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MAD Families, $P^+(\mathcal{I})$-Ideals and Ideal Convergence

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Abstract. Let $\mathcal{I}$ be an ideal on ω, the notion of $\mathcal{I}$-AD family was introduced in [3]. Analogous to the well studied ideal $\mathcal{I}(\mathcal{A})$ generated by almost disjoint families, we introduce and investigate the ideal $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$. It turns out that some properties of $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ depends on the structure of $\mathcal{I}$. Denoting by $\mathfrak{a}(\mathcal{I})$ the minimum of the cardinalities of infinite $\mathcal{I}$-MAD families, several characterizations for $\mathfrak{a}(\mathcal{I}) \geq \omega_1$ will be presented. Motivated by the work in [23], we introduce the cardinality $\mathfrak{s}_{\omega,\omega}(\mathcal{I})$, and obtain a necessary condition for $\mathfrak{s}_{\omega,\omega}(\mathcal{I}) = \mathfrak{s}(\mathcal{I})$. As an application, we show finally that if $\mathfrak{a}(\mathcal{I}) \geq \mathfrak{s}(\mathcal{I})$, then BW property coincides with Helly property.

1. Introduction

Let ω denote the set of all natural numbers, and we are implicitly identifying a natural number $n \in \omega$ with the set $\{0, 1, \cdots, n-1\}$. An ideal on ω is a family of subsets of ω closed under taking finite unions and subsets of its elements. By *Fin* we denote the ideal of all finite subsets of ω. If not explicitly said we assume that all considered ideals are proper (not equal to $\mathcal{P}(\omega)$) and contain *Fin*. For convenience, we fix some notations: $\mathcal{I}^+ = \{A \subseteq \omega : A \notin \mathcal{I}\}$; $\mathcal{I}^* = \{A \subseteq \omega : \omega \setminus A \in \mathcal{I}\}$; for each $A \in \mathcal{I}^+$, let $\mathcal{I}|A = \{I \cap A : I \in \mathcal{I}\}$; $A \subseteq^{\mathcal{I}} B$ if $A \setminus B \in \mathcal{I}$, where A, B are subsets of ω.

A family $\mathcal{A}$ of infinite subsets of ω is called *almost disjoint* (AD-family, in short) if for any different elements $A, B \in \mathcal{A}$, $A \cap B$ is finite. Moreover, if for any infinite $X \subseteq \omega$, there is $A \in \mathcal{A}$ such that $A \cap X$ is infinite, then $\mathcal{A}$ is called a *maximal almost disjoint family* (MAD-family, in short).

The following notions are generalizations of almost disjoint families and maximal almost disjoint families, respectively. They were introduced by Farkas and Soukup, and were extensively studied in, e.g., [4, 14, 17, 21].

Definition 1.1. ([3]) Let $\mathcal{I}$ be an ideal on ω, and let $\mathcal{A} \subseteq \mathcal{I}^+$ be an infinite family.

- $\mathcal{A}$ is called an $\mathcal{I}$-almost disjoint family ($\mathcal{I}$-AD, in short) if $(\forall A, B \in \mathcal{A})(A \cap B \in \mathcal{I})$.
- $\mathcal{A}$ is an $\mathcal{I}$-maximal almost disjoint family ($\mathcal{I}$-MAD, in short) if it is an $\mathcal{I}$-AD family and not properly included in any larger $\mathcal{I}$-AD family or equivalently, $(\forall X \in \mathcal{I}^+)(\exists A \in \mathcal{A})(X \cap A \in \mathcal{I}^+)$.

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Denoting by $\mathfrak{a}(\mathcal{I})$ the minimum of the cardinalities of infinite $\mathcal{I}$-MAD families. In addition, if $\mathcal{I}$ is an analytic P-ideal, let $\bar{\mathfrak{a}}(\mathcal{I})$ be the minimum of cardinalities of uncountable $\mathcal{I}$-MAD families.

The motivation of this note is to investigate the influence of $\mathcal{I}$-AD families on ideal convergence. To be specific, we consider the relation among $\mathcal{I}$-AD families, ideal version of Bolzano-Weierstrass property and ideal version of Helly property.

Definition 1.2. ([24]) Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, and let X be a topological space. We say that X has the $(\mathcal{I},\mathcal{J})$-*BW* property if for any sequence $\langle x_n : n \in \omega\rangle$ from X, there exists $A \in \mathcal{I}^+$ such that $\langle x_n : n \in A\rangle$ is $\mathcal{J}$-convergent (i.e, there is x such that for each open neighborhood U of x, $\{n \in A : x_n \notin U\} \in \mathcal{J}$).

Most of time, we are considering $X = [0, 1]$. In such case, if $[0, 1]$ has the $(\mathcal{I},\mathcal{I})$-*BW* property, we write $\mathcal{I} \in BW$. If $[0, 1]$ has the $(\mathcal{I}, Fin)$-*BW* property, we write $\mathcal{I} \in FinBW$. These notations were introduced first in [6].

Recall that $\mathcal{S} \subseteq [\omega]^\omega$ is an (ω,ω)-splitting family if for any countable family $\{X_n : n \in \omega\} \subseteq [\omega]^\omega$, there exists $S \in \mathcal{S}$ such that both of $\{n : |S \cap X_n| = \omega\}$ and $\{n : |X_n \cap (\omega \setminus S)| = \omega\}$ are infinite. Denoting by $\mathfrak{s}_{\omega,\omega}$ the smallest size of (ω,ω)-splitting families [23]. For the cardinality $\mathfrak{s}$ and its variation $\mathfrak{s}(\mathcal{I})$, one may refer to [7]. By proving $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$, Mildenberger, Raghavan and Steprāns partially answer an open question of Shelah, one can refer to [22] for details.

In Section 3, the cardinality $\mathfrak{s}_{\omega,\omega}(\mathcal{I})$ will be introduced. We obtain a necessary condition for $\mathfrak{s}_{\omega,\omega}(\mathcal{I}) = \mathfrak{s}(\mathcal{I})$ by showing that for any ideal $\mathcal{I}$, if $\mathcal{I} \notin BW$, then $\mathfrak{s}_{\omega,\omega}(\mathcal{I}) \neq \mathfrak{s}(\mathcal{I})$ (see Theorem 3.7).

An ideal $\mathcal{I}$ is called *selective* if for every $\subseteq$-decreasing family $\{Y_n : n \in \omega\} \subseteq \mathcal{I}^+$ there is $Y = \{x_n : n \in \omega\} \in \mathcal{I}^+$ such that $Y \subseteq Y_0$ and $Y \setminus (x_n + 1) \subseteq Y_{x_n}$ (Y is called a *diagonalization* of $\{Y_n : n \in \omega\}$). It is well known that for every AD-family $\mathcal{A}$, $\mathcal{I}(\mathcal{A})$ is selective. This result is due to Mathias [19]. We are interested in the question that is there some analogous results for $\mathcal{I}$-AD families. In Section 4, we show that the answer depends on the construction of $\mathcal{I}$. In particular, we exam the relation between $P^+(\mathcal{I})$-ideals and the $P^+((\mathcal{I}\text{-}\mathcal{A}))$-ideals (see Theorem 4.2).

The classic Helly theorem asserts that for any sequence of real-valued functions $\langle f_n : n \in \omega\rangle$ that is uniformly bounded and monotone, there is a subsequence $\langle f_{n_k} : k \in \omega\rangle$ which is pointwise convergent. The ideal version of Helly theorem was considered by Filipów, Mrożek, Recław and Szuca in [5]. They showed that for any ideal $\mathcal{I}$ on ω, if $\mathcal{I}$ can be extended to an F_σ-ideal or maximal P-ideal, then for any sequence of real value functions $\langle f_n : n \in \omega\rangle$ that is uniformly bounded and monotone, there exists $A \in \mathcal{I}^+$ such that the subsequence $\langle f_n : n \in A\rangle$ is pointwise convergent ([5], Theorem 5.8). Note that every analytic P-ideal with the *BW* property can be extended to an F_σ-ideal ([6], Theorem 4.2). Thus, for every analytic P-ideal $\mathcal{I}$ with the *BW* property, the ideal version of Helly theorem holds.

Let $\mathbb{R}^\mathbb{R}$ be the set of all functions: $\mathbb{R} \to \mathbb{R}$ endowed with the Tychonoff product topology, and let $UBM(\mathbb{R})$ be the set of all sequences from $\mathbb{R}^\mathbb{R}$ that are uniformly bounded and monotone.

Definition 1.3. Let $\mathcal{I}$ be an ideal on ω. We say that $\mathcal{I}$ has the Helly property, and write $\mathcal{I} \in Helly$, if for every sequence $\langle f_n : n \in \omega\rangle$ from $UBM(\mathbb{R})$, there exists $A \in \mathcal{I}^+$ such that $\langle f_n : n \in A\rangle$ is $\mathcal{I}$-convergent. Moreover, if for each $A \in \mathcal{I}^+$, $\mathcal{I}|A \in Helly$, then we say $\mathcal{I}$ has hereditarily Helly property, and write $\mathcal{I} \in hHelly$.

According to these notations, the Helly theorem can be reformed as $Fin \in Helly$, and the ideal version of Helly theorem can be restated as follows: If $\mathcal{I}$ can be extended to an F_σ-ideal or maximal P-ideal then $\mathcal{I} \in Helly$.

It is well known that $\mathcal{I} \in hBW$ if, and only if $\mathcal{I} \in hHelly$ ([5], Theorem 5.9), and we are asked that if $\mathcal{I} \in BW \Rightarrow \mathcal{I} \in Helly$([5], Problem 5.10). In Section 5, we consider this question, and one of our main results can be viewed as a very partial answer to this question (see Theorem 5.6).

2. Preliminaries

We use the standard notions of Set theory. For a nonempty set X, let $|X|$ be the cardinality of X. Let $[X]^{<\omega}$ be the set of all finite subsets of X, and let $\mathcal{P}(X)$ be the power set of X.

A family $\mathcal{S}$ of infinite subsets of ω is called an $\mathcal{I}$-splitting if for every $A \in \mathcal{I}^+$ there exists $S \in \mathcal{S}$ such that $A \cap S \in \mathcal{I}^+$ and $A \setminus S \in \mathcal{I}^+$ [6]. Denoting by $\mathfrak{s}(\mathcal{I})$ the smallest size of $\mathcal{I}$-splitting families, it has been showed that $\mathcal{I} \in BW$ if and only if $\mathfrak{s}(\mathcal{I}) \geq \omega_1$ ([6], Theorem 5.1).

2.1. The Ideal $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$

Let $\mathcal{I}$ be an ideal on ω, and let $\mathcal{A}$ be an infinite $\mathcal{I}$-AD family. Put

$$\mathcal{I}(\mathcal{I}\text{-}\mathcal{A}) = \{I \subset \omega : \exists \mathcal{B} \in [\mathcal{A}]^{<\omega}(I \subseteq^{\mathcal{I}} \bigcup \mathcal{B})\},$$

it is easy to see that $\mathcal{I} \subset \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ and $\mathcal{A} \subseteq \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$. Note that for any $A, B \in \mathcal{I}^*$, $A \cap B \in \mathcal{I}^+$, so every $\mathcal{I}$-AD family disjoint with $\mathcal{I}^*$, and so there is no single $A \in \mathcal{A}$ such that $\omega \subseteq^{\mathcal{I}} A$. Indeed, we have the following result that says $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ is an ideal that strictly extends $\mathcal{I}$.

Lemma 2.1. *Let $\mathcal{I}$ be an ideal on ω, and let $\mathcal{A}$ be an infinite $\mathcal{I}$-AD family. Then $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ is an ideal on ω.*

Proof. It is easy to see that $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ is closed under taking subsets and finite unions. Suppose that $\omega \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$, we may assume there are $A, B \in \mathcal{A}$ such that $\omega = A \cup B$. Note that $\mathcal{A}$ is infinite, there exists $C \in \mathcal{A} \setminus \{A, B\}$. According to the definition of $\mathcal{I}$-AD family, both of $C \cap A$ and $C \cap B$ belong to $\mathcal{I}$. Thus, $C = (C \cap A) \cup (C \cap B) \in \mathcal{I}$, contradiction. □

Corollary 2.2. *For any ideal $\mathcal{I}$ on ω, neither $\mathcal{I}^+$ nor $\mathcal{I}^*$ is an $\mathcal{I}$-AD family.*

2.2. Submeasure

Recall that a submeasure on ω is a map ϕ: $\mathcal{P}(\omega) \to [0, \infty]$ that satisfying the following conditions:

(1) $\phi(\emptyset) = 0$;

(2) $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$ holds for every $A, B \subset \omega$.

Moreover, if for every $A \subset \omega$,

(3) $\phi(A) = limit_{n\to\infty}\phi(A \cap n)$,

then ϕ is called *lower semicontinuous* (*lsc*, in short). For any given *lsc* submeasure ϕ, define

$$Fin(\phi) = \{A \subset \omega : \phi(A) \text{ is } finite\}.$$

It is easy to see that $Fin(\phi)$ is an ideal. Mazur showed that every F_σ-ideal has the following useful characterization via lower semicontinuous submeasures.

Theorem 2.3. ([18]) *Let $\mathcal{I}$ be an ideal on ω. Then $\mathcal{I}$ is an F_σ-ideal if and only if $\mathcal{I} = Fin(\phi)$ for some lsc submeasure ϕ on ω.*

3. Splitting Families

An ideal $\mathcal{I}$ is called *dense* (or, *tall*) if for any $X \in [\omega]^\omega$ there exists $B \subseteq X$ such that $B \in \mathcal{I}$ and $B \in [\omega]^\omega$. Analogously, we introduce the following general notion.

Definition 3.1. Let $\mathcal{A}, \mathcal{B}$ be subsets of $\mathcal{P}(\omega)$. We say that $\mathcal{B}$ is $\mathcal{A}$-dense if for each $A \in \mathcal{A}$, there exists $B \subseteq A$ such that $B \in \mathcal{A}$ and $B \in \mathcal{B}$.

Let $\mathcal{A}$ be the set $[\omega]^\omega$, and let $\mathcal{B}$ be an ideal on ω. Then $\mathcal{B}$ being $\mathcal{A}$-dense coincides with $\mathcal{B}$ being dense.

Definition 3.2. Let $\mathcal{I}$ be an ideal on ω, and let $\mathcal{A}$ be an $\mathcal{I}$-AD family. Define

- $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++} = \{X \subseteq \omega : (\exists \mathcal{B} \in [\mathcal{A}]^\omega)(\forall B \in \mathcal{B})(X \cap B \in \mathcal{I}^+)\}$.

- $(\mathcal{I}\text{-}\mathcal{A})^\perp = \{X \subset \omega : (\forall A \in \mathcal{A})(X \cap A \in \mathcal{I})\}$.

Definition 3.3. ([12]) Let $\mathcal{I}$ be an ideal on ω, $\mathcal{I}$ is called *decomposable* if there is an infinite partition $\{A_n : n \in \omega\} \subset \mathcal{I}^+$ of ω such that for every $A \subseteq \omega$, $A \in \mathcal{I}$ if and only if $A \cap A_n \in \mathcal{I}$ for all $n \in \omega$. $\mathcal{I}$ is called *indecomposable* if it is not decomposable.

Lemma 3.4. *Let $\mathcal{I}$ be an ideal on ω, the following conditions are equivalent:*

(1) *$\mathcal{I}$ is decomposable;*

(2) *There exists an infinite countable $\mathcal{I}$-AD family such that $\mathcal{I} = (\mathcal{I}\text{-}\mathcal{A})^\perp$;*

(3) *$\mathfrak{a}(\mathcal{I}) = \omega$.*

Proof. (1) $\Leftrightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Assume that there exists an $\mathcal{I}$-AD family $\mathcal{A} = \{A_n : n \in \omega\}$ such that $\mathcal{I} = (\mathcal{I}\text{-}\mathcal{A})^\perp$. It is easy to see that $\mathcal{A}$ is an $\mathcal{I}$-MAD family, this implies $\mathfrak{a}(\mathcal{I}) = \omega$. Indeed, for any $A \in \mathcal{I}^+$, $A \notin (\mathcal{I}\text{-}\mathcal{A})^\perp$. So there is $n \in \omega$ such that $A \cap A_n \in \mathcal{I}^+$. This show that $\mathcal{A}$ is maximal.

(3) $\Rightarrow$ (2) Assume that $\mathcal{A} = \{A_n : n \in \omega\}$ is an $\mathcal{I}$-MAD family. $\mathcal{I} \subseteq (\mathcal{I}\text{-}\mathcal{A})^\perp$ is clear. If $A \in (\mathcal{I}\text{-}\mathcal{A})^\perp$, then $A \cap A_n \in \mathcal{I}$ for each $n \in \omega$. By the maximality of $\mathcal{A}$, we have that $A \in \mathcal{I}$. □

The following observations are evident.

Proposition 3.5. *Let $\mathcal{I}$ be an ideal on ω, and let $\mathcal{A}$ be an $\mathcal{I}$-AD family. Then*

(1) *$(\mathcal{I}\text{-}\mathcal{A})^\perp \bigcap \mathcal{I}^+ \subseteq \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$;*

(2) *If $A \subseteq B \in (\mathcal{I}\text{-}\mathcal{A})^\perp$ then $A \in (\mathcal{I}\text{-}\mathcal{A})^\perp$.*

The following properties $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ are analogous to that of the ideal $\mathcal{I}(\mathcal{A})$ ([9], Lemma 18).

Lemma 3.6. *Let $\mathcal{I}$ be an ideal on ω, and let $\mathcal{A}$ be an $\mathcal{I}$-AD family. Then*

(1) *$\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++} \subseteq \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$;*

(2) *$\mathcal{A}$ is an $\mathcal{I}$-MAD family if and only if $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ is $\mathcal{I}^+$-dense.*

(3) *$\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++} = \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$ if and only if $\mathcal{A}$ is an $\mathcal{I}$-MAD family.*

Proof. (1) is obvious.

(2) Assume that $\mathcal{A}$ is an $\mathcal{I}$-MAD family. For every $X \in \mathcal{I}^+$, by the maximality of $\mathcal{A}$, there exists $A \in \mathcal{A}$ such that $X \cap A \in \mathcal{I}^+$. Clearly, $X \cap A \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$.

If $X \in \mathcal{I}^+$, since $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ is $\mathcal{I}^+$-dense, there exists $B \subset X$ such that $B \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ and $B \in \mathcal{I}^+$. So there exists a finite $\mathcal{B} \in [\mathcal{A}]^{<\omega}$ such that $B \subseteq^{\mathcal{I}} \bigcup \mathcal{B}$. We may assume that $\mathcal{B} = \{B_{n_i} : i \leq k\}$ for some $k \in \omega$, then there exists some $i \leq k$ such that $B_{n_i} \cap X \in \mathcal{I}^+$, and then $X \notin \mathcal{A}$. This implies the maximality of $\mathcal{A}$.

(3) Assume that $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++} = \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$. By the item (2), we need to show that $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ is $\mathcal{I}^+$-dense. Note that for any $X \in \mathcal{I}^+$, if $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$, we need to do nothing, so we may assume that $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$, and so there is an infinite set $\{X_n : n \in \omega\} \subseteq \mathcal{A}$ such that $X \cap X_n \in \mathcal{I}^+$ for all $n \in \omega$. Hence, $X \cap X_n \in \mathcal{I}^+ \cap \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ for each $n \in \omega$.

Now we assume that $\mathcal{A}$ is an $\mathcal{I}$-MAD family. Let $X \notin \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$, and let $\mathcal{B} = \{A \in \mathcal{A} : A \cap B \in \mathcal{I}^+\}$. Then $\mathcal{B}$ is finite, according to this, we may assume that $\mathcal{B}$ can be enumerated as $\{A_i : i \leq n\}$. Let $Y = X \setminus \bigcup_{i \leq n} A_i$, then $Y \in (\mathcal{I}\text{-}\mathcal{A})^\perp$. Thanks to the assumption that $\mathcal{A}$ is an $\mathcal{I}$-MAD family, we have that $Y \in \mathcal{I}$, and so $X \subseteq^{\mathcal{I}} \bigcup_{i \leq n} A_i$. This implies that $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$. □

The following definitions are motivated by (ω, ω)-splitting families and $\mathfrak{s}_{\omega,\omega}$ mentioned previously.

Definition 3.7. Let $\mathcal{I}$ be an ideal on ω. Define

- $\mathcal{S} \subseteq [\omega]^\omega$ is an $\mathcal{I}$-(ω,ω)-splitting family if for every countable collection $\{X_n : n \in \omega\} \subset \mathcal{I}^+$ there exists $S \in \mathcal{S}$ such that both of $\{n : X_n \cap S \in \mathcal{I}^+\}$ and $\{n : X_n \cap (\omega \setminus S) \in \mathcal{I}^+\}$ are infinite.

- $\mathfrak{s}_{\omega,\omega}(\mathcal{I}) = \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega \wedge \mathcal{S}$ *is an* $\mathcal{I}$-(ω,ω)-*splitting family*$\}$.

Theorem 3.8. *Let $\mathcal{I}$ be an ideal on ω. If $\mathfrak{s}_{\omega,\omega}(\mathcal{I}) = \mathfrak{s}(\mathcal{I})$, then $\mathcal{I} \in BW$.*

Proof. Let $\mathcal{S}$ be an $\mathcal{I}$-(ω,ω)-splitting family such that $|\mathcal{S}| = \mathfrak{s}_{\omega,\omega}(\mathcal{I})$.

Claim 3.9. *For every $\mathcal{I}$-AD family $\mathcal{A} \subset \mathcal{I}^+$, $\mathcal{S}$ is an $\mathcal{I}(\mathcal{I}$-$\mathcal{A})$-splitting family.*

Proof. **Case 1** If $\mathcal{A}$ is an $\mathcal{I}$-MAD family. For $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$, there exists $\{X_n : n \in \omega\} \subseteq \mathcal{A}$ such that $\{X \cap X_n : n \in \omega\} \subset \mathcal{I}^+$. Since $\mathcal{S}$ is an $\mathcal{I}$-(ω,ω)-splitting family, there exists $S \in \mathcal{S}$ such that $\{n : S \cap (X \cap X_n) \in \mathcal{I}^+\}$ and $\{n : (\omega \setminus S) \cap (X \cap X_n) \in \mathcal{I}^+\}$ are infinite. Thus, both of $S \cap X$ and $X \cap (\omega \setminus S)$ are in $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$.

Case 2. If $\mathcal{A}$ not is an $\mathcal{I}$-MAD family, for $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$, there are two subcases:

Subcase 1 $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$. In this case we just do with the same argument as the Case 1.

Subcase 2 If $X \notin \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$, we can extend $\mathcal{A}$ to be an $\mathcal{I}$-MAD family $\mathcal{A}'$ such that $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A}')^+$ as follows: note that $X \notin \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$, there exists a finite family $\{A_0, A_1, \cdots, A_n\} \subset \mathcal{A}$ such that for each $A \in \mathcal{A} \setminus \{A_0, A_1, \cdots, A_n\}$, $A \cap X \in \mathcal{I}$. Take

$$\tilde{X} = X \setminus \bigcup_{k \le n} A_k.$$

Since $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$, $\tilde{X} \in \mathcal{I}^+$. Let $\{Y_n : n \in \omega\} \subseteq \mathcal{I}^+$ be a partition of $\tilde{X}$. Clearly, $\mathcal{A} \bigcup \{Y_n : n \in \omega\}$ is also an $\mathcal{I}$-AD family. Extending it to an $\mathcal{I}$-MAD family $\mathcal{A}'$, we have that $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A}')^{++}$ because of $Y_n \cap X \in \mathcal{I}^+$ for each $n \in \omega$. By the Case 1, there exists $S \in \mathcal{S}$ such that $X \cap S \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A}')^+$, and $X \cap (\omega \setminus S) \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A}')^+$. Notice that $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A}')^+ \subseteq \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$, we finish the proof of the Claim. □

Let $\mathcal{A}$ be an $\mathcal{I}$-AD family that is not maximal. By Lemma 3.5(2), $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ is not dense, and then it has BW property ([7], Lemma 3.5). According to the Claim above, $\mathcal{S}$ is an $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$-splitting family. But Theorem 5.1 in [6] tell us that for any ideal $\mathcal{I}$, it has BW property if, and only if there is no countable $\mathcal{I}$-splitting family. So,

$$\mathfrak{s}(\mathcal{I}) = \mathfrak{s}_{\omega,\omega}(\mathcal{I}) = |\mathcal{S}| > \omega.$$

Again, by Theorem 5.1 in [6] mentioned above, $\mathcal{I} \in BW$. □

Remark 3.10. It has been proved in [22] that $\mathfrak{s}_{\omega,\omega} = \mathfrak{s}$, but how about the $\mathfrak{s}_{\omega,\omega}(\mathcal{I})$ and $\mathfrak{s}(\mathcal{I})$. Our result shows that the if $\mathcal{I} \notin BW$, then $\mathfrak{s}_{\omega,\omega}(\mathcal{I}) \neq \mathfrak{s}(\mathcal{I})$.

4. $P^+(\mathcal{I})$-Ideals

Definition 4.1. Let $\mathcal{I}$ be an ideal on ω. $\mathcal{I}$ is called a $P^+(\mathcal{I})$-ideal if for any $\subseteq$-decreasing sequence $\langle A_n : n \in \omega \rangle$ from $\mathcal{I}^+$ there exists $A \in \mathcal{I}^+$ such that $A \setminus A_n \in \mathcal{I}$ for every $n \in \omega$.

It is easy to see that the $P^+(\mathcal{I})$-ideal coincides with the notion of σ-closed in $\mathcal{P}(\omega)/\mathcal{I}$ (see [12]), and coincides with the notion of $P(\mathcal{I})$-coideal defined in [5].

Let $\mathcal{I}$ be an ideal on ω, the game $G_3(\mathcal{I})$ is defined as follows: In the step n, Player I chooses $X_n \in \mathcal{I}^+$, and Player II chooses $F_n \in [X_n]^{<\omega}$. Player II wins if $\bigcup_{n \in \omega} F_n \in \mathcal{I}^+$. Otherwise, the Player I wins (see [16]).

Theorem 4.2. *Let $\mathcal{I}$ be an ideal on ω, $\mathcal{A}$ being an $\mathcal{I}$-AD family. Consider the following conditions:*

(1) *$\mathcal{I}$ is an F_σ-ideal;*

(2) *Player II has a winning strategy in $G_3(\mathcal{I})$;*

(3) $\mathcal{I}$ *is a* P^+*-ideal;*

(4) $\mathcal{I}$ *is a* $P^+(\mathcal{I})$*-ideal;*

(5) $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$ *is a* $P^+(\mathcal{I}(\mathcal{I}\text{-}\mathcal{A}))$*-ideal;*

(6) $[0,1]$ *has the* $(\mathcal{I}(\mathcal{I}\text{-}\mathcal{A}),\mathcal{I})$*-BW property;*

(7) $\mathfrak{a}(\mathcal{I}) > \omega$.

(1) ⇒ (2) ⇒ (3) ⇒ (4) ⇔ (5) ⇔ (6) ⇔ (7).

Before giving proofs, we point out that if $\mathcal{I}$ is analytic, then (2) implies (1) ([20], Theorem 3.2.13). If $\mathcal{I}$ is a P^+_{tower}-ideal, then (4) ⇒ (3) ([12], Theorem 3.8 (1)).

Proof. (1) ⇒ (2) see Theorem 3.2.13 in [20], we present here its proof for the sake of completeness. Let $\mathcal{I}$ be an F_σ-ideal, by Theorem 2.2, there exists a lower semicontinuous submesure ϕ such that $\mathcal{I} = \{A \subset \omega : \phi(A) < \infty\}$. We define a strategy σ for Player II as the form

I	X_0		X_1		$\cdots$	X_n		$\cdots$
II		$\sigma(X_0)$		$\sigma(X_0, X_1)$	$\cdots$		$\sigma(X_0, \cdots, X_n)$	$\cdots$

such that for each $n \in \omega$,

- $X_n \in \mathcal{I}^+$;
- $\sigma(X_0, \cdots, X_n) \in [X_n]^{<\omega}$;
- $\phi(\sigma(X_0, \cdots, X_n)) \geq n$.

The last item is possible since $\phi(X_n) = \infty$ and ϕ is lower semicontinuous. It is easy to check that the Player II will win according to this strategy.

(2) ⇒ (3) Assume that σ is a winning strategy for the Player II. Let $\{X_n : n \in \omega\} \subseteq \mathcal{I}^+$ such that $X_0 \supseteq X_1 \supseteq \cdots$. We define a run of Player I in $G_3(\mathcal{I})$ as form:

I	X_0		X_1		$\cdots$	X_n		$\cdots$
II		$\sigma(0)$		$\sigma(1)$	$\cdots$		$\sigma(n)$	$\cdots$

such that for each $n \in \omega$, $\sigma(n) \in [X_n]^{<\omega}$. Since the Player II win this run, $\bigcup_{n\in\omega} \sigma(n) \in \mathcal{I}^+$. In addition, it is obvious that $\bigcup_{n\in\omega} \sigma(n) \subseteq^* X_n$ for all $n \in \omega$.

(3) ⇒ (4) is evident.

(4) ⇒ (5) Let $\{Y_n : n \in \omega\} \subset \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$ such that $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$. There are two possible cases.

Case 1 If there are infinitely many n, $Y_n \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$, we may assume that for each $n \in \omega$, $Y_n \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$. Otherwise, we remove off these not in $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$. For Y_0, there is a countable family $\{A_n : n \in \omega\}$ such that $Y_0 \cap A_n \in \mathcal{I}^+$ for each $n \in \omega$. Assume that the family $\{A_n : n \in \omega\}$ covers ω, we shall construct inductively a $\subseteq$-decreasing family $\{Z_n : n \in \omega\}$ such that for each $n \in \omega$,

- $Z_n \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$;
- $Z_n \subseteq Y_n$;
- $Z_n \cap A_k = \emptyset$ for each $k < n$.

Put $Z_0 = A_0$, let $n_1 = \min\{k : Y_1 \cap A_k \in \mathcal{I}^+\}$, and define

$$Z_1 = Y_1 \setminus \textstyle\bigcup_{k\leq n_1} A_k.$$

Thanks to $Y_1 \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$, n_1 is well defined, and $Z_1 \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$. With the same manner, we finish the construction. Note that $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++} \subset \mathcal{I}^+$, by the item (4), there exists $Z \in \mathcal{I}^+$ such that $Z \subseteq^{\mathcal{I}} Z_n$ for each $n \in \omega$. It is enough to show that $Z \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$, and this follows from the following Claim.

Claim 4.3. *There are infinitely many k such that $Z \cap A_k \in \mathcal{I}^+$.*

Proof. Suppose that there exists n such that for each $k > n$, $Z \cap A_k \in \mathcal{I}$. According to the assumption of $\{A_n : n \in \omega\}$ covering ω, we have that $Z \bigcap \bigcup_{k \le n} A_k \in \mathcal{I}^+$. Note that

$$Z \bigcap \bigcup_{k \le n} A_k \subseteq Z \setminus Z_k.$$

So $Z \setminus Z_k \in \mathcal{I}^+$, this contradict to the fact that $Z \subseteq^{\mathcal{I}} Z_k$. □

Case 2 If for all but finitely many n, $Y_n \notin \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$, we may assume that $\{Y_n : n \in \omega\} \subset \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{+} \setminus \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$ since it does no matter to removing off finitely many Y_n which belong to $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$.

Claim 4.4. *Let $\mathcal{A}$ be an infinite $\mathcal{I}$-AD family. For any $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{+} \setminus \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$, there is a family $\{Y_n : n \in \omega\}$ such that $Y_n \cap X \in \mathcal{I}^+$ for each $n \in \omega$, and $\mathcal{A} \bigcup \{Y_n : n \in \omega\}$ is also an $\mathcal{I}$-AD family.*

Proof. Note that $X \notin \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{++}$, there exists $\{A_0, A_1, \cdots, A_n\} \subset \mathcal{A}$ such that for each $A \in \mathcal{A} \setminus \{A_0, A_1, \cdots, A_n\}$, $A \cap X \in \mathcal{I}$. Put

$$\tilde{X} = X \setminus \bigcup_{k \le n} A_k.$$

Since $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{+}$, $\tilde{X} \in \mathcal{I}^+$. Let $\{Y_n : n \in \omega\} \subseteq \mathcal{I}^+$ be a partition of $\tilde{X}$. Clearly, $Y_n \cap X \in \mathcal{I}^+$ for each $k \in \omega$. In addition, this is also an $\mathcal{I}$-AD family. Therefore, the family $\mathcal{A} \bigcup \{Y_k : k \in \omega\}$ is desired. □

According to the previous claim, we can inductively construct a sequence $\{\mathcal{A}_n : n \in \omega\}$ of $\mathcal{I}$-AD families such that

- $\mathcal{A}_0 = \mathcal{A}$;
- $\mathcal{A}_n \subseteq \mathcal{A}_m$ for $n < m$;
- $Y_n \cap A \in \mathcal{I}^+$ for all $A \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n$.

The last term implies that $Y_n \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A}_{n+1})^{++}$. We extend the union $\bigcup_{n \in \omega} \mathcal{A}_n$ to an $\mathcal{I}$-MAD family $\mathcal{B}$. Note that for each $n \in \omega$,

$$Y_n \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A}_{n+1})^{++} \subseteq \mathcal{I}(\mathcal{I}\text{-}\mathcal{B})^{++},$$

so $\{Y_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{I}\text{-}\mathcal{B})^{++}$. With the same argument as the Case 1, we obtain $X \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{B})^{+} \subseteq \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{+}$ such that $X \subseteq^{\mathcal{I}} Y_n$ for each $n \in \omega$.

(5) ⇒ (6) The Corollary 5.6 in [5] asserts that if $\mathcal{I}$ is a $P^+(\mathcal{I})$-ideal, then $\mathcal{I} \in BW$. By the item (5), $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A}) \in BW$. As we mentioned previous, $\mathcal{I} \subset \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$, so $[0, 1]$ has the $(\mathcal{I}(\mathcal{I}\text{-}\mathcal{A}), \mathcal{I})$-$BW$ property.

(6) ⇒ (7) For the sake of contradiction, suppose that $\mathfrak{a}(\mathcal{I}) = \omega$ and $\mathcal{A} = \{A_n : n \in \omega\} \subset \mathcal{I}^+$ be an $\mathcal{I}$-MAD family. We may assume that $A_n \cap A_m = \emptyset$ for any different $n, m \in \omega$. Otherwise, we can shrink them to be pairwise disjoint via replacing A_n by $A_n \setminus \bigcup_{i<n} A_i$. Define $\{x_n : n \in \omega\}$ by

$$x_n = 1/k \text{ if } n \in A_k.$$

Since $\mathcal{A}$ is an $\mathcal{I}$-MAD family, by Lemma 3.5(3), for any $A \in \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{+}$ there are infinite set $\{n_k : k \in \omega\}$ such that $A \cap A_{n_k} \in \mathcal{I}^+ \setminus \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^{+}$ for each $k \in \omega$. The subsequence $\{x_n : n \in A\}$ cannot be $\mathcal{I}$-convergent since it has infinitely many cluster points. Indeed, for each $k \in \omega$, $1/n_k$ is a cluster point of this subsequence. This contradict to the the item (6).

(7) ⇒ (4) Recall that $\mathcal{I}$ is a $P^+(\mathcal{I})$-ideal if, and only if $\mathcal{I}$ is indecomposable ([12], Theorem 3.8(2)), this implication follows from Lemma 3.4 above. □

Remark 4.5. Let h be a function from ω to $\mathbb{R}^+$ satisfying

$$\sum_{n\in\omega} h(n) = \infty.$$

Let

$$\mathcal{I}_h = \{A \subset \omega : \sum_{n\in A} h(n) < \infty\}.$$

It was showed in [3] that for any summable ideal $\mathcal{I}_h$, $\mathfrak{a}(\mathcal{I}_h) > \omega$. Note that every summable ideal is F_σ, so this result can be viewed as a special case of Theorem 4.2.

Remark 4.6. In [10], it is shown that if $\mathcal{I}$ is a nowhere prime $P^+(\mathcal{I})$-ideal then $\mathfrak{a}(\mathcal{I}) > \omega$ ([10], Proposition 2.9). Theorem 4.2 improves this result.

Remark 4.7. We should point out that the implication (1) ⇒ (3) was probably first proved by Just and Krawczyk in [13], see also [5].

Definition 4.8. Let $\langle P_n : n \in \omega\rangle$ be a decomposition of ω into pairwise disjoint nonempty finite sets, $\overrightarrow{\mu} = \langle \mu_n : n \in \omega\rangle$ being a sequence of probability measures μ_n: $\mathcal{P}(P_n) \to [0, 1]$. Let

$$\mathcal{Z}_{\overrightarrow{\mu}} = \{A \subset \omega : lim_n \mu_n(A \cap P_n)\} = 0.$$

$\mathcal{Z}_{\overrightarrow{\mu}}$ is an ideal called the density ideal generated by $\overrightarrow{\mu}$, it was introduced by Farah in [2].

Corollary 4.9. *Let $\mathcal{I}$ be an ideal on ω.*

(1) *If $\mathcal{I}$ is not dense, then $\mathcal{I}$ is a $P^+(\mathcal{I})$-ideal.*

(2) $\mathfrak{a}(\mathcal{Z}_{\overrightarrow{u}}) = \omega$*([3], Theorem 2.2 (2)).*

(3) *If $\mathcal{I}$ is an analytic P-ideal, then $\bar{\mathfrak{a}}(\mathcal{I}) = \mathfrak{a}(\mathcal{I})$ if and only if $\mathcal{I}$ is a $P^+(\mathcal{I})$-ideal.*

Proof. (1) It is enough to show that $\mathfrak{a}(\mathcal{I}) > \omega$. Since $\mathcal{I}$ is not dense, it is easy to see that $\mathcal{I} \leq_K Fin$ (i.e. there exists $f: \omega \to \omega$ such that $f^{-1}(I) \in Fin$ if $I \in \mathcal{I}$ [15]).

Claim 4.10. *Let $\mathcal{I} \leq_K Fin$ that witnessed by $f: \omega \to \omega$. If $\mathcal{A}$ is an $\mathcal{I}$-MAD family then $\{f^{-1}(A) : A \in \mathcal{A}\}$ is a MAD family.*

Proof. Let $\mathcal{A}$ be an $\mathcal{I}$-MAD family, it is easy to see that $\{f^{-1}(A) : A \in \mathcal{A}\}$ is a *Fin*-AD family. We show that it is maximal. For any $X \in [\omega]^\omega$, $f(X) \in \mathcal{I}^+$. So there exists $A \in \mathcal{A}$ such that $A \cap f(X) \in \mathcal{I}^+$, and so $f^{-1}(A \cap f(X)) \in [\omega]^\omega$. Note that $f^{-1}(A \cap f(X)) \subseteq f^{-1}(A) \cap X$. Thus, $f^{-1}(A) \cap X \in [\omega]^\omega$. □

The Claim 4 implies that if $\mathcal{I}$ do not dense , then $\mathfrak{a}(\mathcal{I}) \geq \mathfrak{a} > \omega$, and then we obtain the item (1).

(2) Note that $\mathcal{Z}_{\overrightarrow{u}}$ does not have the *BW* property (see [6] or [20]), so it not be a $P^+(\mathcal{Z}_{\overrightarrow{u}})$-ideal. The item (2) followed by the equivalence between (4) and (7) in Theorem 4.2.

(3) Recall that for any analytic P-ideal $\mathcal{I}$, $\bar{\mathfrak{a}}(\mathcal{I})$ be the minimum of cardinalities of uncountable $\mathcal{I}$-MAD families. If $\bar{\mathfrak{a}}(\mathcal{I}) = \mathfrak{a}(\mathcal{I})$, then $\mathfrak{a}(\mathcal{I}) > \omega$, and this implies that $\mathcal{I}$ is a $P^+(\mathcal{I})$-ideal. It's the same the other way round. □

Corollary 4.11. *Let $\mathcal{I}$ be an ideal on ω, and let $\mathcal{A}$ be an $\mathcal{I}$-AD family.*

(1) *If $\mathcal{I}$ is a P^+-ideal, then so is the $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$;*

(2) *If $\mathcal{I}$ is selective, then so is the $\mathcal{I}(\mathcal{I}\text{-}\mathcal{A})$.*

Proof. Both proofs are the same as that of (4) ⇒ (5) in Theorem 4.2, and we just consider the Case 1 since the other case is analogous. Let $\{Y_n : n \in \omega\} \subset \mathcal{I}(\mathcal{I}\text{-}\mathcal{A})^+$ such that $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$. With the same notations as we have used, we obtain $Z \in \mathcal{I}^+$ such that $Z \subseteq^* Z_n$ for each $n \in \omega$. Thus, Z is desired. □

Recall that *Fin* is selective, so we have the following well known result mentioned in Section 1 ([9], Lemma 19).

Corollary 4.12. *(Mathias) For any AD-family $\mathcal{A}$, $\mathcal{I}(\mathcal{A})$ is selective.*

5. $P^+_{tower}(\mathcal{I})$-Ideals and Comments

Definition 5.1. Let $\mathcal{I}$ be an ideal, we say that $\mathcal{I}$ is a $P^+_{tower}(\mathcal{I})$-ideal if for every decreasing sequences $\langle A_n : n \in \omega \rangle$ that fulfills $X_n \setminus X_{n+1} \in \mathcal{I}$ for all $n \in \omega$, there exists $X \subset \omega$ such that for each $n \in \omega$ $X \subseteq^{\mathcal{I}} X_n$.

The notion of $P^+_{tower}(\mathcal{I})$-ideal is a generalization of the P^+_{tower}-ideal introduced in [12].

Definition 5.2. ([6]) $\mathcal{I}$ has the *hereditary BW*property (write as $\mathcal{I} \in hBW$) if for any $A \in \mathcal{I}^+$, $\mathcal{I}|A \in BW$. The *hFinBW* property was defined analogously.

Recall that $\mathcal{I}$ is a P-ideal if for every countable family $\{A_n : n \in \omega\} \subseteq \mathcal{I}$, there exists $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for each $n \in \omega$. It is well known that for any P-ideal $\mathcal{I}$, $\mathcal{I} \in hBW$ is coincides with $\mathcal{I} \in hFinBW$.

The goal of this section is to show the following diagram.

$$\begin{array}{ccccc} P^+ & \longrightarrow & P^+(\mathcal{I}) & \longrightarrow & P^+_{tower}(\mathcal{I}) \\ & & \downarrow & \nearrow & \\ & & hBW & & \end{array}$$

The implication of $P^+(\mathcal{I}) \Rightarrow hBW$ follows from the fact that if $\mathcal{I}$ is a $P^+(\mathcal{I})$-ideal, then for each $A \in \mathcal{I}^+$, $\mathcal{I}|A$ is a $P^+(\mathcal{I}|A)$-ideal.

For $s \in 2^{<\omega}$, $lh(s)$ denotes the length of s. For $i \in \{0, 1\}$, $s \frown i$ denotes the sequence $\langle s(0), \cdots, s(lh(s) - 1), i\rangle$. In order to prove $hBW \Rightarrow P^+_{tower}(\mathcal{I})$, we need the following result, which is the Proposition 2.9 in [7].

Lemma 5.3. *Let $r \in \omega$, and let $\mathcal{I}$ be an ideal. $\mathcal{I}$ has BW property if and only if for every family $\{A_s : s \in r^{<\omega}\}$ that fulfills the following conditions:*

S_1 $A_\emptyset = \omega$;

S_2 $A_s = A_{s\frown 0} \cup A_{s\frown 1}$;

S_3 $A_{s\frown 0} \cap A_{s\frown 1} = \emptyset$.

There exists $x \in 2^\omega$ and $B \subset \omega$ such that

- $B \in \mathcal{I}^+$;
- $B \setminus A_{x|n} \in \mathcal{I}$ *for all* $n \in \omega$.

It is easy to check the following result.

Lemma 5.4. *Let $r \in \omega$, and let $\mathcal{I}$ be an ideal. $\mathcal{I}$ has the hBW property if and only if for every $X \in \mathcal{I}^+$, and for every family $\{A_s : s \in r^{<\omega}\}$ that fulfils the following conditions:*

S_1 $A_\emptyset = X$;

S_2 $A_s = A_{s\frown 0} \cup A_{s\frown 1}$;

S_3 $A_{s\frown 0} \cap A_{s\frown 1} = \emptyset$.

There exists $x \in 2^\omega$ and $B \subset \omega$ such that

- $B \in \mathcal{I}^+$;
- $B \setminus A_{x|n} \in \mathcal{I}$ *for all* $n \in \omega$.

If $\mathcal{I}$ is a weak Q-ideal such that $\mathcal{I} \in hFinBW$, then the set B is a diagonalization of the sequence $\{A_{x|n} : n \in \omega\}$ ([7], Theorem 3.16).

Theorem 5.5. *Let $\mathcal{I}$ be an ideal on ω. If $\mathcal{I} \in hBW$, then it is a $P^+_{tower}(\mathcal{I})$-ideal.*

Proof. Let $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ be a decreasing sequence from $\mathcal{I}^+$ such that $X_n \setminus X_{n+1} \in \mathcal{I}$ for all $n \in \omega$. We construct a family $\{A_s : s \in 2^{<\omega}\}$ such that

(1) $A_\emptyset = X_0$;

(2) $A_{s^\frown 0} = X_{lh(s)}$, $A_{s^\frown 1} = X_{lh(s)} \setminus X_{lh(s)+1}$ for all $s \in \{0\}^{<\omega}$.

Since $\mathcal{I}$ has the hBW property, according to Lemma 5.4 above, there exists $r \in 2^\omega$, $B \in \mathcal{I}^+$ such that $B \setminus A_{r|n} \in \mathcal{I}$ for all $n \in \omega$. The condition of $X_n \setminus X_{n+1} \in \mathcal{I}$ actually force $r = 0^\omega$. So $X \subseteq^{\mathcal{I}} X_n$ for all $n \in \omega$. $\square$

Note that $\mathcal{I} \in BW$ is equal to $\mathfrak{s}(\mathcal{I}) \geq \omega_1$, and $\mathcal{I} \in hBW$ is equal to $\mathcal{I}$ having the hereditarily $\mathcal{I}$-Helly property ([5], Theorem 5.9). We observe the following result.

Theorem 5.6. *Let $\mathcal{I}$ be an ideal on ω. If $\mathfrak{a}(\mathcal{I}) \geq \mathfrak{s}(\mathcal{I})$, then the following conditions are equivalent:*

(1) *$\mathcal{I} \in BW$;*

(2) *$\mathcal{I}$ is a $P^+(\mathcal{I})$-ideal;*

(3) *$\mathcal{I} \in hBW$;*

(4) *$\mathcal{I} \in hHelly$;*

(5) *$\mathcal{I} \in Helly$.*

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