive and T is injective, then* $\bigcup_{x \in V} 0 \circ x = \ker T = \{\underline{0}\} = 0 \circ \underline{0} = \Omega_V$.

Proof. 1) Suppose $t \in 0 \circ x$, for some $x \in V$. Then $T(t) \in T(0 \circ x) = 0 \cdot T(x) = 0_K = 0 \cdot 0_K = \Omega_K$. Thus $t \in \ker T$.

2) If T is injective and $x \in \ker T$, then $T(x) \in \Omega_K = 0 \cdot 0_K = \{0_K\}$. Thus $T(x) = 0 = T(\underline{0})$ and so $x = \underline{0}$. Conversely, if $x, y \in V$ such that $T(x) = T(y)$, then $x - y \in \ker T$. Thus $x = y$.

3) If $t \in \ker T$, then by (2) $t = \underline{0}$. Thus $t \in 0 \circ x$, for all $x \in V$. Hence by (1) $\ker T = \bigcup_{x \in V} 0 \circ x$. Also $\underline{0} \in 0 \circ \underline{0}$, and so by (2) $\ker T \subseteq 0 \circ \underline{0}$. But $x \in 0 \circ \underline{0}$ implies that $T(x) = 0 \cdot T(\underline{0}) = 0_K$ and so $x \in \ker T$. Therefore $\ker T = 0 \circ \underline{0}$, which completes the proof. $\square$

Theorem 5.4. *Let V be a finite-dimensional strongly left distributive and invertible hypervector space over the field K. Then $V \cong V^{**}$.*

Proof. We show that the mapping $\phi : V \longrightarrow V^{**}$ defined by $\phi(x)(T) = L_x(T) = T(x)$, for all $x \in V$ and $T \in V^*$, is an isomorphism. For any $x \in V$, by Lemma 5.1 the function L_x is a linear functional, so ϕ is well-defined. Suppose $x, y \in V$, $a \in K$ and $T \in V^*$. Then

$$\begin{aligned}
\phi(x + y)(T) &= L_{x+y}(T) = T(x + y) \\
&= T(x) + T(y) = L_x(T) + L_y(T) \\
&= \phi(x)(T) + \phi(y)(T) \\
&= (\phi(x) + \phi(y))(T),
\end{aligned}$$

and so $\phi(x + y) = \phi(x) + \phi(y)$. Also for any $t \in a \circ x$,

$$\begin{aligned}
\phi(t)(T) &= L_t(T) = T(t) \\
&= a \cdot T(x) = a \cdot L_x(T) \\
&= a \cdot \phi(x)(T) = (a \cdot \phi(x))(T),
\end{aligned}$$

so $\phi(t) = a \cdot \phi(x)$, which implies that $\phi(a \circ x) = a \cdot \phi(x)$. Hence ϕ is a good transformation from V into V^{**}. Now if $x, y \in V$, such that $\phi(x) = \phi(y)$, then

$$\begin{aligned}
& \forall T \in V^*;\ \phi(x)(T) = \phi(y)(T) \\
\Longrightarrow\ & \forall T \in V^*;\ L_x(T) = L_y(T) \\
\Longrightarrow\ & \forall T \in V^*;\ L_{x-y}(T) = 0 \\
\Longrightarrow\ & L_{x-y} = 0 \\
\Longrightarrow\ & x - y = 0 \ \ (\textit{by Lemma}\ 5.3) \\
\Longrightarrow\ & x = y.
\end{aligned}$$

Thus ϕ is one-to-one. But if $x \in \ker\phi$, then $\phi(x) \in \Omega_{V^{**}} = \{0_{V^{**}}\}$ and by injectivity of ϕ, $x = \underline{0}$. Thus $x \in 0 \circ \underline{0}$, since V is strongly left distributive. Hence $\ker\phi \subseteq 0 \circ \underline{0}$. On the other hand, if $x \in 0 \circ \underline{0}$, then $\phi(x) \in 0 \cdot \phi(\underline{0}) = 0 \cdot 0_{V^{**}} = \Omega_{V^{**}}$ and so $x \in \ker\phi$. Hence $\ker\phi = 0 \circ \underline{0} = \Omega_V$. Therefore $\dim\ker\phi = \dim\Omega_V = 0$ ([15] If V is strongly left distributive, then $\dim\Omega_V = 0$). Then by Theorems 2.8 and 3.5, $\dim\phi(V) = \dim V = \dim V^{**}$. Consequently, by Proposition 2.5, $\phi(V) = V^{**}$ and so ϕ is onto, which completes the proof. □

Corollary 5.5. *Let V be a finite-dimensional strongly left distributive and invertible hypervector space over the field K. If L is a linear functional on the dual space V^* of V, then there exists a unique vector x in V such that $L(T) = T(x)$, for all $T \in V^*$.*

Proof. ϕ in the proof of Theorem 5.4 is onto. □

Theorem 5.6. *Let V be a finite-dimensional strongly left distributive and invertible hypervector space over the field K. Then any basis for V^* is dual of some basis for V.*

Proof. Let $\beta^* = \{T_1, \dots, T_n\}$ be a basis for V^*. Then by Theorem 3.6, there exists a basis $\beta^{**} = \{L_1, \dots, L_n\}$ for V^{**} such that $L_i(T) = a_i$, $1 \le i \le n$, where $T = a_1T_1 + \cdots + a_nT_n$; in other words, $L_i(T_j) = \delta_{ij}$. By Corollary 5.5 for any $1 \le i \le n$ there exists a unique vector $x_i \in V$ such that $L_i(x) = T(x_i)$, for all $T \in V^*$, i.e. $L_i = L_{x_i}$. We show that $\beta = \{x_1, \dots, x_n\}$ is a basis for V:

By Theorem 5.4, the mapping $\phi : V \longrightarrow V^{**}$ defined by $\phi(x)(T) = L_x(T) = T(x)$ is an isomorphism. Thus for all $x \in V$, $\phi(x) = a_1L_1 + \cdots + a_nL_n = a_1L_{x_1} + \cdots + a_nL_{x_n}$, for some $a_1, \dots, a_n \in K$. Hence

$$\begin{aligned}
\phi(x)(T) &= (a_1L_{x_1} + \cdots + a_nL_{x_n})(T)\\
&= (a_1L_{x_1})(T) + \cdots + (a_nL_{x_n})(T)\\
&= a_1L_{x_1}(T) + \cdots + a_nL_{x_n}(T)\\
&= L_{x_1}(a_1T) + \cdots + L_{x_n}(a_nT)\\
&= (a_1T)(x_1) + \cdots + (a_nT)(x_n)\\
&= a_1 \cdot T(x_1) + \cdots + a_n \cdot T(x_n)\\
&= T(a_1 \circ x_1 + \cdots + a_n \circ x_n)\\
&= \phi(a_1 \circ x_1 + \cdots + a_n \circ x_n)(T).
\end{aligned}$$

Hence $\phi(x) = \phi(a_1 \circ x_1 + \cdots + a_n \circ x_n)$ and so $\phi(x) = \phi(t)$, for some $t \in a_1 \circ x_1 + \cdots + a_n \circ x_n$. By injectivity of ϕ, $x \in a_1 \circ x_1 + \cdots + a_n \circ x_n$. Therefore β generates V. Now let $\underline{0} \in a_1 \circ x_1 + \cdots + a_n \circ x_n$, for some $a_1, \dots, a_n \in K$. Then

$$0_K = T(\underline{0}) \in T(a_1 \circ x_1 + \cdots + a_n \circ x_n) = a_1T(x_1) + \cdots + a_nT(x_n),$$

for any $T \in V^*$. Thus

$$\begin{aligned}
&\forall T \in V^*;\ a_1L_1(T) + \cdots + a_nL_n(T) = 0_K\\
\Longrightarrow\ &\forall T \in V^*;\ (a_1L_1 + \cdots + a_nL_n)(T) = 0_K\\
\Longrightarrow\ &a_1L_1 + \cdots + a_nL_n = 0_K.
\end{aligned}$$

Hence $a_1 = \cdots = a_n = 0$, which implies that β is independent and so it is a basis for V. Finally we show that β^* is the dual of β. Let $\{\acute{T}_1, \dots, \acute{T}_n\}$ be the dual of β, i.e. $\acute{T}_i(x) = a_i$, for all $x \in V$, such that $x \in a_1 \circ x_1 + \cdots + a_n \circ x_n$. Then

$$\begin{aligned}
T_i(x) &\in T_i(a_1 \circ x_1 + \cdots + a_n \circ x_n)\\
&= a_1T_i(x_1) + \cdots + a_nT_i(x_n)\\
&= a_1L_{x_1}(T_i) + \cdots + a_nL_{x_n}(T_i)\\
&= a_1L_1(T_i) + \cdots + a_nL_n(T_i)\\
&= a_1\delta_{1i} + \cdots + a_n\delta_{ni}\\
&= a_i\\
&= \acute{T}_i(x),
\end{aligned}$$

for all $1 \leq i \leq n$. Thus $T_i = \acute{T}_i$, for all $1 \leq i \leq n$. Hence $\{T_1, \ldots, T_n\} = \{\acute{T}_1, \ldots, \acute{T}_n\}$. $\square$

Theorem 5.7. *Let V be a finite-dimensional strongly left distributive and invertible hypervector space over the field K. If $S \subseteq V$, then $S^{\circ\circ} = \langle S \rangle$, up to isomorphism.*

Proof. Suppose $W = \langle S \rangle$. By Proposition 4.9(4), $W^\circ = S^\circ$. Thus it is sufficient to prove that $W = W^{\circ\circ}$, up to isomorphism. By Theorem 4.10, $\dim W + \dim W^\circ = \dim V$ and $\dim W^\circ + \dim W^{\circ\circ} = \dim V^*$ (by Proposition 4.9(1), $S^\circ \leqslant V^*$). Also by Theorem 3.5 $\dim V = \dim V^*$, so $\dim W = \dim W^{\circ\circ}$. By considering the isomorphism $\phi : V \longrightarrow W^*$ in Theorem 5.4, with $\phi(x)(T) = L_x(T) = T(x)$, for all $x \in V$ and $T \in V^*$, it follows that $\phi(w)(T) = L_w(T) = T(w) = 0$, for all $w \in W$ and $T \in W^\circ$. Then $\phi(w) \in W^{\circ\circ}$, i.e. $\phi(W) \subseteq W^{\circ\circ}$, such that $\dim W = \dim \phi(W) = \dim W^{\circ\circ}$. Thus $W \cong \phi(W) = W^{\circ\circ}$, by Proposition 2.5. $\square$

The notion of a superhyperspace of finite-dimensional hypervector space was defined in Definition 4.1, which it can be defined for arbitrary hypervector spaces in the following way:

Definition 5.8. *If V is a hypervector space over K, then a superhyperspace in V is a maximal proper subhyperspace of V.*

Theorem 5.9. *Let V be a strongly left distributive hypervector space over the field K and T be a non-zero linear functional on V. Then* $\ker T$ *is a superhyperspace in V. Conversely, if V is strongly right distributive and invertible and W is a superhyperspace in V, then there exists a non-zero linear functional T on V such that* $\ker T = W + 0 \circ v$, *for some $v \in V \setminus W$.*

Proof. By Proposition 2.7(2), $\ker T \leqslant V$. Clearly $\ker T$ is a proper subhyperspace of V. Now let $\ker T \leqslant W \leqslant V$ such that $\ker T \neq W$ and $x \in W \setminus \ker T$, i.e. $T(x) \neq 0$. We shall show that $V = \langle \ker T \cup \{x\} \rangle$.

Suppose $t \in V$. Then

$$\begin{aligned} t &\in t + 0 \circ x \\ &= t + (-T(t)T(x)^{-1} + T(t)T(x)^{-1}) \circ x \\ &\subseteq [t - (T(t)T(x)^{-1}) \circ x] + (T(t)T(x)^{-1}) \circ x, \end{aligned}$$

such that

$$\begin{aligned} T[t - (T(t)T(x)^{-1}) \circ x] &= T(t) - T(T(t)T(x)^{-1} \circ x) \\ &= T(t) - T(t)T(x)^{-1}T(x) \\ &= 0, \end{aligned}$$

i.e. $t - (T(t)T(x)^{-1}) \circ x \subseteq \ker T$. Thus $V = \langle \ker T \cup \{x\} \rangle$. Hence

$$\langle \ker T \cup \{x\} \rangle \subseteq W \subseteq V = \langle \ker T \cup \{x\} \rangle,$$

and so $W = V$. Therefore $\ker T$ is a maximal subhyperspace of V.

Conversely, if W is a subhyperspace of V, then similar to above $V = \langle W \cup \{v\} \rangle$, for some fix vector $v \in V \setminus W$. Then every $x \in V$ is belonging to $w + a \circ v$, for some $w \in W$ and $a \in K$. Define a mapping $T : V \longrightarrow K$ by $g(x) = a$. If $x \in w + a \circ v$ and $x \in \acute{w} + \acute{a} \circ v$, then $x = w + y$ and $x = \acute{w} + \acute{y}$, with $y \in a \circ v$ and $\acute{y} \in \acute{a} \circ v$. Thus $\acute{w} - w = y - \acute{y} \in a \circ v - \acute{a} \circ v = (a - \acute{a}) \circ v$. But if $a - \acute{a} \neq 0$, then $v \in (a - \acute{a})^{-1} \circ (\acute{w} - w) \subseteq W$, which is a contradiction. Hence $a = \acute{a}$ and so T is well-defined.
Now suppose $x, y \in V$. Then $x \in w_1 + a \circ v$ and $y \in w_2 + b \circ v$, for some $w_1, w_2 \in W$ and $a, b \in K$. Thus $x + y \in w_1 + a \circ v + w_2 + b \circ v = (w_1 + w_2) + (a + b) \circ v$, so $T(x + y) = a + b = T(x) + T(y)$. Also if $x \in V$, $a \in K$ and $t \in a \circ x$, then $x \in w + \acute{a} \circ v$, for some $w \in W$ and $\acute{a} \in K$ and so $t \in a \circ (w + \acute{a} \circ v) \subseteq a \circ w + (a\acute{a}) \circ v$. Hence $T(t) = a\acute{a} = aT(x)$, which implies that $T(a \circ x) = aT(x)$. Therefore T is a linear functional in V. Finally,

$$\begin{aligned} \ker T &= \{x \in V : T(x) = 0_K\} \\ &= \{w + t : w \in W, y \in a \circ v, a = 0\} \\ &= W + 0 \circ v. \end{aligned}$$

$\square$

Lemma 5.10. *Let V be a strongly left distributive hypervector space over the field K. If T_1 and T_2 are linear functionals on V, then T_2 is a scalar multiple of T_1 if and only if* $\ker T_1 \subseteq \ker T_2$.

Proof. Note that $a \odot T = \{\acute{T} \in V^*; \acute{T}(x) \in T(a \circ x), \forall x \in V\} = aT$, for all $a \in K$ and $T \in V^*$. Now if $T_2 = aT_1$, for some $a \in K$, then clearly $\ker T_1 \subseteq \ker T_2$. Conversely, let $\ker T_1 \subseteq \ker T_2$. If $T_1 = 0$, then $T_2 = 0$. If $T_1 \neq 0$, then similar to the proof of Theorem 5.9, $V = \langle \ker T_1 \cup \{x\} \rangle$, for some $x \in V \setminus \ker T_1$ and $T = T_2 - T_1(x)^{-1}T_2(x)T_1$ is a linear functional on V such that $T(t) = T_2(t) - T_1(x)^{-1}T_2(x)T_1(t) = 0$, for all $t \in \ker T_1$, and $T(x) = T_2(x) - T_1(x)^{-1}T_2(x)T_1(x) = 0$. Thus for all $v \in t + a \circ x$, where $t \in \ker T_1$, $T(v) = T(t) + aT(x) = 0$. Hence $T = 0$ and so $T_2 = T_1(x)^{-1}T_2(x)T_1$. □

Theorem 5.11. *Let $T, T_1, \ldots, T_n$ be linear functionals on strongly left distributive hypervector space V over the field K. Then T is a linear combination of $T_1, \ldots, T_n$ if and only if* $\ker T_1 \cap \cdots \cap \ker T_n \subseteq \ker T$.

Proof. If $T = a_1T_1 + \cdots + a_nT_n$ and $x \in \ker T_1 \cap \cdots \cap \ker T_n$, then $T_1(x) = \cdots = T_n(x) = 0$ and so $T(x) = a_1T_1(x) + \cdots + a_nT_n(x) = 0$. Thus $x \in \ker T$.
We prove the converse by induction on n. The case $n = 1$ is hold by Lemma 5.10. Suppose the result is true for $n = k - 1$ and let $T_1, \ldots, T_k$ be linear functionals such that $\ker T_1 \cap \cdots \cap \ker T_k \subseteq \ker T$. Let $\acute{T}, \acute{T}_1, \ldots, \acute{T}_{k-1}$ be the restrictions of $T, T_1, \ldots, T_{k-1}$ to the subhyperspace $\ker T_k$. Then $\acute{T}, \acute{T}_1, \ldots, \acute{T}_{k-1}$ are linear functionals on the hypervector space $\ker T_k$, such that $\ker \acute{T}_1 \cap \cdots \cap \ker \acute{T}_{k-1} \subseteq \ker \acute{T}$. Thus if $x \in \ker \acute{T}_1 \cap \cdots \cap \ker \acute{T}_{k-1} \cap \ker T_k$, then $x \in \ker T_1 \cap \cdots \cap \ker T_{k-1} \cap \ker T_k$, which implies that $x \in \ker T$, i.e. $T(x) = 0$. Also by the induction hypothesis, $\acute{T} = a_1\acute{T}_1 + \cdots + a_{k-1}\acute{T}_{k-1}$, for some $a_1, \ldots, a_{k-1} \in K$. Now suppose

$$S = T - a_1T_1 - \cdots - a_{k-1}T_{k-1}.$$

Then S is a linear functional on V such that $S(x) = 0$ for all $x \in \ker T_k$. Hence by Lemma 5.10, $S = a_kT_k$, for some $a_k \in K$. Therefore

$$T = a_1T_1 + \cdots + a_kT_k,$$

which completes the proof. □

6. Conclusion

The motivation of this paper was to generalize the notion of linear functionals over vector spaces into hypervector spaces. In this regards, we investigated some essential concepts and properties about linear functionals on hypervector spaces under special conditions such as:
- dual basis of a given basis of a finite-dimensional hypervector space,
- relation between linear functionals and subhyperspaces,
- superhyperspace of V and annihilator of a subset of V,
- $\dim V^* = \dim V$,
- $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$,
- $\dim W + \dim W^\circ = \dim V$,
- every superhyperspace is the kernel of a linear functional,
- whether every basis for V^* is the dual of some basis for V.

One can follow this paper and study some concepts from the algebraic and analytic points of view, such as transpose of linear transformations between hypervector spaces, linear functionals on an inner product hyperspace, linear functionals on convex hypervector spaces specially on normed hypervector spaces. Also investigation the fuzzy case of above concluded results is an idea for studying in the future.

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