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Ribbon Entwining Datum

Xiaohui Zhang[a], Wei Wang[b], Xiaofan Zhao[c], Shengxiang Wang[d]

[a]*School of Mathematical Sciences, Qufu Normal University, Qufu Shandong 273165, P. R. China*
[b]*Nanjing Research Institute of Electronic Engineering, Nanjing, 210007, P. R. China*
[c]*College of Mathematics and Information Science, Henan Normal University, Xinxiang Henan 453007, P. R. China*
[d]*School of Mathematics Sciences, Chuzhou University, Chuzhou Anhui 239000, P. R. China*

Abstract. Let (C, A, φ) be an entwining structure over a field k. In this paper, we introduce the notion of the ribbon entwined datum to generalize the definition of (co)ribbon structures, and give several necessary and sufficient conditions for the category of entwined modules to be a ribbon category. We also discuss the ribbon structures in the Long dimodule category and Yetter-Drinfel'd category for applications.

1. Introduction

Entwining structures were proposed by Brzezinski and Majid in [7] to define coalgebra principal bundles. An entwining structure over a monoidal category $\mathcal{C}$ consists of an algebra A, a coalgebra C and a morphism $\varphi : C \otimes A \to A \otimes C$ satisfying some axioms. The entwining modules are both A-modules and C-comodules, with compatibility relation given by φ. Note that the definition of entwined modules generalizes lots of important modules such as Hopf modules, Doi-Hopf modules, and Yetter-Drinfel'd modules ([9], [17]). Further researches on entwining structures can be found in [1], [21], [23], and so on.

Monoidal category theory played an important role in the theory of knots and links and the theory of quantum groups. Through the reconstruction theory and Tannakian duality ([12], [20]), quantum groups and monoidal categories are correspondent with each other. There are many kinds of monoidal categories with additional structure - braided, rigid, pivotal, balanced, ribbon, etc., and many of them have an associated form in low dimensional topology theory and knot theory. For example, ribbon category (see [18]) is based on the isotopy invariants of framed tangles; spherical category (see [2] and [10]) is based on the Turaev-Virostate sum model invariant of a closed piecewise-linear 3-manifold. From the reconstruction theoretical point of view, a ribbon (resp. pivotal) category is equivalent to the category of (co)modules over a (co)ribbon (resp. pivotal) Hopf algebras (or its generalizations)(see [3], [4], and [22] - [26]).

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Corresponding author: Xiaohui Zhang
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Email addresses: zxhhhhh@hotmail.com (Xiaohui Zhang), weiwang2012spring@yahoo.com (Wei Wang), zhaoxiaofan8605@gmail.com (Xiaofan Zhao), wangsx-math@163.com (Shengxiang Wang)

The motivations of our paper is raised from the study of how to get the ribbon structure in the center of the category of modules of a finite dimensional Hopf algebra.

A ribbon structure (see [18], and also see [13]) in a rigid braided category is a self-dual twist (or a self-dual balanced structure), which is a natural isomorphism from the identity functor to itself and compatible with the duality and the braiding. In 1993, Kauffman and Radford got a necessary and sufficient condition for a finite-dimensional Drinfel'd double to be a ribbon Hopf algebra (see [14]). As well-known, when a Hopf algebra H is finite-dimensional we have that the category of modules of Drinfel'd double $D(H)$ is actually the Yetter-Drinfel'd category $\mathcal{YD}_H^H$. Therefore one is prompted to ask the following question: is there any other approach to get the ribbon structures in the Yetter-Drinfel'd categories? When does $\mathcal{YD}_H^H$ becomes a ribbon category from the point of view of the category theory? To figure out these mentioned questions necessitates the following discussion about the process of the emergence of the ribbon structures in the category of entwined modules.

The paper is organized as follows. In Section 2 we recall some notions of entwining structures, and ribbon categories. Section 3 is concerned about the presentation of Hopf algebras induced by $\mathcal{C}_A^C(\varphi)$, and about the exhibition of its (co)representation category is monoidal identified to $\mathcal{C}_A^C(\varphi)$. In Section 4, we mainly give a necessary and sufficient condition for $\mathcal{C}_A^C(\varphi)$ to be a ribbon category. Finally, we consider the Yetter-Drinfel'd category and the category of generalized Long dimodules as applications.

2. Preliminaries

Throughout the paper, we let k be a fixed field and $char(k) = 0$ and Vec_k be the category of finite dimensional k-spaces. All the algebras and coalgebras, modules and comodules are supposed to be in Vec_k. For the comultiplication Δ of a k-module C, we use the Sweedler-Heyneman's notation: $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$. τ means the flip map $\tau(a \otimes b) = b \otimes a$.

2.1. *Entwining structure and entwined modules*

In this part we first review several definitions related to entwined modules (see [7] or [11]).

Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra and (A, m_A, η_A) an algebra over k. A map $\varphi : C \otimes A \to A \otimes C$, $\varphi(c \otimes a) = \sum a_\varphi \otimes c^\varphi$, is called an *entwining map* if the following identities hold

$$\begin{cases} (E1)\ \sum (ab)_\varphi \otimes c^\varphi = \sum a_\varphi b_\psi \otimes c^{\varphi\psi}; \\ (E2)\ \sum a_\varphi \otimes (c^\varphi)_1 \otimes (c^\varphi)_2 = \sum a_{\varphi\psi} \otimes (c_1)^\psi \otimes (c_2)^\varphi; \\ (E3)\ \sum (1_A)_\varphi \otimes c^\varphi = 1_A \otimes c; \\ (E4)\ \sum a_\varphi \varepsilon_C(c^\varphi) = a\varepsilon_C(c), \end{cases}$$

where $a, b \in A$, $c \in C$, $\psi = \varphi$. Furthermore, (C, A, φ) is called *a right-right entwining structure*.

Let $\varphi : C \otimes A \to A \otimes C$ be an entwining map, $M \in \mathcal{C}$, (M, ϱ_M) be a right A-module, (M, ρ^M) be a right C-comodule. If the diagram

$$\begin{array}{ccccc} M \otimes A & \xrightarrow{\varrho_M} & M & \xrightarrow{\rho^M} & M \otimes C \\ \downarrow{\scriptstyle \rho^M \otimes id_A} & & & & \uparrow{\scriptstyle \varrho_M \otimes id_C} \\ M \otimes C \otimes A & & \xrightarrow{id_M \otimes \varphi} & & M \otimes A \otimes C \end{array} \qquad (E0)$$

is commutative, then we call the triple (M, ϱ_M, ρ^M) an *entwined module*.

The morphism between entwined modules is called *entwined module morphism* if it is both A-linear and C-colinear. The category of entwined modules is denoted by $\mathcal{C}_A^C(\varphi)$.

Recall from [16], a k-linear map $f : C \to A$ is called *convolution invertible* if there exists $f^{-1} : C \to A$ such that $f(x_1)f^{-1}(x_2) = f^{-1}(x_1)f(x_2) = \varepsilon_C(x)1_A$ for any $x \in C$.

Recall from [6] and [11], for any $g, f \in hom_k(C, A)$, one can define their *entwined convolution product* $g \star f \in hom_k(C, A)$ via

$$(g \star f)(c) := \sum \underline{f(c_2)}_\varphi g(\underline{c_1}^\varphi), \ \ c \in C,$$

and hence $hom_k(C, A)$ is an algebra. Note that the unit is $\eta_A \circ \varepsilon_C$.

Similarly, $hom_k(C \otimes C, A \otimes A)$ is also an algebra with the following *entwined convolution product*:

$$(g' \star f') := m_{A\otimes A}(A \otimes A \otimes g')(A \otimes \tau_{C,A} \otimes C)(\varphi \otimes \varphi)(C \otimes \tau_{C,A} \otimes A)(C \otimes C \otimes f')\Delta_{C\otimes C},$$

where $g', f' \in hom_k(C \otimes C, A \otimes A)$. Note that the unit is $(\eta_A \otimes \eta_A) \circ (\varepsilon_C \otimes \varepsilon_C)$.

Recall from [[5], Corollary 3.4,] (or see [11]) that $C \otimes A$ is an object in $\mathcal{C}_A^C(\varphi)$ via

$$\begin{aligned} (c \otimes a) \cdot x &= c \otimes ax; \\ (c \otimes a)_0 \otimes (c \otimes a)_1 &= \sum c_1 \otimes a_\varphi \otimes c_2{}^\varphi, \end{aligned}$$

where $a, x \in A$ and $c \in C$.

$A \otimes C$ is also an object in $\mathcal{C}_A^C(\varphi)$ via

$$\begin{aligned} (a \otimes c) \cdot x &= \sum a x_\varphi \otimes c^\varphi; \\ (a \otimes c)_0 \otimes (a \otimes c)_1 &= a \otimes c_1 \otimes c_2. \end{aligned}$$

Furthermore, for any right A-module M, $M \otimes C$ is also an entwined module by

$$\begin{aligned} (m \otimes c) \cdot a &= \sum m \cdot a_\varphi \otimes c^\varphi; \\ (m \otimes c)_0 \otimes (m \otimes c)_1 &= m \otimes c_1 \otimes c_2. \end{aligned}$$

This defines a right adjoint functor for the underlying functor $U : \mathcal{C}_A^C(\varphi) \to \mathcal{M}_A$.

2.2. *Monoidal entwining datum and double quantum group*

Suppose that C and A are two bialgebras over k such that (C, A, φ) is an entwining structure. Recall that (C, A, φ) is called a *monoidal entwining datum* if the following equations hold

$$\begin{cases} (E5) \ \sum (a_\varphi)_1 \otimes (a_\varphi)_2 \otimes (cd)^\varphi = \sum (a_1)_\varphi \otimes (a_2)_\psi \otimes c^\varphi d^\psi; \\ (E6) \ \sum \varepsilon_A(a_\varphi)(1_C)^\varphi = \varepsilon_A(a)1_C, \end{cases}$$

where $a \in A$, $c \in C$.

Recall from [[11], Theorem 4.1] that $\mathcal{C}_A^C(\varphi)$ is a monoidal category such that the forgetful functors are strict monoidal if and only if (C, A, φ) is a monoidal entwining datum. Further, for any $M, N \in \mathcal{C}_A^C(\varphi)$, the A-action and the C-coaction on $M \otimes N$ are given by

$$\begin{aligned} (m \otimes n) \cdot a &= m \cdot a_1 \otimes n \cdot a_2; \\ (m \otimes n)_0 \otimes (m \otimes n)_1 &= m_0 \otimes n_0 \otimes m_1 n_1, \end{aligned}$$

where $m \in M$, $n \in N$, $a \in A$. Moreover, the tensor unit in $\mathcal{C}_A^C(\varphi)$ is $(k, id_k \otimes \varepsilon_A, id_k \otimes \eta_C)$.

Recall that a pair of bialgebras C and A together with a monoidal entwining map φ (such that $\mathcal{C}_A^C(\varphi)$ is a monoidal category) and together with a k-linear morphism $R : C \otimes C \to A \otimes A$ is called a *double quantum group* if the following identities hold for any $a \in A$, $c, d, x, y, z \in C$:

$$\begin{cases} (E7) \ \sum R(c_1 \otimes d_1) \otimes c_2 d_2 = \underline{R^{(1)}(c_2 \otimes d_2)}_\varphi \otimes \underline{R^{(2)}(c_2 \otimes d_2)}_\psi \otimes d_1{}^\varphi c_1{}^\psi; \\ (E8) \ \sum a_{2\psi} R^{(1)}(c^\varphi \otimes d^\psi) \otimes a_{1\varphi} R^{(2)}(c^\varphi \otimes d^\psi) = R^{(1)}(c \otimes d)a_1 \otimes R^{(2)}(c \otimes d)a_2; \\ (E9) \ \sum \underline{R^{(1)}(x \otimes yz)}_1 \otimes \underline{R^{(1)}(x \otimes yz)}_2 \otimes R^{(2)}(x \otimes yz) \\ \qquad = r^{(1)}(x_2 \otimes y) \otimes R^{(1)}(x_1{}^\varphi \otimes z) \otimes \underline{r^{(2)}(x_2 \otimes y)}_\varphi R^{(2)}(x_1{}^\varphi \otimes z); \\ (E10) \ \sum R^{(1)}(xy \otimes z) \otimes \underline{R^{(2)}(xy \otimes z)}_1 \otimes \underline{R^{(2)}(xy \otimes z)}_2 \\ \qquad = \underline{r^{(1)}(y \otimes z_2)}_\varphi R^{(1)}(x \otimes z_1{}^\varphi) \otimes R^{(2)}(x \otimes z_1{}^\varphi) \otimes r^{(2)}(y \otimes z_2); \\ (E11) \ R \in hom_k(C \otimes C, A \otimes A) \text{ is invertible under the entwined convolution,} \end{cases}$$

where $R(m \otimes n) := \sum R^{(1)}(m \otimes n) \otimes R^{(2)}(m \otimes n) \in A \otimes A$.

Recall from [[11], Theorem 5.5] that $\mathcal{C}_A^C(\varphi)$ is a braided monoidal category if and only if (C, A, φ, R) is a double quantum group. Further, the braiding $\mathbf{C}$ in $\mathcal{C}_A^C(\varphi)$ is given by

$$\mathbf{C}_{M,N} : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto \sum n_0 \cdot R^{(2)}(m_1 \otimes n_1) \otimes m_0 \cdot R^{(1)}(m_1 \otimes n_1), \tag{2.1}$$

where $R(m \otimes n) := \sum R^{(1)}(m \otimes n) \otimes R^{(2)}(m \otimes n)$, $M, N \in \mathcal{C}_A^C(\varphi)$.

Lemma 2.1. *Assume that (C, A, φ, R) is a double quantum group. If the following identity holds*

$$\sum a_\varphi \otimes (1_C)^\varphi = a \otimes 1_C, \quad \textit{for any } c \in C,\ a \in A,$$

then $(A, R(1_C \otimes 1_C))$ is a quasitriangular Hopf algebra.

Dually, if the following identity holds

$$\sum \varepsilon_A(a_\varphi) c^\varphi = \varepsilon_A(a)c, \quad \textit{for any } c \in C,\ a \in A,$$

then $(C, (\varepsilon_A \otimes \varepsilon_A) \circ R)$ is a coquasitriangular Hopf algebra.

Proof. Straightforward. □

2.3. Ribbon category

In this section, we will review several definitions and notations related to ribbon structures.

Let $(C, \otimes, I)$ be a strict monoidal category. Recall from [13] or [3] that for an object $V \in C$, a *left dual* of V is a triple $(V^*, ev_V, coev_V)$, where V^* is an object, $ev_V : V^* \otimes V \to I$ and $coev_V : I \to V \otimes V^*$ are morphisms in C, satisfying

$$(V \otimes ev_V)(coev_V \otimes V) = id_V, \ \text{ and } \ (ev_V \otimes V^*)(V^* \otimes coev_V) = id_{V^*}.$$

Similarly, a *right dual* of V is a triple $({}^*V, \widetilde{ev}_V, \widetilde{coev}_V)$, where ${}^*V$ is an object, $\widetilde{ev}_V : V \otimes {}^*V \to I$ and $\widetilde{coev}_V : I \to {}^*V \otimes V$ are morphisms in C, satisfying

$$(\widetilde{ev}_V \otimes V)(V \otimes \widetilde{coev}_V) = id_V, \ \text{ and } \ ({}^*V \otimes \widetilde{ev}_V)(\widetilde{coev}_V \otimes {}^*V) = id_{{}^*V}.$$

If each object in C admits a left dual (respectively a right dual, respectively both a left dual and a right dual), then C is called a *left rigid category* (respectively a *right rigid category*, respectively a *rigid category*).

Assume that C is a left rigid category. $X, Y \in C$, for a morphism $g : Y \to X$ define its *transpose* as follows:

$$g^* := X^* \xrightarrow{id_{X^*} \otimes coev_Y} X^* \otimes Y \otimes Y^* \xrightarrow{id_{X^*} \otimes g \otimes id_{Y^*}} X^* \otimes X \otimes Y^* \xrightarrow{ev_X \otimes id_{Y^*}} Y^* . \qquad (TR1)$$

Then it is easy to get the following commutative diagrams

$$\begin{array}{ccc} X^* \otimes Y & \xrightarrow{g^* \otimes id_Y} & Y^* \otimes Y \\ {\scriptstyle id_{X^*} \otimes g}\downarrow & & \downarrow{\scriptstyle ev_Y} \\ X^* \otimes X & \xrightarrow[ev_X]{} & I, \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{coev_Y} & Y \otimes Y^* \\ {\scriptstyle coev_X}\downarrow & & \downarrow{\scriptstyle g \otimes id^*_Y} \\ X \otimes X^* & \xrightarrow[id_X \otimes g^*]{} & X \otimes Y^*. \end{array} \qquad (TR2)$$

Further, this defines a bijection between $Hom_C(X^*, Y^*)$ and $Hom_C(Y, X)$.

Lemma 2.2. *Let C be a left rigid category, U, V, W be objects in C. Then*
(1). $V^* \otimes U^* \cong (U \otimes V)^*$;
(2). if $f : V \to W$ and $g : U \to V$ are morphisms in C, then we have $(f \circ g)^* = g^* \circ f^*$, and $(1_V)^* = 1_{V^*}$;
(3). $I^* = I$.

Let $(C, \otimes, I, a, l, r, \mathbf{C})$ be a braided monoidal category with the braiding $\mathbf{C}$. Recall from [13] (or [19]) that a *twist* (or a *balanced structure*) on C is a family $\theta_V : V \to V$ of natural isomorphisms indexed by the objects V of C satisfying

$$\theta_{V \otimes W} = \mathbf{C}_{W,V} \mathbf{C}_{V,W} (\theta_V \otimes \theta_W), \quad \text{where } V, W \in C.$$

A twist θ on an autonomous category C is *self-dual* if $\theta_{V^*} = (\theta_V)^*$ (or equivalently, $\theta_{{}^*V} = {}^*(\theta_V)$).

A *ribbon category* is a braided autonomous category endowed with a self-dual twist.

3. Entwined smash product

3.1. The entwined smash product

Definition 3.1. *Let A, B be algebras in a monoidal category $\mathcal{C}$. A morphism $\Phi : B \otimes A \to A \otimes B$ in $\mathcal{C}$ is called an* algebra distributive law *if Φ satisfying*

$$\begin{array}{ccccc} B \otimes B \otimes A & & \xrightarrow{m_B \otimes id_A} & & B \otimes A \\ \downarrow{\scriptstyle id_B \otimes \Phi} & & & & \downarrow{\scriptstyle \Phi} \\ B \otimes A \otimes B & \xrightarrow[\Phi \otimes id_B]{} & A \otimes B \otimes B & \xrightarrow[id_A \otimes m_B]{} & A \otimes B, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_B \otimes id_A} & B \otimes A \\ & {\scriptstyle id_A \otimes \eta_B} \searrow & \downarrow{\scriptstyle \Phi} \\ & & A \otimes B, \end{array}$$

$$\begin{array}{ccccc} B \otimes A \otimes A & & \xrightarrow{id_B \otimes m_A} & & B \otimes A \\ \downarrow{\scriptstyle \Phi \otimes id_A} & & & & \downarrow{\scriptstyle \Phi} \\ A \otimes B \otimes A & \xrightarrow[id_A \otimes \Phi]{} & A \otimes A \otimes B & \xrightarrow[m_A \otimes id_B]{} & A \otimes B, \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{id_B \otimes \eta_A} & B \otimes A \\ & {\scriptstyle \eta_A \otimes id_B} \searrow & \downarrow{\scriptstyle \Phi} \\ & & A \otimes B. \end{array}$$

Similar to [[9], Theorem 8], we have the following property.

Lemma 3.2. *Let C be a finite dimensional coalgebra and A a finite dimensional algebra. Then give an entwining map $\varphi : C \otimes A \to A \otimes C$ is identified to give an algebra distributive law $\Phi : A \otimes C^{*op} \to C^{*op} \otimes A$.*

Proof. If there is an entwining map $\varphi : c \otimes a \mapsto \sum a_\varphi \otimes c^\varphi$, one can define a linear map $\Phi : A \otimes C^{*op} \to C^{*op} \otimes A$ by

$$\Phi(a \otimes p) = \sum p^\Phi \otimes a_\Phi := \sum p({e_i}^\varphi) e^i \otimes a_\varphi,$$

where $c \in C$, $a \in A$, $p \in C^*$, e_i and e^i are dual bases of C and C^*. It is obvious to see that Φ is an algebra distributive law.

Conversely, if is an algebra distributive law $\Phi : a \otimes p \mapsto \sum p^\Phi \otimes a_\Phi$, one can define a linear map $\varphi : C \otimes A \to A \otimes C$ by

$$\varphi(c \otimes a) = \sum a_\varphi \otimes c^\varphi := \sum {e^i}^\Phi(c) a_\Phi \otimes e_i.$$

Also it can be easily checked that φ is an entwining map. □

Recall from [[8], Theorem 2.5], if there is an algebra distributive law $\Phi : B \otimes A \to A \otimes B$, then $(A \otimes B, (m_A \otimes m_B) \circ (id_A \otimes \Phi \otimes id_B), \eta_A \otimes \eta_B)$ is also an algebra.

Now we suppose that (C, A, φ) is a monoidal entwining datum over k where C, A are two Hopf algebras with bijective antipodes.

Definition 3.3. *The* entwined smash product $C^{*op} \otimes A$ *of the entwining structure (C, A, φ), in a form containing C^{*op} and A, is a Hopf algebra under the following structures:*

- *the multiplication $\widehat{m}$ is given by*

$$(p \otimes a)(q \otimes b) := \sum p *^{op} e^i \otimes a_\varphi b q({e_i}^\varphi) = \sum e^i * p \otimes a_\varphi b q({e_i}^\varphi),$$

where $a, b \in A$, $p, q \in C^{*op}$, e_i and e^i are dual bases of C and C^*;

- *the unit is $\widehat{\eta}(1_k) = \varepsilon_C \otimes 1_A$;*
- *the comultiplication is given by*

$$\widehat{\Delta}(p \otimes a) := (p_1 \otimes a_1) \otimes (p_2 \otimes a_2);$$

- *the counit is given by*

$$\widehat{\varepsilon}(p \otimes a) := p(1_C)\varepsilon_A(a);$$

- *the antipode is given by*

$$\widehat{S}(p \otimes a) := \sum p(S_C^{-1}({e_i}^\varphi)) e^i \otimes S_A(a)_\varphi.$$

Proof. Firstly, since Lemma 3.2, $C^{*op}\otimes A$ is an algebra.

Next we will show $C^{*op}\otimes A$ is a bialgebra. Obviously, $C^{*op}\otimes A$ is a coalgebra under the given comultiplication. We only need check that $\widehat{\Delta}$ and $\widehat{\varepsilon}$ are algebra maps.

For $p,q\in C^{*op}$, $a,b\in A$, we compute

$$\begin{aligned}&\widehat{\Delta}(p\otimes a)\widehat{\Delta}(q\otimes b)\\ =&((p_1\otimes a_1)\otimes(p_2\otimes a_2))((q_1\otimes b_2)\otimes(q_1\otimes b_2))\\ =&\sum p_1 *^{op} e^i\otimes a_{1\varphi}b_1q_1(e_i{}^{\varphi})\otimes p_2 *^{op} o^i\otimes a_{2\psi}b_2q_2(o_i{}^{\psi}).\end{aligned}$$

Thus for any $c,d\in C$, we have

$$\begin{aligned}&\sum(p_1 *^{op} e^i)(c)\otimes a_{1\varphi}b_1q_1(e_i{}^{\varphi})\otimes(p_2 *^{op} o^i)(d)\otimes a_{2\psi}b_2q_2(o_i{}^{\psi})\\ =&\sum p(c_2d_2)\otimes a_{1\varphi}b_1\otimes q(c_1{}^{\varphi}d_1{}^{\psi})\otimes a_{2\psi}b_2.\end{aligned}$$

Also we have

$$\begin{aligned}\widehat{\Delta}((p\otimes a)(q\otimes b)) &= \widehat{\Delta}(\sum p *^{op} e^i\otimes a_{\varphi}bq(e_i{}^{\varphi}))\\ &= \sum p_1 *^{op} e^i{}_1\otimes a_{\varphi1}b_1\otimes p_2 *^{op} e^i{}_2\otimes a_{\varphi2}b_2q(e_i{}^{\varphi}),\end{aligned}$$

Then for $c,d\in C$, we obtain

$$\begin{aligned}&\sum(p_1 *^{op} e^i{}_1)(c)\otimes a_{\varphi1}b_1\otimes(p_2 *^{op} e^i{}_2)(d)\otimes a_{\varphi2}b_2q(e_i{}^{\varphi})\\ =&\sum p(c_2d_2)\otimes a_{\varphi1}b_1\otimes q(\underline{c_1d_1}^{\varphi})\otimes a_{\varphi2}b_2\\ \overset{(E5)}{=}&\sum p(c_2d_2)\otimes a_{1\varphi}b_1\otimes q(c_1{}^{\varphi}d_1{}^{\psi})\otimes a_{2\psi}b_2,\end{aligned}$$

which implies $\widehat{\Delta}((p\otimes a)(q\otimes b))=\widehat{\Delta}(p\otimes a)\widehat{\Delta}(q\otimes b)$.

Since $\widehat{\varepsilon}$ preserves multiplication, $C^{*op}\otimes A=(C^{*op}\otimes A,\widehat{m},\varepsilon_C\otimes 1_A,\widehat{\Delta},\widehat{\varepsilon})$ is a bialgebra.

In order to prove $\widehat{S}$ is the antipode of $C^{*op}\otimes A$, we compute

$$\begin{aligned}&\widehat{S}((p\otimes a)_1)(p\otimes a)_2\\ =&\sum p_1(S_C^{-1}(e_i{}^{\varphi}))(e^i\otimes S_A(a_1)_{\varphi})(p_2\otimes a_2)\\ =&\sum p(S_C^{-1}(e_i{}^{\varphi})o_i{}^{\psi})e^i *^{op} o^i\otimes S_A(a_1)_{\varphi\psi}a_2.\end{aligned}$$

For any $c\in C$, we have

$$\begin{aligned}&\sum p(S_C^{-1}(e_i{}^{\varphi})o_i{}^{\psi})(e^i *^{op} o^i)(c)\otimes S_A(a_1)_{\varphi\psi}a_2\\ =&\sum p(S_C^{-1}(c_2{}^{\varphi})c_1{}^{\psi})\otimes S_A(a_1)_{\varphi\psi}a_2\\ \overset{(E2)}{=}&\sum p(S_C^{-1}(c^{\varphi}{}_2)c^{\varphi}{}_1)\otimes S_A(a_1)_{\varphi}a_2\\ =&\ p(1_C)\varepsilon_C(c)\otimes S_A(a_1)a_2=p(1_C)\varepsilon_C(c)\otimes\varepsilon_A(a)1_A.\end{aligned}$$

Thus $\widehat{S}*id=\widehat{\eta\varepsilon}$. Similarly, one can show that $id*\widehat{S}=\widehat{\eta\varepsilon}$. Hence $(C^{*op}\otimes A,\widehat{S})$ is a Hopf algebra. □

Be similar with [[9], Theorem 9], we have the following property.

Proposition 3.4. *The category of entwined modules $\mathcal{C}_A^C(\varphi)$ is monoidal isomorphic to the representation category of $C^{*op}\otimes A$.*

Proof. For any object (M, θ_M, ρ^M), and morphism $\lambda : M \to N$ in $\mathcal{C}_A^C(\varphi)$, one can define a functor Γ from $\mathcal{C}_A^C(\varphi)$ to the category of right $C^{*op} \otimes A$-modules via

$$\Gamma(M) := M \text{ as } k\text{-spaces}, \quad \Gamma(f) := f,$$

where the $C^{*op} \otimes A$-module structure on M is given by

$$m \leftharpoonup (p \otimes a) := p(m_1)m_0 \cdot a, \ \text{ for all } m \in M,\ p \in C^*,\ a \in A.$$

First of all, we claim that Γ is well-defined. In fact, for any $m \in M$, $p, q \in C^*$, $a, b \in A$, we have

$$m \leftharpoonup (\varepsilon_C \otimes 1_A) = \varepsilon_C(m_1)m_0 \cdot 1_A = m.$$

Also, we can get

$$\begin{aligned}(m \leftharpoonup (p \otimes a)) \leftharpoonup (q \otimes b) &= p(m_1)(m_0 \cdot a) \leftharpoonup (q \otimes b)\\ &= \sum p(m_2)e^i(m_1)m_0 \cdot a_\varphi b q({e_i}^\varphi)\\ &= m \leftharpoonup (p \otimes a)(q \otimes b),\end{aligned}$$

where e_i and e^i are dual bases of C and C^* respectively. Hence $(M, \leftharpoonup)$ is a right $C^{*op} \otimes A$-module.

For the morphism $\lambda : M \to N$, it is a direct computation to check $\Gamma(\lambda)$ is $C^{*op} \otimes A$-linear. Thus Γ is well-defined.

Conversely, we define the functor Λ from the representation category of $C^{*op} \otimes A$ to $\mathcal{C}_A^C(\varphi)$ by

$$\Lambda(U) := U \text{ as } k\text{-spaces}, \quad \Lambda(\lambda) := \lambda,$$

where $(U, \leftharpoonup)$ is a right $C^{*op} \otimes A$-module, $\lambda : U \to V$ is a morphism of $C^{*op} \otimes A$-modules. Further, the A-action on U is defined by

$$u \cdot a := u \leftharpoonup (\varepsilon_C \otimes a), \quad \text{for any } u \in U,\ a \in A,$$

and the C-coaction on U is given by

$$\rho^U(u) = u_0 \otimes u_1 := \sum (u \leftharpoonup (e^i \otimes 1_A)) \otimes e_i.$$

Next we will show that Λ is well defined. It is straightforward to show $(U, \cdot)$ is an A-module and (U, ρ^U) is a C-comodule. We only check U satisfies Diagram (E0).

Since for any $a \in A$, we have

$$\begin{aligned}\rho^U(u \cdot a) &= \sum (u \cdot a \leftharpoonup (e^i \otimes 1_A)) \otimes e_i\\ &= \sum (u \leftharpoonup (e^i({o_i}^\varphi)o^i \otimes a_\varphi)) \otimes e_i\\ &= \sum (u \leftharpoonup (e^i \otimes 1_A)(\varepsilon_C \otimes a_\varphi)) \otimes {e_i}^\varphi\\ &= \sum u_0 \cdot a_\varphi \otimes {u_1}^\varphi,\end{aligned}$$

hence $U \in \mathcal{C}_A^C(\varphi)$.

Since $\Lambda(\lambda) : U \to V$ are both A-linear and C-colinear, Λ is well-defined, as desired.

Obviously Γ is a strict monoidal functor, and Λ is the inverse of Γ. This completes the proof. □

Remark 3.5. *Be similar with Lemma 3.2 and Proposition 3.4, for any finite dimensional k-algebras A and B, if* $\Phi : B \otimes A \to A \otimes B$, $b \otimes a \mapsto \sum a^\Phi \otimes b_\Phi$, *is an algebra distributive law, then there is an entwining map* $\varphi : A^{*cop} \otimes B \to B \otimes A^{*cop}$, *defined by*

$$\varphi(\gamma \otimes b) := \sum \gamma({e_i}^\Phi)b_\Phi \otimes e^i, \ \textit{where } b \in B,\ \gamma \in A^*,\ e_i \textit{ and } e^i \textit{ are dual bases of } A \textit{ and } A^*.$$

Conversely, if there an entwining map $\varphi : A^{*cop} \otimes B \to B \otimes A^{*cop}$, $\gamma \otimes b \mapsto \sum b_\varphi \otimes \gamma^\varphi$, then one can define an algebra distributive law $\Phi : B \otimes A \to A \otimes B$ via

$$\Phi(b \otimes a) := \sum e^{i\varphi}(a) e_i \otimes b_\varphi.$$

Moreover, the category of $A \otimes B$-modules is identified to $C_B^{A^{*cop}}(\varphi)$, the category of entwined modules.

Example 3.6. *(1) If we define $\varphi : H \otimes H \to H \otimes H$ by $\varphi(x \otimes y) = y_1 \otimes xy_2$, where $x, y \in H$ and H is a finite dimensional Hopf algebra, then the category $C_H^H(\varphi)$ is the category of Hopf modules. Recall from Lemma 3.2 and Theorem 3.4, if we define the following multiplication on $H^{*op} \otimes H$*

$$(\delta \otimes a)(\gamma \otimes b) := \gamma(?a_2)\delta \otimes a_1 b,$$

then $H^{*op} \otimes H$ is an associative algebra, and the category of Hopf modules of H is identified to the category of $H^{*op} \otimes H$-modules.

(2) Let H be a finite dimensional Hopf algebra and A a finite dimensional left H-module algebra. Recall that the multiplication on $A \sharp H$, the usual smash product of A and H is

$$(a\sharp x)(b\sharp y) = a(x_1 \cdot b)\sharp x_2 y, \quad \text{where } a, b \in A, \quad x, y \in H,$$

which implies there is an algebra distributive law $\Phi : H \otimes A \to A \otimes H$, $\Phi(h \otimes a) = h_1 \cdot a \otimes h_2$. Thus from Remark 3.5, there exists an entwining map $\varphi : A^{*cop} \otimes H \to H \otimes A^{*cop}$, defined by

$$\varphi(\gamma \otimes h) := \sum \gamma(h_1 \cdot e_i) h_2 \otimes e^i, \quad \text{where } h \in H, \quad \gamma \in A^*, \quad e_i \text{ and } e^i \text{ are dual bases of } A \text{ and } A^*.$$

Furthermore, the representation of $A\sharp H$ is isomorphic to the category $C_H^{A^{*cop}}(\varphi)$.

3.2. *The dual case*

The definition and results in this section are dual to the corresponding results in Section 3.1, so we will not give the complete proof.

Definition 3.7. *Let C, D be coalgebras in a monoidal category $\mathcal{C}$. A morphism $\Psi : C \otimes D \to D \otimes C$ in $\mathcal{C}$ is called a* coalgebra distributive law *if Ψ satisfying*

$$\begin{array}{ccccc} C \otimes D & & \xrightarrow{\quad\Psi\quad} & & D \otimes C \\ \downarrow{\scriptstyle id_C \otimes \Delta_D} & & & & \downarrow{\scriptstyle \Delta_D \otimes id_C} \\ C \otimes D \otimes D & \xrightarrow[\Psi \otimes id_D]{} & D \otimes C \otimes D & \xrightarrow[id_D \otimes \Psi]{} & D \otimes D \otimes C, \end{array} \qquad \begin{array}{ccc} C \otimes D & \xrightarrow{\Psi} & D \otimes C \\ & {\scriptstyle id_C \otimes \varepsilon_D}\searrow & \downarrow{\scriptstyle \varepsilon_D \otimes id_C} \\ & & C, \end{array}$$

$$\begin{array}{ccccc} C \otimes D & & \xrightarrow{\quad\Psi\quad} & & D \otimes C \\ \downarrow{\scriptstyle \Delta_C \otimes id_D} & & & & \downarrow{\scriptstyle id_D \otimes \Delta_C} \\ C \otimes C \otimes D & \xrightarrow[id_C \otimes \Psi]{} & C \otimes D \otimes C & \xrightarrow[\Psi \otimes id_C]{} & D \otimes C \otimes C, \end{array} \qquad \begin{array}{ccc} C \otimes D & \xrightarrow{\Psi} & D \otimes C \\ & {\scriptstyle \varepsilon_C \otimes id_D}\searrow & \downarrow{\scriptstyle id_D \otimes \varepsilon_C} \\ & & D. \end{array}$$

Similar to [[9], Theorem 12], we have the following property.

Lemma 3.8. *Let C be a finite dimensional coalgebra and A a finite dimensional algebra over k. Then give an entwining map $\varphi : C \otimes A \to A \otimes C$ is identified to give a coalgebra distributive law $\Psi : A^{*cop} \otimes C \to C \otimes A^{*cop}$.*

Now suppose that (C, A, φ) is a monoidal entwining datum over k where C, A are two Hopf algebras with bijective antipodes.

Definition 3.9. *The* entwined smash coproduct $A^{*cop} \otimes C$ *of* (C, A, φ)*, in a form containing* A^{*cop} *and* C*, is a Hopf algebra with the following structures:*

• *the multiplication* $\overline{m}$ *is given by*

$$(\gamma \otimes c)(\delta \otimes d) := \gamma * \delta \otimes ab,$$

where $c, d \in A$, $\gamma, \delta \in A^{*cop}$;

• *the unit is* $\overline{\eta}(1_k) = \varepsilon_A \otimes 1_C$;

• *the comultiplication is given by*

$$\overline{\Delta}(\gamma \otimes c) := \sum (\gamma_1(e_{i\varphi})\gamma_2 \otimes c_1{}^{\varphi}) \otimes (e^i \otimes c_2),$$

where e_i *and* e^i *are dual bases of* A *and* A^*

• *the counit is given by*

$$\overline{\varepsilon}(\gamma \otimes c) := \gamma(1_A)\varepsilon_C(c);$$

• *the antipode is given by*

$$\overline{S}(\gamma \otimes c) := \sum \gamma(e_{i\varphi})S_{A^*}^{-1}(e^i) \otimes S_C(c^{\varphi}).$$

Be similar with [[9], Theorem 13], we have the following property.

Proposition 3.10. $C_A^C(\varphi)$ *is monoidal isomorphic to the corepresentation category of* $A^{*cop} \otimes C$.

Corollary 3.11. *For any* $a \in A$, $c \in C$*, the following identities hold*

$$S_A^{-1}(a) \otimes S_C(c) = \sum \underline{S_A^{-1}(a_{\varphi})}_{\psi} \otimes \underline{S_C(m_1{}^{\psi})}^{\varphi}; \tag{3.1}$$

$$S_A(a) \otimes S_C^{-1}(c) = \sum \underline{S_A(a_{\varphi})}_{\psi} \otimes \underline{S_C^{-1}(c^{\psi})}^{\varphi}. \tag{3.2}$$

Proof. We only prove Eq.(3.2). For any $a \in A$, $c \in C$, $p \in C^*$, $\gamma \in A^*$, since $\widehat{S}$ is the antipode of $C^{*op} \otimes A$, we have

$$\widehat{S}((\varepsilon_c \otimes a)(p \otimes 1_A)) = \widehat{S}(p \otimes 1_A)\widehat{S}(\varepsilon_c \otimes a).$$

For one thing, we compute

$$\begin{aligned} \widehat{S}((\varepsilon_c \otimes a)(p \otimes 1_A)) &= \widehat{S}(e^i \otimes a_{\varphi})p(e_i{}^{\varphi}) \\ &= \sum p(\underline{S_C^{-1}(o_i{}^{\psi})}^{\varphi})o_i \otimes S_A(a_{\varphi})_{\psi}, \end{aligned}$$

where e_i (o_i) and e^i (o^i) are dual bases of C and C^* respectively.

For another, we have

$$\begin{aligned} \widehat{S}(p \otimes 1_A)\widehat{S}(\varepsilon_c \otimes a) &= (p(S_C^{-1}(e_i))e^i \otimes 1_A)(\varepsilon_C \otimes S_A(a)) \\ &= \sum p(S_C^{-1}(e_i))e^i \otimes S_A(a). \end{aligned}$$

Thus $\sum p(S_C^{-1}(o_i{}^{\psi})^{\varphi})o_i \otimes S_A(a_{\varphi})_{\psi} = \sum p(S_C^{-1}(e_i))e^i \otimes S_A(a)$. Indeed, we can easily get

$$\sum p(\underline{S_C^{-1}(o_i{}^{\psi})}^{\varphi})o_i(c) \otimes \gamma(S_A(a_{\varphi})_{\psi}) = \sum p(S_C^{-1}(e_i))e^i(c) \otimes \gamma(S_A(a)),$$

i.e.

$$\sum p(\underline{S_C^{-1}(c^{\psi})}^{\varphi}) \otimes \gamma(S_A(a_{\varphi})_{\psi}) = \sum p(S_C^{-1}(c)) \otimes \gamma(S_A(a)),$$

which implies Eq.(3.2).

Similarly, since

$$(\overline{S} \otimes \overline{S}) \circ \overline{\Delta}^{cop} = \overline{\Delta} \circ \overline{S},$$

Eq.(3.1) holds. □

Assume that C and A are two Hopf algebras with bijective antipodes over k, and $\varphi : C\otimes A \to A\otimes C$ is a k-linear map such that (C, A, φ) is a monoidal entwining datum.

For any $(M, \varrho_M, \rho^M) \in \mathcal{C}_A^C(\varphi)$, set $M^* = {}^*M = hom_k(M, k)$ as spaces, and define the A-action and C-coaction on M^* and ${}^*M$ by

$$\begin{aligned}
&\varrho_{M^*} : M^*\otimes A \longrightarrow M^*, \ \ (f\cdot a)(m) := f(m\cdot S_A^{-1}(a)),\\
&\rho^{M^*} : M^* \longrightarrow M^*\otimes C, \ \ f_0(m)\otimes f_1 := f(m_0)\otimes S_C(m_1),\\
&\varrho_{{}^*M} : {}^*M\otimes A \longrightarrow {}^*M, \ \ (f\cdot a)(m) := f(m\cdot S_A(h)),\\
&\rho^{{}^*M} : {}^*M \longrightarrow {}^*M\otimes C, \ \ f_0(m)\otimes f_1 := f(m_0)\otimes S_C^{-1}(m_1),
\end{aligned}$$

where $f \in hom_k(M, k)$, $a \in A$, $m \in M$, and define the evaluation map and coevaluation map by

$$\begin{aligned}
&ev_M : f\otimes m \longmapsto f(m); \quad coev_M : 1_k \longmapsto \sum_i e_i\otimes e^i;\\
&\widetilde{ev}_M : m\otimes f \longmapsto f(m); \quad \widetilde{coev}_M : 1_k \longmapsto \sum_i e^i\otimes e_i,
\end{aligned}$$

where e_i and e^i are dual bases in M and M^*. It is easy to check that M^* and ${}^*M$ are all both A-modules and C-comodules. Further, $ev, coev, \widetilde{ev}, \widetilde{coev}$ are all both A-linear and C-colinear maps.

Theorem 3.12. *$\mathcal{C}_A^C(\varphi)$ is a rigid category.*

Proof. We only show that $\mathcal{C}_A^C(\varphi)$ admit left duality, i.e., M^* is an object in $\mathcal{C}_A^C(\varphi)$ for any entwined module M. Actually, for any $\mu \in M^*$, $m \in M$, $a \in A$, we have

$$\begin{aligned}
(\mu\cdot a)_0(m)\otimes(\mu\cdot a)_1 &= \mu(m_0\cdot S_A^{-1}(a))\otimes S_C(m_1)\\
&\overset{(3.1)}{=} \sum \mu(m_0\cdot \underline{S_A^{-1}(a_\varphi)}_{\psi})\otimes \underline{S_C({m_1}^{\psi})}^{\varphi}\\
&= \sum \mu_0(m\cdot S_A^{-1}(a_\varphi))\otimes {\mu_1}^{\varphi} = \sum(\mu_0\cdot a_\varphi)(m)\otimes(\mu_1)^{\varphi}.
\end{aligned}$$

Thus $\mathcal{C}_A^C(\varphi)$ is a left rigid category.

Similarly, one can check that $\mathcal{C}_A^C(\varphi)$ is a right rigid category by using Eq.(3.2). □

4. The ribbon structure in the category of entwined modules

Now suppose that (C, A, φ, R) is a double quantum group over k, thus $\mathcal{C}_A^C(\varphi)$ is a braided category with the braiding which is defined by Eq.(2.1). We also assume that $Nat(F, F)$ means the collection of natural transformations from the forgetful functor $F : \mathcal{C}_A^C(\varphi) \to Vec_k$ to itself. Then we have the following property.

Proposition 4.1. *There is a bijective map between the algebra $Nat(F, F)$ and $hom_k(C, A)$.*

Proof. See [[11], Theorem 2.1 and Proposition 2.4]

Actually, one can define a map $\Pi : Nat(F, F) \to hom_k(C, A)$ by

$$\Pi(\theta) : C \to A, \quad c \mapsto \sum(\varepsilon_C\otimes A)\theta_{C\otimes A}(c\otimes 1_A),$$

where $\theta \in Nat(F\circ id, F\circ id)$, $c \in C$. Define $\Sigma : hom_k(C, A) \to Nat(F, F)$ by

$$\Sigma(g)_M : M \to M, \ \ m \mapsto m_0\cdot g(m_1),$$

where $g \in hom_k(C, A)$, $M \in \mathcal{C}_A^C(\varphi)$, $m \in M$. It is a direct computation to check that Π and Σ are well-defined and inverse with each other. □

From now on, assume that $\theta \in Nat(F, F)$ and $g \in hom_k(C, A)$ are in correspondence with each other.

Lemma 4.2. *θ is a natural isomorphism if and only if g is invertible under the entwined convolution.*

Proof. Straightforward from Proposition 4.1. □

Lemma 4.3. *For any $(M, \varrho_M, \rho^M) \in \mathcal{C}_A^C(\varphi)$, θ_M is A-linear if and only if g satisfies*

$$g(c)a = \sum a_\varphi g(c^\varphi), \qquad \text{for any } \ c \in C,\ a \in A. \tag{4.1}$$

Proof. ⇐: Since we have

$$\begin{aligned}\theta_M(m \cdot a) &= \sum m_0 \cdot a_\varphi g(m_1{}^\varphi)\\ &\overset{(4.1)}{=} \sum m_0 \cdot g(m_1)a = \theta_M(m)\cdot a,\end{aligned}$$

where $m \in M$, $a \in A$. Hence θ_M is an A-module morphism.

⇒: Conversely, since $\theta_{A\otimes C}$ is A-linear, for any $c \in C$ and $a \in A$, we have the following commute diagram

$$\begin{array}{ccc}(1_A \otimes c)\otimes a & \xrightarrow{\ \varrho_{A\otimes C}\ } & \sum a_\varphi \otimes c^\varphi\\ \Big\downarrow{\scriptstyle \theta_{A\otimes C}\otimes id_A} & & \Big\downarrow{\scriptstyle \theta_{A\otimes C}}\\ \sum(\underline{g(c_2)}_\varphi \otimes c_1{}^\varphi)\otimes a & \xrightarrow{\ \varrho_{A\otimes C}\ } & \sum \underline{a_\varphi \otimes c^\varphi}_0 \cdot g(\underline{a_\varphi \otimes c^\varphi}_1)\end{array}$$

which implies

$$\sum a_\varphi \underline{g(c^\varphi{}_2)}_\psi \otimes \underline{c^\varphi{}_1}^\psi = \sum \underline{g(c_2)}_\varphi a_\psi \otimes \underline{c_1}^{\varphi\psi}.$$

Take $id_A \otimes \varepsilon_C$ to action at the both side of the above equation, we immediately get Eq.(4.1). □

Lemma 4.4. *For any $(M, \varrho_M, \rho^M) \in \mathcal{C}_A^C(\varphi)$, θ_M is C-colinear if and only if g satisfies*

$$g(c_1)\otimes c_2 = \sum g(c_2)_\varphi \otimes c_1{}^\varphi, \qquad \text{for any } \ c \in C. \tag{4.2}$$

Proof. Be similar with Lemma 4.3. □

Lemma 4.5. *If θ is both A-linear and C-colinear, then g is invertible under the entwined convolution if and only if g is convolution invertible, i.e., there exists a k-linear map $g' : C \to A$, such that*

$$g(c_1)g'(c_2) = g'(c_1)g(c_2) = \varepsilon_C(c)1_A, \quad \forall c \in C.$$

Proof. ⇒: Suppose that $g^{\star -1}$ is the inverse of g under the entwined convolution. Then for any $c \in C$, we have

$$\begin{aligned}g(c_1)g^{\star-1}(c_2) &\overset{(4.1)}{=} \sum \underline{g^{\star-1}(c_2)}_\varphi g(\underline{c_1}^\varphi)\\ &= (g \star g^{\star-1})(c) = \varepsilon_C(c)1_A,\end{aligned}$$

and similarly we can obtain $g^{\star-1}(c_1)g(c_2) = \varepsilon_C(c)1_A$ from Eq.(4.2).

⇐: Conversely, if g is convolution invertible and g' is its inverse, then it is easily to get that g' is the inverse of g under the entwined convolution. □

Remark 4.6. *From Lemma 4.2 - 4.5, we immediately get that g is convolution invertible and satisfies Eqs.(4.1)-(4.2) if and only if θ is a natural isomorphism in $\mathcal{C}_A^C(\varphi)$.*

Lemma 4.7. *Suppose that θ is a natural transformation in $C_A^C(\varphi)$, then θ is a twist if and only if for any $x, y \in C$, g satisfies*

$$\Delta_A(g(xy)) = \sum g(x_1)\underline{r^{(2)}(x_3\otimes y_3)}_{\psi}R^{(1)}(\underline{y_2}^{\varphi}\otimes\underline{x_2}^{\psi})\otimes g(y_1)\underline{r^{(1)}(x_3\otimes y_3)}_{\varphi}R^{(2)}(\underline{y_2}^{\varphi}\otimes\underline{x_2}^{\psi}). \tag{4.3}$$

Proof. $\Leftarrow$: For any $M, N \in C_A^C(\varphi)$ and $m \in M$, $n \in N$, we compute that

$$\begin{aligned}
&(\mathbf{C}_{N,M}\circ\mathbf{C}_{M,N}\circ(\theta_M\otimes\theta_N))(m\otimes n)\\
=\;&(\mathbf{C}_{N,M}\circ\mathbf{C}_{M,N})(m_0\cdot g(m_1)\otimes n_0\cdot g(n_1))\\
\overset{(4.2)}{=}\;&\mathbf{C}_{N,M}(\sum(n_{00}\cdot\underline{g(n_1)}_{\varphi}\otimes m_{00}\cdot\underline{g(m_1)}_{\psi})\cdot R(\underline{m_{01}}^{\psi}\otimes\underline{n_{01}}^{\varphi}))\\
\overset{(E1)}{=}\;&\sum m_0\cdot\underline{g(m_2)}_{\varphi}\underline{R^{(2)}(m_3\otimes n_3)}_{\phi}r^{(1)}(\underline{n_1}^{\psi\chi}\otimes\underline{m_1}^{\varphi\phi})\otimes n_0\cdot\underline{g(n_2)}_{\psi}\underline{R^{(1)}(m_3\otimes n_3)}_{\chi}r^{(2)}(\underline{n_1}^{\psi\chi}\otimes\underline{m_1}^{\varphi\phi})\\
\overset{(4.2)}{=}\;&\sum m_0\cdot g(m_1)\underline{R^{(2)}(m_3\otimes n_3)}_{\varphi}r^{(1)}(\underline{n_2}^{\psi}\otimes\underline{m_2}^{\varphi})\otimes n_0\cdot g(n_1)\underline{R^{(1)}(m_3\otimes n_3)}_{\psi}r^{(2)}(\underline{n_2}^{\psi}\otimes\underline{m_2}^{\varphi})\\
\overset{(4.3)}{=}\;&(m_0\otimes n_0)\cdot(g(m_1n_1)),
\end{aligned}$$

which implies θ is a twist.

$\Rightarrow$: Conversely, for the entwined modules $C\otimes A$ and $A\otimes C$, since θ is a twist, then for any $x, y\in C$ we have

$$\begin{aligned}
&(\mathbf{C}_{A\otimes C,C\otimes A}\circ\mathbf{C}_{C\otimes A,A\otimes C}\circ(\theta_{C\otimes A}\otimes\theta_{A\otimes C}))((x\otimes 1_A)\otimes(1_A\otimes y))\\
&=\theta_{C\otimes A,A\otimes C}((x\otimes 1_A)\otimes(1_A\otimes y)),
\end{aligned}$$

Since

$$\begin{aligned}
&(\mathbf{C}_{A\otimes C,C\otimes A}\circ\mathbf{C}_{C\otimes A,A\otimes C}\circ(\theta_{C\otimes A}\otimes\theta_{A\otimes C}))((x\otimes 1_A)\otimes(1_A\otimes y))\\
\overset{(4.2)}{=}\;&(\mathbf{C}_{A\otimes C,C\otimes A}\circ\mathbf{C}_{C\otimes A,A\otimes C})((x_1\otimes g(x_2))\otimes(g(y_1)\otimes y_2))\\
=\;&\mathbf{C}_{A\otimes C,C\otimes A}(\sum(g(y_1)\underline{R^{(1)}(x_3\otimes y_3)}_{\varphi}\otimes\underline{y_2}^{\varphi})\otimes(x_1\otimes g(x_2)R^{(2)}(x_3\otimes y_3)))\\
\overset{(E1)}{=}\;&\sum x_1\otimes g(x_3)_{\psi}\underline{R^{(2)}(x_4\otimes y_3)}_{\phi}r^{(1)}(y_2{}^{\varphi}{}_2\otimes x_2{}^{\psi\phi})\otimes g(y_1)\underline{R^{(1)}(x_4\otimes y_3)}_{\varphi}\underline{r^{(2)}(y_2{}^{\varphi}{}_2\otimes x_2{}^{\psi\phi})}_{\chi}\otimes y_2{}^{\varphi}{}_1{}^{\chi},
\end{aligned}$$

and

$$\begin{aligned}
&\theta_{C\otimes A,A\otimes C}((x\otimes 1_A)\otimes(1_A\otimes y))\\
=\;&x_1\otimes\underline{g(x_1y_1)}_1\otimes\underline{g(x_1y_1)}_2\otimes y_1,
\end{aligned}$$

we have

$$\begin{aligned}
&\sum x_1\otimes g(x_3)_{\psi}\underline{R^{(2)}(x_4\otimes y_3)}_{\phi}r^{(1)}(y_2{}^{\varphi}{}_2\otimes x_2{}^{\psi\phi})\otimes g(y_1)\underline{R^{(1)}(x_4\otimes y_3)}_{\varphi}\underline{r^{(2)}(y_2{}^{\varphi}{}_2\otimes x_2{}^{\psi\phi})}_{\chi}\otimes y_2{}^{\varphi}{}_1{}^{\chi}\\
&=x_1\otimes\underline{g(x_1y_1)}_1\otimes\underline{g(x_1y_1)}_2\otimes y_1.
\end{aligned}$$

Take $\varepsilon_C\otimes id_A\otimes id_A\otimes\varepsilon_C$ to action at the both side of the above equation, we immediately get Eq.(4.3). □

Recall from Theorem 3.12, we get that $C_A^C(\varphi)$ is a rigid category. Then we get the following property.

Lemma 4.8. *θ is self-dual in $C_A^C(\varphi)$ if and only if g satisfies*

$$g(c) = \sum\underline{S_A^{-1}gS_C(c^{\varphi})}_{\varphi},\quad \textit{for any } c\in C. \tag{4.4}$$

Or equivalently,

$$g(c) = \sum a_{i\varphi}a^i(S_A^{-1}gS_C(c^{\varphi})), \quad \text{for any } \gamma \in A^*,\ c \in C, \tag{4.5}$$

where a_i and a^i are bases of A and A^* respectively, dual to each other.

Proof. $\Leftarrow$: For any object $M \in \mathcal{C}_A^C(\varphi)$, suppose that o_i and o^i are dual bases of M and M^*, a_i and a^i are dual bases of A and A^*, $\mu \in M^*$, $m \in M$, then we have

$$\begin{aligned}
\theta_{M^*}(\mu)(m) &= (\mu_0 \cdot g(\mu_1))(m) = \mu_0(m \cdot S_A^{-1}g(\mu_1)) \\
&\overset{(TR2)}{=} \sum (\varrho_M)^*(\mu_0)(m \otimes S_A^{-1}g(\mu_1)) \\
&\overset{(TR1)}{=} \sum \mu_0(o_i \cdot a_i)a^i(S_A^{-1}g(\mu_1))o^i(m) \\
&= \sum \mu(o_{i0} \cdot a_{i\varphi})a^i(S_A^{-1}gS_C(o_{i1}{}^{\varphi}))o^i(m) \\
&= \sum \mu(m_0 \cdot \underline{S_A^{-1}gS_C(m_1{}^{\varphi})}_{\varphi}) \\
&\overset{(4.4)}{=} \mu(m_0 \cdot g(m_1)) = (\theta_M)^*(\mu)(m).
\end{aligned}$$

Thus θ is self-dual in $\mathcal{C}_A^C(\varphi)$.

$\Rightarrow$: Conversely, for $(C \otimes A)^* \in \mathcal{C}_A^C(\varphi)$, since θ is self-dual, we have

$$\theta_{(C\otimes A)^*}(\gamma \otimes \varepsilon_C)(c \otimes 1_A) = (\theta_{C\otimes A})^*(\gamma \otimes \varepsilon_C)(c \otimes 1_A), \quad \text{where } \gamma \in A^*,\ c \in C.$$

For one thing, consider that

$$\begin{aligned}
(\theta_{C\otimes A})^*(\gamma \otimes \varepsilon_C)(c \otimes 1_A) &= (\gamma \otimes \varepsilon_C)((c_1 \otimes 1_A) \cdot g(c_2)) \\
&= \gamma(g(c)).
\end{aligned}$$

For another, we compute

$$\begin{aligned}
&\theta_{(C\otimes A)^*}(\gamma \otimes \varepsilon_C)(c \otimes 1_A) \\
\overset{(TR2)}{=} &\sum (\varrho_{C\otimes A})^*((\gamma \otimes \varepsilon_C)_0)((c \otimes 1_A) \otimes S_A^{-1}g((\gamma \otimes \varepsilon_C)_1)) \\
\overset{(TR1)}{=} &\sum (\gamma \otimes \varepsilon_C)_0((c_i \otimes b_i) \cdot a_i)a^i(S_A^{-1}g((\gamma \otimes \varepsilon_C)_1))(b^i \otimes c^i)(c \otimes 1_A) \\
= &\sum (\gamma \otimes \varepsilon_C)(c_{i1} \otimes \underline{b_i a_i}_{\varphi})a^i(S_A^{-1}gS_C(c_{i2}{}^{\varphi}))c^i(c)b^i(1_A) \\
= &\sum \gamma(a_{i\varphi})a^i(S_A^{-1}gS_C(c^{\varphi})) = \gamma(\underline{S_A^{-1}gS_C(c^{\varphi})}_{\varphi}),
\end{aligned}$$

where c_i and c^i are dual bases of C and C^*, a_i and a^i, b_i and b^i are two dual bases of A and A^*. Hence Eq.(4.4) holds. □

Definition 4.9. *Assume that C, A are two Hopf algebras with bijective antipodes over a field k, and (C, A, φ, R) is a double quantum group. If there exists a k-linear map $g : C \to A$, such that g is convolution invertible, and Eqs.(4.1)-(4.4) are satisfied, then g is called an* entwined ribbon morphism *over (C, A, φ, R). Further, (C, A, φ, R, g) is called a* ribbon entwined datum.

Combining Proposition 4.1 - Lemma 4.8, we get our main theorem below.

Theorem 4.10. *Assume that C, A are two Hopf algebras with bijective antipodes over k, $\varphi : C \otimes A \to A \otimes C$ and $R : C \otimes C \to A \otimes A$ are two k-linear maps such that (C, A, φ, R) is a double quantum group. Then $\mathcal{C}_A^C(\varphi)$ is a ribbon category if and only if there is an entwined ribbon morphism $g \in Hom_k(C, A)$. Moreover, the ribbon structure θ in $\mathcal{C}_A^C(\varphi)$ is defined by*

$$\theta_M : M \to M, \quad \theta_M(m) = m_0 \cdot g(m_1), \quad \text{where } m \in M$$

for any $(M, \theta_M, \rho^M) \in \mathcal{C}_A^C(\varphi)$.

Theorem 4.11. *Suppose that (C, A, φ, R) is a double quantum group where R is a map from $C \otimes C$ to $A \otimes A$. Then there exists a k-linear map $g : C \to A$ such that (C, A, φ, R, g) is a ribbon entwined datum if and only if $C^{*op} \otimes A$ is a ribbon Hopf algebra.*

Proof. $\Rightarrow$: If (C, A, φ, R, g) is a ribbon entwined datum, then the R-matrix of $C^{*op} \otimes A$ is $\sum c^i \otimes R^{(2)}(c_i \otimes e_i) \otimes e^i \otimes R^{(1)}(c_i \otimes e_i)$, where e_i and e^i, c_i and c^i are all dual bases of C and C^* respectively. Further, the ribbon element in $C^{*op} \otimes A$ is $\sum e^i \otimes g(e_i)$.

$\Leftarrow$: Conversely, if $C^{*op} \otimes A$ is a ribbon Hopf algebra with the ribbon element $L = \sum L^{(1)} \otimes L^{(2)} \in C^{*op} \otimes A$, then the entwined ribbon morphism of $\mathcal{C}_A^C(\varphi)$ is $c \mapsto \sum L^{(1)}(c)L^{(2)}$. □

Theorem 4.12. *Suppose that (C, A, φ, R) is a double quantum group where R is a map from $C \otimes C$ to $A \otimes A$. Then there exists a k-linear map $g : C \to A$ such that (C, A, φ, R, g) is a ribbon entwined datum if and only if $A^{*cop} \otimes C$ is a coribbon Hopf algebra.*

Proof. If (C, A, φ, R, g) is a ribbon entwined datum, then the coquasitriangular structure on $A^{*cop} \otimes C$ is

$$\zeta : (A^{*cop} \otimes C) \otimes (A^{*cop} \otimes C) \to k, \;\; \zeta((\gamma' \otimes c) \otimes (\gamma \otimes d)) \mapsto (\gamma' \otimes \gamma)R(c \otimes d).$$

And the coribbon form on $A^{*cop} \otimes C$ is $\gamma \otimes c \mapsto \gamma(g(c))$.

Conversely, if $A^{*cop} \otimes C$ is a coribbon Hopf algebra with the coribbon form $\Theta \in (A^{*cop} \otimes C)^*$, then the entwined ribbon morphism of $\mathcal{C}_A^C(\varphi)$ is $c \mapsto \sum \Theta(e^i \otimes c)e_i$, where e_i and e^i are dual bases of A and A^*, respectively. □

Example 4.13. *If $C = k$, $\varphi = id_A$, then the double quantum group (C, A, φ, R) becomes a quasitriangular Hopf algebra (A, R), where R means the R-matrix in A. And the entwined ribbon morphism becomes an invertible element $g \in A$ satisfies*

$$\begin{cases} (1)\ g \text{ is in the center of } A; \\ (2)\ \Delta(g) = (g \otimes g)R_{21}R; \\ (3)\ g = S(g), \end{cases}$$

which implies g is a usual ribbon element in A, thus A is a ribbon Hopf algebra.

Example 4.14. *Let k be a field and H_4 be the Sweedler's 4-dimensional Hopf algebra $H_4 = k\{1_H, e, x, y | e^2 = 1_H, x^2 = 0, y = ex = -xe\}$ with the following structure*

$$\Delta(e) = e \otimes e,\ \Delta(x) = x \otimes 1_H + e \otimes x,\ \Delta(y) = y \otimes e + 1_H \otimes y,$$
$$\varepsilon(e) = 1,\ \varepsilon(x) = \varepsilon(y) = 0,\ S(e) = e,\ S(x) = -y,\ S(y) = x.$$

Since the triangular structure in H_4 is

$$R = \frac{1}{2}(1_H \otimes 1_H + 1_H \otimes e + e \otimes 1_H - e \otimes e), \tag{4.6}$$

we immediately get that H_4 is a ribbon Hopf algebra with the ribbon element 1_H.

Example 4.15. *If $A = k$, $\varphi = id_C$, then the double quantum group (C, A, φ, R) becomes a coquasitriangular Hopf algebra (C, R). And the entwined ribbon morphism is a convolution invertible k-linear character $g \in C^*$, satisfies*

$$\begin{cases} (1)\ g(c_1)c_2 = c_1 g(c_2); \\ (2)\ g(cd) = g(c_1)g(d_1)R(c_2 \otimes d_2)R(d_3 \otimes c_3); \\ (3)\ g(c) = g(S(c)), \end{cases}$$

for any $c, d \in C$, which implies g is a coribbon form on C, thus C is a coribbon Hopf algebra.

Example 4.16. *Let k be a field and H_4 be the Sweedler's 4-dimensional Hopf algebra. Since the cotriangular structure on H_4 is*

β	1_H	g	x	y
1_H	1	1	0	0
g	1	−1	0	0
x	0	0	0	0
y	0	0	0	0

(4.7)

we immediately get that H_4 is a coribbon Hopf algebra with the coribbon form ε.

Example 4.17. *Assume that (C, A, φ, R, g) is a ribbon entwined datum. If the following identity hold*

$$\sum a_\varphi \otimes (1_C)^\varphi = a \otimes 1_C, \quad \textit{for any } a \in A,$$

then recall from Lemma 2.1 that $(A, R(1_C \otimes 1_C))$ is a quasitriangular Hopf algebra. Further, $(A, g(1_C))$ is a ribbon Hopf algebra.

Dually, if the following identity hold

$$\sum \varepsilon_A(a_\varphi)c^\varphi = \varepsilon_A(a)c, \quad \textit{for any } c \in C,\ a \in A,$$

then $(C, (\varepsilon_A \otimes \varepsilon_A) \circ R)$ is a coquasitriangular Hopf algebra. Further, $(C, \varepsilon_A \circ g)$ is a coribbon Hopf algebra.

5. Applications

5.1. Generalized Long dimodules

Suppose that H and B are both finite dimensional Hopf algebras over k, M is at the same time a right H-module and a right B-comodule. Recall that M is called a *generalized right-right Long dimodule* (see [15]) if

$$\rho(m \cdot h) = \sum m_{(0)} \cdot h \otimes m_{(1)}$$

for all $m \in M$ and $h \in H$. The category of generalized right-right Long dimodules and H-linear B-colinear homomorphisms is denoted by $\mathcal{L}_H^B$. If we define $\tau : B \otimes H \to H \otimes B$ as the flip map in Vec_k, then obviously (B, H, τ) is a monoidal entwining datum, and $\mathcal{L}_H^B = \mathcal{C}_H^B(\tau)$.

Then from Theorem 3.4, we immediately get that the category of generalized Long dimodules is identified to the representations of the Hopf algebra $B^{*op} \otimes H$. Here the bialgebra structure of $B^{*op} \otimes H$ is the ordinary bialgebra structure which is induced by the tensor product of B^{*op} and H, and the antipode is defined by

$$\overline{S}(p \otimes a) = S_{B^*}^{-1}(p) \otimes S_H(a), \ \text{ where } p \in B^*,\ a \in H.$$

Proposition 5.1. *$\mathcal{L}_H^B$ is a braided category if and only if H is a quasitriangular Hopf algebra and B is a coquasitriangular Hopf algebra.*

Proof. We only need to prove the existence of the double quantum group $(B, H, \tau, \mathbf{R})$ is equivalent to the fact that H is quasitriangular and B is coquasitriangular. Consider the following sets

$$\mathcal{P} = \{\mathbf{R} \in Hom_k(B \otimes B, H \otimes H) \mid (B, H, \tau, \mathbf{R}) \text{ is a double quantum group}\},$$

and

$$\mathcal{Q} = \{(\mathrm{R}, \beta) | \text{ where R is the quasitriangular structure in } H, \text{ and } \beta \text{ is the coquasitriangular structure on } B\}.$$

Define the map $\mathfrak{F} : \mathcal{P} \to \mathcal{Q}$ by

$$\mathfrak{F}(\mathbf{R}) = (\mathbf{R}(1_C \otimes 1_C), (\varepsilon_A \otimes \varepsilon_A) \circ \mathbf{R}), \quad \text{for any } \mathbf{R} \in \mathcal{P}.$$

Clearly $\mathfrak{F}$ is well-defined because of Lemma 2.1. Further, $\mathfrak{F}$ is invertible, and its inverse is given by $\mathfrak{F}' : \mathcal{Q} \to \mathcal{P}$,

$$\mathfrak{F}'(\mathrm{R}, \beta) : B \otimes B \longrightarrow H \otimes H$$
$$a \otimes b \longmapsto \sum \beta(a, b)\mathrm{R}^{(1)} \otimes \mathrm{R}^{(2)},$$

where $(\mathrm{R}, \beta) \in \mathcal{Q}$. Thus the conclusion holds. □

Example 5.2. *Let k be a field and H_4 be the Sweedler's 4-dimensional Hopf algebra. Recall from Example 4.14 and Example 4.16 that $\mathcal{L}_{H_4}^{H_4}$ is a braided category with the braiding:*

$$C_{M,N}(m\otimes n)=\sum\beta(m_{(1)},n_{(1)})R^{(2)}\cdot n_{(0)}\otimes R^{(1)}\cdot m_{(0)},\quad \textit{where } m\in M,\ n\in N,\ M,N\in\mathcal{L}_{H_4}^{H_4}.$$

Theorem 5.3. *$\mathcal{L}_H^B$ is a ribbon category if and only if H is a ribbon Hopf algebra and B is a coribbon Hopf algebra.*

Proof. $\Leftarrow$: Suppose the ribbon element in H is ξ, the coribbon form on B is ζ. Define a k-linear map $g: B\to H$ by

$$g(b):=\zeta(b)\xi,\quad \text{for any } b\in B,$$

it is easy to check that g satisfies Eqs.(4.1)-(4.4). Since Theorem 4.10, the conclusion hold.

$\Rightarrow$: Straightforward from Example 4.17. □

Example 5.4. *Let k be a field and H_4 be the Sweedler's 4-dimensional Hopf algebra. Recall from Example 4.14 and Example 4.16 that $\mathcal{L}_{H_4}^{H_4}$ is a ribbon category and its ribbon structure is id.*

5.2. ***Yetter-Drinfel'd modules***

Let H be a finite dimensional Hopf algebra over k. Recall that if M is both a right H-module and a right H-comodule, and satisfies

$$\rho(m\cdot h)=\sum m_{(0)}\cdot h_2\otimes S(h_1)m_{(1)}h_3$$

for any $h\in H, m\in M$, then M is a *right-right Yetter-Drinfel'd module*. The category of Yetter-Drinfel'd modules and H-linear H-colinear homomorphisms is denoted by $\mathcal{YD}_H^H$.

If we define

$$\begin{aligned}\ddot{\varphi}: H\otimes H&\longrightarrow H\otimes H\\ c\otimes a&\longmapsto\sum a_{\ddot{\varphi}}\otimes c^{\ddot{\varphi}}:=a_2\otimes S(a_1)ca_3.\end{aligned}$$

It is straightforward to show $\ddot{\varphi}$ is a right-right entwining structure, and $C_H^H(\ddot{\varphi})=\mathcal{YD}_H^H$. Further, it is easy to see that the entwined smash product $H^{*op}\otimes H$ is the Drinfel'd double of H, and the entwined smash coproduct of $(H,H,\ddot{\varphi})$ is the *Drinfel'd codouble* (see [16], Section 10) of H.

Since $\mathcal{YD}_H^H$ is a braided category with the braiding

$$t_{M,N}: M\otimes N\to N\otimes M,\ \ m\otimes n\mapsto n_{(0)}\otimes m\cdot n_{(1)},\ \text{where } M,N\in\mathcal{YD}_H^H,\ m\in M,\ n\in N,$$

we immediately get that $(H,H,\ddot{\varphi},\mathbf{R})$ is a double quantum group, where $\mathbf{R}$ is defined by

$$\begin{aligned}\mathbf{R}: H\otimes H&\longrightarrow H\otimes H\\ a\otimes b&\longmapsto 1_H\otimes\varepsilon(a)b.\end{aligned}$$

Then we have the following result from Theorem 4.10.

Theorem 5.5. *$\mathcal{YD}_H^H$ is a ribbon category if and only if there is a k-linear map $g: H\to H$, which is convolution invertible and satisfies the following identities for any $a,b\in H$:*

$$\begin{cases}(1)\ g(b)a=a_2g(S(a_1)ba_3);\\ (2)\ g(a_1)\otimes a_2=\underline{g(a_2)}_2\otimes S(\underline{g(a_2)}_1)a_1\underline{g(a_2)}_3;\\ (3)\ \Delta(g(ab))=g(a_1)b_3\otimes g(b_1)S(b_2)a_2b_4;\\ (4)\ g(b)=\sum\underline{a_i}_2a^i(S^{-1}gS(S(\underline{a_i}_1)b\underline{a_i}_3)),\end{cases}$$

where a_i and a^i are bases of A and A^* respectively, dual to each other.

Recall from [16] that if (H,R) is a quasitriangular Hopf algebra, then any $M \in \mathcal{M}_H$ can be seen as an object in $\mathcal{YD}_H^H$ by the coaction defined by

$$\rho^M(m) = \sum m \cdot R^{(2)} \otimes R^{(1)}, \text{ for any } m \in M.$$

Hence $\mathcal{M}_H \subseteq \mathcal{YD}_H^H$. We denote this subcategory by $\mathcal{MYD}_H^H$. Dually, if (H,β) is a coquasitriangular Hopf algebra, then $\mathcal{M}^H \subseteq \mathcal{YD}_H^H$ by the following H-action

$$h \cdot m = \beta(h, m_{(1)})m_{(0)}, \text{ for any } M \in \mathcal{M}^H,\ h \in H,\ m \in M.$$

We denote this subcategory of $\mathcal{YD}_H^H$ by $\mathcal{CYD}_H^H$.

Proposition 5.6. *(1) If (H,R,ξ) is a ribbon Hopf algebra, then $\mathcal{MYD}_H^H$ is a ribbon category;*
(2) If (H,β,ζ) is a coribbon Hopf algebra, then $\mathcal{CYD}_H^H$ is a ribbon category.

Proof. (1) If the ribbon element in H is ξ, then we can define a k-linear map $g : H \to H$ via $g(x) = \varepsilon(x)\xi$. It is a direct computation to check that g is an entwined ribbon morphism of $\mathcal{MYD}_H^H$.

(2) Similarly, if the coribbon form on H is ζ, then we can define define $g' : H \to H$ by $g'(x) = \zeta(x)1_H$. It is easy to check that g' also satisfies Eqs.(4.1)-(4.4). □

Example 5.7. *Let H_4 be the Sweedler's 4-dimensional Hopf algebra. After a direct calculation, we get that $\mathcal{YD}_{H_4}^{H_4}$ is not a ribbon category. However, $\mathcal{MYD}_{H_4}^{H_4}$ and $\mathcal{CYD}_{H_4}^{H_4}$ are both ribbon categories.*

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