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Sharp Z-Eigenvalue Inclusion Set-Based Method for Testing the Positive Definiteness of Multivariate Homogeneous Forms

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Abstract. In this paper, we establish a sharp Z-eigenvalue inclusion set for even-order real tensors by Z-identity tensor and prove that new Z-eigenvalue inclusion set is sharper than existing results. We propose some sufficient conditions for testing the positive definiteness of multivariate homogeneous forms via new Z-eigenvalue inclusion set. Further, we establish upper bounds on the Z-spectral radius of weakly symmetric nonnegative tensors and estimate the convergence rate of the greedy rank-one algorithms. The given numerical experiments show the validity of our results.

1. Introduction

Consider the following multivariate homogeneous forms with spherical constraint:

$$\begin{aligned} f_{\mathcal{A}}(x) = \mathcal{A}x^m = \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m} \\ s.t. \quad x^\top x = 1, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, m, n \geq 2, f_{\mathcal{A}}(x)$ is a multivariate homogeneous form of degree m with n variables, and $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is an m-order n-dimensional real tensor with entries [12, 14]

$$a_{i_1 \ldots i_m} \in \mathbb{R}, i_j \in N = \{1, \ldots, n\}, j = 1, \ldots, m.$$

Clearly, the critical points of (1) satisfy the following equations for some $\lambda \in \mathbb{R}$:

$$\mathcal{A}x^{m-1} = \lambda x \text{ and } x^\top x = 1, \tag{2}$$

where $(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}$. The real number λ and the real vector x satisfying with (2) are called Z-eigenvalue and Z-eigenvector, respectively.

The multivariate homogeneous form $f_{\mathcal{A}}(x)$ is positive definite, which plays important roles in signal processing [15] and the stability study of nonlinear autonomous systems via Lyapunov's direct method in

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automatic control [3, 4, 13]. Note that $f_{\mathcal{A}}(x)$ is positive definite if and only if tensor $\mathcal{A}$ is positive definite, and that an even-order real symmetric tensor is positive definite if and only if all of its Z-eigenvalues are positive [14]. Some effective algorithms for finding Z-eigenvalue and the corresponding eigenvector have been implemented [5–9, 11, 16, 18, 21–26], but it is difficult to compute all the Z-eigenvalues and judge the positive definiteness of an even-order real symmetric tensor. Very recently, Li *et al.* [10] proposed Gershgorin-type Z-eigenvalue inclusion set with parameters by Z-identity tensor, which can identify the positive-definiteness of an even-order real symmetric tensor. It is remarkable that Brauer-type inclusion set is tighter than Gershgorin-type inclusion set [20]. As a continuation of the article [20], we shall establish sharp Brauer-type Z-eigenvalue localization set and propose some sufficient conditions for the positive definiteness of multivariate homogeneous forms.

To end this section, we introduce Z-identity tensor in [8, 10] and important results proposed in [10].

Definition 1.1. *Assume that m is even. We call $I_Z \in \mathcal{R}^{[m,n]}$ a Z-identity tensor if*

$$I_Z x^{m-1} = x, \ \ x^\top x = 1, \ \forall x \in \mathbb{R}^n.$$

It is worth noting that the even-order n dimension Z-identity tensor is not unique in general. For instance, each even tensor in the following is a Z-identity tensor:

Case I: $(I_Z)_{iii_2i_2\ldots i_ki_k} = 1, \forall k \in N$ and $m = 2k$;

Case II (Property 2.4 of [8]): $(I_Z)_{i_1i_2\ldots i_m} = \frac{1}{m!}\sum\limits_{p\in II_m} \delta_{i_{p(1)}}\delta_{i_{p(2)}}\ldots\delta_{i_{p(m-1)}}\delta_{i_{p(m)}}$, where δ is the standard Kronecker, i.e.,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.2. *(Theorem 2 of [10]) Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with m being even. Let $\sigma_Z(\mathcal{A})$ be the set of all Z-eigenvalues of $\mathcal{A}$. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, then*

$$\sigma_Z(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{A}, \alpha) = \bigcup_{i\in N} \mathcal{G}_i(\mathcal{A}, \alpha) := \{z \in \mathbb{R} : |z - \alpha_i| \le R_i(\mathcal{A}, \alpha_i)\},$$

where $R_i(\mathcal{A}, \alpha_i) = \sum\limits_{i_2\ldots i_m\in N} |a_{ii_2\ldots i_m} - \alpha_i (I_Z)_{ii_2\ldots i_m}|$. Furthermore, $\sigma_Z(\mathcal{A}) \subseteq \bigcap\limits_{\alpha\in\mathbb{R}^n} \mathcal{G}(\mathcal{A}, \alpha)$.

2. A sharp Z-eigenvalue inclusion set for even-order real tensors

In this section, we establish new Z-eigenvalue inclusion set for even-order tensors. To this end, we define

$$\Theta_j = \Big\{(i_2, i_3, \ldots, i_m) : i_k = j \text{ for some } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in N\Big\},$$

$$\overline{\Theta}_j = \Big\{(i_2, i_3, \ldots, i_m) : i_k \ne j \text{ all any } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in N\Big\},$$

$$r_i^{\Theta_j}(\mathcal{A}, \alpha_i) = \sum_{\{i_2,\ldots,i_m\}\in\Theta_j} |a_{ii_2\ldots i_m} - \alpha_i (I_Z)_{ii_2\ldots i_m}|, \ r_i^{\overline{\Theta}_j}(\mathcal{A}, \alpha_i) = \sum_{\{i_2,\ldots,i_m\}\in\overline{\Theta}_j} |a_{ii_2\ldots i_m} - \alpha_i (I_Z)_{ii_2\ldots i_m}|.$$

Obviously, $R_i(\mathcal{A}, \alpha_i) = r_i^{\Theta_j}(\mathcal{A}, \alpha_i) + r_i^{\overline{\Theta}_j}(\mathcal{A}, \alpha_i)$.

Theorem 2.1. *Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with m being even. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top \in \mathbb{R}^n$, then*

$$\sigma_Z(\mathcal{A}) \subseteq \mho(\mathcal{A}, \alpha) = \bigcup_{i\in N} \bigcap_{j\in N, j\ne i} \mho_{i,j}(\mathcal{A}, \alpha),$$

where $\mho_{i,j}(\mathcal{A}, \alpha) = \Big\{z \in R : (|\lambda - \alpha_i| - r_i^{\overline{\Theta}_j}(\mathcal{A}, \alpha_i)\Big)|\lambda - \alpha_j| \le r_i^{\Theta_j}(\mathcal{A}, \alpha_i) R_j(\mathcal{A}, \alpha_j)\}$. Furthermore, $\sigma_Z(\mathcal{A}) \subseteq \bigcap\limits_{\alpha\in\mathbb{R}^n} \mho(\mathcal{A}, \alpha)$.

Proof. Let (λ, x) be a Z-eigenpair of $\mathcal{A}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x = \lambda I_Z x^{m-1}, \;\; x^\top x = 1. \tag{3}$$

Assume $|x_t| = \max_{i\in N} |x_i|$, then $0 < |x_t|^{m-1} \le |x_t| \le 1$.

On one hand, taking the t-th equation from (3), for any $j \in N$, $j \ne t$, we have

$$\sum_{i_2,\dots,i_m\in N} \lambda (I_Z)_{ti_2\dots i_m} x_{i_2}\dots x_{i_m} = \sum_{i_2,\dots,i_m\in N} a_{ti_2\dots i_m} x_{i_2}\dots x_{i_m}. \tag{4}$$

Hence, for any real number α_t, it follows that

$$\begin{aligned}(\lambda-\alpha_t)x_t &= \sum_{i_2,\dots,i_m\in N} (\lambda-\alpha_t)(I_Z)_{ti_2\dots i_m} x_{i_2}\dots x_{i_m} = \sum_{i_2,\dots,i_m\in N} \Big(a_{ti_2\dots i_m} - \alpha_t (I_Z)_{ti_2\dots i_m}\Big) x_{i_2}\dots x_{i_m} \\ &= \sum_{\{i_2,\dots,i_m\}\in\Theta_j} \Big(a_{ti_2\dots i_m} - \alpha_t (I_Z)_{ti_2\dots i_m}\Big) x_{i_2}\dots x_{i_m} + \sum_{\{i_2,\dots,i_m\}\in\overline{\Theta}_j} \Big(a_{ti_2\dots i_m} - \alpha_t (I_Z)_{ti_2\dots i_m}\Big) x_{i_2}\dots x_{i_m}\end{aligned} \tag{5}$$

Taking modulus in (5) and using the triangle inequality give

$$\begin{aligned}|\lambda-\alpha_t||x_t| &\le \sum_{\{i_2,\dots,i_m\}\in\Theta_j} |a_{ti_2\dots i_m} - \alpha_t (I_Z)_{ti_2\dots i_m}||x_{i_2}|\dots|x_{i_m}| + \sum_{\{i_2,\dots,i_m\}\in\overline{\Theta}_j} |a_{ti_2\dots i_m} - \alpha_t (I_Z)_{ti_2\dots i_m}||x_{i_2}|\dots|x_{i_m}| \\ &\le r_t^{\Theta_j}(\mathcal{A},\alpha_t)|x_j| + r_t^{\overline{\Theta}_j}(\mathcal{A},\alpha_t)|x_t|,\end{aligned} \tag{6}$$

i.e.,

$$\Big(|\lambda-\alpha_t| - r_t^{\overline{\Theta}_j}(\mathcal{A},\alpha_t)\Big)|x_t| \le r_t^{\Theta_j}(\mathcal{A},\alpha_t)|x_j|. \tag{7}$$

On the other hand, for $t \ne j \in N$, taking the j-th equation from (3), we obtain

$$(\lambda-\alpha_j)x_j = \sum_{i_2,\dots,i_m\in N} (\lambda-\alpha_j)(I_Z)_{ji_2\dots i_m} x_{i_2}\dots x_{i_m} = \sum_{i_2,\dots,i_m\in N} \Big(a_{ji_2\dots i_m} - \alpha_j (I_Z)_{ji_2\dots i_m}\Big) x_{i_2}\dots x_{i_m}. \tag{8}$$

Taking modulus in (8) and using the triangle inequality, one has

$$|\lambda-\alpha_j||x_j| \le R_j(\mathcal{A},\alpha_j)|x_t|. \tag{9}$$

If $|x_j| = 0$, by (7), we obtain

$$|\lambda-\alpha_t| \le r_t^{\overline{\Theta}_j}(\mathcal{A},\alpha_t).$$

Thus, $\lambda \in \mho_{t,j}(\mathcal{A},\alpha) \subseteq \mho(\mathcal{A},\alpha)$.

Otherwise, $|x_j| > 0$. Multiplying (7) with (9) yields

$$\Big(|\lambda-\alpha_t| - r_t^{\overline{\Theta}_j}(\mathcal{A},\alpha_t)\Big)|\lambda-\alpha_j||x_j||x_t| \le r_t^{\Theta_j}(\mathcal{A},\alpha_t)R_j(\mathcal{A},\alpha_j)|x_j||x_t|,$$

equivalently,

$$\Big(|\lambda-\alpha_t| - r_t^{\overline{\Theta}_j}(\mathcal{A},\alpha_t)\Big)|\lambda-\alpha_j| \le r_t^{\Theta_j}(\mathcal{A},\alpha_t)R_j(\mathcal{A},\alpha_j),$$

which implies $\lambda \in \mho_{t,j}(\mathcal{A},\alpha)$. From the arbitrariness of j, we have $\lambda \in \bigcup_{i\in N}\bigcap_{j\in N, j\ne i} \mho_{i,j}(\mathcal{A},\alpha)$. Further, $\sigma_Z(\mathcal{A}) \subseteq \bigcap_{\alpha\in\mathbb{R}^n} \mho(\mathcal{A},\alpha)$ by the arbitrariness of α. □

Corollary 2.2. *Let $\mathcal{A} = (a_{i_1 i_2\dots i_m}) \in \mathbb{R}^{[m,n]}$ with m being even. For any real vector $\alpha = (\alpha_1,\dots,\alpha_n)^\top \in \mathbb{R}^n$, then*

$$\mho(\mathcal{A},\alpha) \subseteq \mathcal{G}(\mathcal{A},\alpha).$$

Proof. For any $\lambda \in \mho(\mathcal{A}, \alpha)$, without loss of generality, there exists $t \in N$ such that $\lambda \in \mho_{t,s}(\mathcal{A})$, that is,

$$\Big(|\lambda - \alpha_t| - r_t^{\overline{\Theta}_s}(\mathcal{A}, \alpha_t)\Big)|\lambda - \alpha_s| \le r_t^{\Theta_s}(\mathcal{A}, \alpha_t)R_s(\mathcal{A}, \alpha_s),\ \forall s \neq t. \tag{10}$$

Next, the following argument is divided into two cases.

Case I: $r_t^{\Theta_s}(\mathcal{A}, \alpha_t)R_s(\mathcal{A}, \alpha_s) = 0$. Since $|\lambda - \alpha_s| \ge 0$, from (10), we deduce $|\lambda - \alpha_t| - r_t^{\overline{\Theta}_s}(\mathcal{A}, \alpha_t) \le 0$. Further, it holds that

$$|\lambda - \alpha_t| \le r_t^{\overline{\Theta}_s}(\mathcal{A}, \alpha_t) \le R_t(\mathcal{A}, \alpha_t),$$

i.e., $\lambda \in \mathcal{G}_t(\mathcal{A}, \alpha)$. So, we have $\mho_{t,s}(\mathcal{A}, \alpha) \subseteq \mathcal{G}_t(\mathcal{A}, \alpha)$.

Case II: $r_t^{\Theta_s}(\mathcal{A}, \alpha_t)R_s(\mathcal{A}, \alpha_s) > 0$. Then dividing both sides by $r_t^{\Theta_s}(\mathcal{A}, \alpha_t)R_s(\mathcal{A}, \alpha_s)$ in (10), we obtain

$$\frac{|\lambda - \alpha_t| - r_t^{\overline{\Theta}_s}(\mathcal{A}, \alpha_t)}{r_t^{\Theta_s}(\mathcal{A}, \alpha_t)} \cdot \frac{|\lambda - \alpha_s|}{R_s(\mathcal{A}, \alpha_s)} \le 1, \tag{11}$$

which implies

$$\frac{|\lambda - \alpha_t| - r_t^{\overline{\Theta}_s}(\mathcal{A}, \alpha_t)}{r_t^{\Theta_s}(\mathcal{A}, \alpha_t)} \le 1 \tag{12}$$

or

$$\frac{|\lambda - \alpha_s|}{R_s(\mathcal{A}, \alpha_s)} \le 1. \tag{13}$$

If (12) holds, then we have $|\lambda - \alpha_t| - r_t^{\overline{\Theta}_s}(\mathcal{A}, \alpha_t) \le r_t^{\Theta_s}(\mathcal{A}, \alpha_t)$, i.e,

$$|\lambda - \alpha_t| \le r_t^{\overline{\Theta}_s}(\mathcal{A}, \alpha_t) + r_t^{\Theta_s}(\mathcal{A}, \alpha_t) = R_t(\mathcal{A}, \alpha_t).$$

So, $\lambda \in \mathcal{G}_t(\mathcal{A}, \alpha)$. Otherwise, (13) holds, we can verify $\lambda \in \mathcal{G}_s(\mathcal{A}, \alpha)$.

From the above two cases, we can get $\mho_{t,s}(\mathcal{A}, \alpha) \subseteq \mathcal{G}_t(\mathcal{A}, \alpha) \bigcup \mathcal{G}_s(\mathcal{A}, \alpha)$. Thus, $\mho(\mathcal{A}, \alpha) \subseteq \mathcal{G}(\mathcal{A}, \alpha)$ for a given parameter α. □

Next, we give a numerical comparison between Theorem 2.1 and Theorem 2 of [10].

Example 2.3. *Consider* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ *defined by*

$$a_{ijkl} = \begin{cases} a_{1111} = 10; a_{1122} = 9; a_{1121} = a_{1211} = -1; \\ a_{2222} = 5; a_{2211} = 6; a_{2122} = a_{2212} = -1; \\ a_{ijkl} = 0, \quad otherwise. \end{cases}$$

All Z-eigenvalues of $\mathcal{A}$ are 5.0000 and 10.0000. We choose different parameters $\alpha_1 = [3, 8]^\top$, $\alpha_2 = [10, 7]^\top$, $\alpha_3 = [9, 5]^\top$ and $\alpha_4 = [9, 5.5]^\top$, respectively. Set $\alpha_1 = [3, 8]^\top$ and $I_Z = (I_{ijkl})$ as Case I of Definition 1.1

$$I_{ijkl} = \begin{cases} I_{1111} = I_{1122} = I_{2211} = I_{2222} = 1; \\ 0, \quad otherwise. \end{cases}$$

Accordingly to Theorem 2.1, we obtain

$$\mho(\mathcal{A}, \alpha_1 = (3, 8)) = [-7.5917, 16.5498] \cup [-3.8102, 15.7178] = [-7.5917, 16.5498];$$

Similarly, we can obtain the following table:

α	$[3,8]^\top$	$[10,7]^\top$	$[9,5]^\top$	$[9,5.5]^\top$
$\mho(\mathcal{A}, \alpha)$	$[-7.5917, 16.5498]$	$[3.5949, 12.6533]$	$[3.6277, 11]$	$[3.6088, 10.6225]$
$\mathcal{G}(\mathcal{A}, \alpha)$	$[-12, 18]$	$[2, 13]$	$[2, 12]$	$[2.5, 12]$

Numerical results show that the bound of Theorem 2.1 is tighter than that of Theorem 2 of [10] and the suitable parameter α has a great influence on the numerical effect.

3. Positive definiteness of multivariate homogeneous forms

In this section, based on the inclusion set $\mho(\mathcal{A}, \alpha)$ in Theorem 2.1, we propose a sufficient condition for the positive definiteness of even-order tensors. Before proceeding further, we introduce the results of [1, 10].

Definition 3.1. *(i) We say that $\mathcal{A}$ is symmetric if*

$$a_{i_1\ldots i_m} = a_{i_{\pi(1)}\ldots i_{\pi(m)}}, \forall \pi \in \Gamma_m,$$

where Γ_m is the permutation group of m indices.

(ii) We say that $\mathcal{A}$ is weakly symmetric if the associated homogeneous polynomial $f_{\mathcal{A}}(x)$ satisfies

$$\nabla f_{\mathcal{A}}(x) = m\mathcal{A}x^{m-1}.$$

Obviously, if tensor $\mathcal{A}$ is symmetric, then $\mathcal{A}$ weakly symmetric. However, the converse result may not hold.

Lemma 3.2. *(Theorem 3 of [10]) Let λ be a Z-eigenvalue of $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with m being even. If there exists a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ such that*

$$\alpha_i > R_i(\mathcal{A}, \alpha_i), \forall i \in N,$$

then $\lambda > 0$. Further, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1) is positive definite.

Theorem 3.3. *Let λ be a Z-eigenvalue of $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with m being even. For $i \in N$, if there exist a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^\top$ and $j \neq i$ such that*

$$(\alpha_i - r_i^{\overline{\Theta}_j}(\mathcal{A}, \alpha_i))\alpha_j > r_i^{\Theta_j}(\mathcal{A}, \alpha_i)R_j(\mathcal{A}, \alpha_j), \tag{14}$$

then $\lambda > 0$. Further, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1) is positive definite.

Proof. Suppose on the contrary that $\lambda \leq 0$. From Theorem 2.1, there exists $t \in N$ with $\lambda \in \mho_{t,j}(\mathcal{A}, \alpha_t)$, i.e.,

$$\big(|\lambda - \alpha_t| - r_t^{\overline{\Theta}_j}(\mathcal{A}, \alpha_t))|\lambda - \alpha_j| \leq r_t^{\Theta_j}(\mathcal{A}, \alpha_t)R_j(\mathcal{A}, \alpha_j), \forall j \neq t.$$

Further, it follows from $\alpha_i > 0$ and $\lambda \leq 0$ that

$$(\alpha_t - r_t^{\overline{\Theta}_j}(\mathcal{A}, \alpha_t))\alpha_j \leq r_t^{\Theta_j}(\mathcal{A}, \alpha_t)R_j(\mathcal{A}, \alpha_j), \forall j \neq t,$$

which contradicts (14). Thus, $\lambda > 0$. When $\mathcal{A}$ is a symmetric tensor and all Z-eigenvalues are positive, $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ defined in (1) is positive definite. □

The following example shows the validity of Theorem 3.3.

Example 3.4. *Consider $f_{\mathcal{A}}(x) = \mathcal{A}x^m$ deduced by symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ as follows*

$$a_{1111} = 1.4; a_{2222} = 3.2; a_{3333} = 2.6; a_{1112} = a_{1121} = a_{1211} = a_{2111} = -0.1;$$
$$a_{1122} = a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2211} = 0.8;$$
$$a_{1133} = a_{1313} = a_{1331} = a_{3113} = a_{3131} = a_{3311} = 1.1;$$
$$a_{1233} = a_{1323} = a_{1332} = a_{2133} = a_{2313} = a_{2331} = -0.1;$$
$$a_{3123} = a_{3132} = a_{3213} = a_{3231} = a_{3312} = a_{3321} = 0.1;$$
$$a_{2223} = a_{2232} = a_{2322} = a_{3222} = 0.1;$$
$$a_{2233} = a_{2323} = a_{2332} = a_{3223} = a_{3232} = a_{3322} = 1.0; a_{ijkl} = 0, \textit{otherwise}.$$

Taking I_Z as Case II (Case I) of Definition 1.1, by simple computations, we cannot find positive real number α_1 such that

$$\alpha_1 > R_1(\mathcal{A}, \alpha_1),$$

which shows that Theorem 3 of [10] cannot check the positive definiteness of $\mathcal{A}$ and $f_{\mathcal{A}}(x)$.

Set $\alpha = (2.85, 3.0, 2.7)$ and let $I_Z = (I_{ijkl})$ be Case II of Definition 1.1

$$I_{ijkl} = \begin{cases} I_{1111} = I_{2222} = I_{3333} = 1; \\ I_{1122} = I_{1212} = I_{1221} = I_{1133} = I_{1313} = I_{1331} = \frac{1}{3}; \\ I_{2112} = I_{2121} = I_{2221} = I_{2233} = I_{2323} = I_{2332} = \frac{1}{3}; \\ I_{3113} = I_{3131} = I_{3311} = I_{3223} = I_{3232} = I_{3322} = \frac{1}{3}; \\ 0, \ \textit{otherwise}. \end{cases}$$

From Theorem 3.3, we can calculate the following corresponding values

	$(\alpha_i - \overline{r}_i^{\Theta_j}(\mathcal{A}, \alpha_i))\alpha_j$	$r_i^{\Theta_j}(\mathcal{A}, \alpha_i)R_j(\mathcal{A}, \alpha_j)$
$i = 1, j = 2$	2.85	1.575
$i = 1, j = 3$	1.755	1.275
$i = 2, j = 1$	4.56	2.065
$i = 2, j = 3$	6.21	2.55
$i = 3, j = 1$	6.27	3.54
$i = 3, j = 2$	6	1.5

From the above table, we verify

$$(\alpha_i - \overline{r}_i^{\Theta_j}(\mathcal{A}, \alpha_i)))\alpha_j > r_i^{\Theta_j}(\mathcal{A}, \alpha_i)R_j(\mathcal{A}, \alpha_j), \forall i \neq j \in N,$$

which implies that $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ is positive definite.

4. Estimations of Z-spectral radius and convergence rate on the greedy rank-one algorithms

As we know, the best rank-one approximation which has numerous applications in wireless communication systems, image processing, data analysis [7, 15–17, 21]. The best rank-one approximation of $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is to find a rank-one tensor $\kappa x^m = (\kappa x_{i_1} x_{i_2} \ldots x_{i_m})$ such that

$$\min_{\kappa \in \mathbb{R}, x} \{\|\mathcal{A} - \kappa x^m\|_F : \ x^T x = 1\},$$

where $\|\mathcal{A}\|_F := \sqrt{\sum\limits_{i_1, i_2, \ldots, i_m \in N} a^2_{i_1 i_2 \ldots i_m}}$. When $\mathcal{A}$ is nonnegative and weakly symmetric, $\rho(\mathcal{A})x_0^m$ is a best rank-one approximation of $\mathcal{A}$, i.e.,

$$\min_{\kappa \in \mathbb{R}, x^T x = 1} \|\mathcal{A} - \kappa x^m\|_F = \|\mathcal{A} - \rho(\mathcal{A})x_0^m\|_F = \sqrt{\|\mathcal{A}\|_F^2 - \rho(\mathcal{A})^2}.$$

Further, Qi [17] defined the quotient on the residual of the best rank-one approximation of tensor $\mathcal{A}$ as follows:

$$\omega = \frac{\|\mathcal{A} - \rho(\mathcal{A})x_0^m\|_F}{\|\mathcal{A}\|_F} = \sqrt{1 - \frac{\rho(\mathcal{A})^2}{\|\mathcal{A}\|_F^2}},$$

which can estimate the convergence rate of the greedy rank-one algorithm [2, 17, 18, 25]. Hence, we shall devote to finding sharp upper bounds of the Z-spectral radius of weakly symmetric nonnegative tensors to estimate the convergence rate of the greedy rank-one algorithms. We recall some fundamental results of nonnegative tensors [1].

Lemma 4.1. *(Theorem 3.11 of [1]) Assume $\mathcal{A}$ is a weakly symmetric nonnegative tensor. Then, $\rho(\mathcal{A}) = \lambda^*$, where λ^* denotes the largest Z-eigenvalue.*

Lemma 4.2. *(Corollary 4.10 of [1]) Assume $\mathcal{A}$ is a weakly symmetric nonnegative tensor. Then,*

$$\rho(\mathcal{A}) \geq \max_{i\in N} a_{i\dots i}.$$

Theorem 4.3. *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor (Case I or Case II) with m being even. For real vector $\alpha = (\alpha_1, \dots, \alpha_n)^\top \in \mathbb{R}^n$ with $\alpha_i \leq \max_{i\in N} a_{i\dots i}$, then*

$$\rho(\mathcal{A}) \leq \max_{i\in N} \min_{j\in N, i\neq j, \alpha\in\mathbb{R}^n} \frac{1}{2}(\alpha_i + \alpha_j + r_i^{\overline{\Theta}_j}(\mathcal{A}, \alpha_i) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A}, \alpha_i)),$$

where $\Lambda_{i,j}(\mathcal{A}) = (\alpha_i - \alpha_j + r_i^{\overline{\Theta}_j}(\mathcal{A}, \alpha_j))^2 + 4 r_i^{\Theta_j}(\mathcal{A}, \alpha_i) R_j(\mathcal{A}, \alpha_j)$.

Proof. From Lemma 4.1, we assume that $\rho(\mathcal{A}) = \lambda^*$ is the largest Z-eigenvalue. It follows from Theorem 2.1 that there exists $t \in N$ such that

$$(|\rho(\mathcal{A}) - \alpha_t| - r_t^{\overline{\Theta}_j}(\mathcal{A}, \alpha_t))|\rho(\mathcal{A}) - \alpha_j| \leq r_t^{\Theta_j}(\mathcal{A}, \alpha_t) R_j(\mathcal{A}, \alpha_j), \forall j \neq t. \tag{15}$$

Since $\mathcal{A}$ is nonnegative and Lemma 4.2 holds, for $\alpha_i \leq \max_{i\in N} a_{i\dots i}$, we have

$$\rho(\mathcal{A}) \geq \alpha_t \text{ and } \rho(\mathcal{A}) \geq \alpha_j.$$

Thus, (15) is equivalent to

$$(\rho(\mathcal{A}) - \alpha_t - r_t^{\overline{\Theta}_j}(\mathcal{A}, \alpha_t))(\rho(\mathcal{A}) - \alpha_j) \leq r_t^{\Theta_j}(\mathcal{A}, \alpha_t) R_j(\mathcal{A}, \alpha_j), \forall j \neq t. \tag{16}$$

Solving for (16), we obtain

$$\rho(\mathcal{A}) \leq \frac{1}{2}(\alpha_j + \alpha_t + r_t^{\overline{\Theta}_j}(\mathcal{A}, \alpha_t) + \Lambda_{t,j}^{\frac{1}{2}}(\mathcal{A}, \alpha_t)),$$

where $\Lambda_{t,s}(\mathcal{A}) = (\alpha_t - \alpha_s + r_t^{\overline{\Theta}_j}(\mathcal{A}, \alpha_t))^2 + 4 r_t^{\Theta_j}(\mathcal{A}, \alpha_t) R_j(\mathcal{A}, \alpha_j)$. Since $j \in N$ and α are chosen arbitrarily, it holds

$$\rho(\mathcal{A}) \leq \min_{j\in N, t\neq j, \alpha\in\mathbb{R}^n} \frac{1}{2}(\alpha_j + \alpha_t + r_t^{\overline{\Theta}_j}(\mathcal{A}, \alpha_t) + \Lambda_{t,j}^{\frac{1}{2}}(\mathcal{A}, \alpha_t)).$$

Consequently,

$$\rho(\mathcal{A}) \leq \max_{i\in N} \min_{j\in N, i\neq j, \alpha\in\mathbb{R}^n} \frac{1}{2}(\alpha_i + \alpha_j + r_i^{\overline{\Theta}_j}(\mathcal{A}, \alpha_i) + \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A}, \alpha_i)).$$

Thus, the conclusion holds. □

The following numerical experiment shows validity of Theorem 4.3 and gives an estimation for the convergence rate of the greedy rank-one algorithms.

Example 4.4. *Consider tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ defined by*

$$a_{ijkl} = \begin{cases} a_{1111} = 1; a_{2222} = 3; a_{1122} = a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2211} = \frac{1}{3}; \\ a_{1112} = a_{1121} = a_{1211} = a_{2111} = \frac{1}{3}; a_{ijkl} = 0, \text{ otherwise.} \end{cases}$$

By simple computation, we obtain $(\rho(\mathcal{A}), x) = (3, (0, 1))$ and $\|\mathcal{A}\|_F = 3.3166$. For this tensor, set $\alpha = (1, 1)$ and let $I_Z = (I_{ijkl})$ be Case II of Definition 1.1. The bounds via different estimations given in the literature are shown in the following table:

References	upper bound	parameter α
Theorem 3.11 of [1]	$\rho(\mathcal{A}) \leq 6.1283$	No
Corollary 4.5 of [19]	$\rho(\mathcal{A}) \leq 4.3333$	No
Theorems 4.5-4.7 of [20]	$\rho(\mathcal{A}) \leq 4.1985$	No
Theorem 7 of [18]	$\rho(\mathcal{A}) \leq 4.0000$	No
Theorem 1 of [10]	$\rho(\mathcal{A}) \leq 3.3333$	$\alpha = (1, 1)$
Theorems 4.1	$\rho(\mathcal{A}) \leq 3.1055$	$\alpha = (1, 1)$

From the table above, it is easy to see that only the upper bound obtained by Theorem 4.1 is smaller than $\|\mathcal{A}\|_F$. Consequently, we have

$$\min_{\kappa\in\mathbb{R},\kappa\in\mathbb{R}^n,x^Tx=1} \|\mathcal{A} - \kappa x^m\|_F = \sqrt{\|\mathcal{A}\|_F^2 - \rho(\mathcal{A})^2} \geq 1.3559.$$

Further, we obtain that the quotient on the residual of the best rank-one approximation of $\mathcal{A}$ is

$$\omega = \frac{\|\mathcal{A} - \rho(\mathcal{A})x_0^m\|_F}{\|\mathcal{A}\|_F} = \sqrt{1 - \frac{\rho(\mathcal{A})^2}{\|\mathcal{A}\|_F^2}} \geq 0.3511,$$

which implies the convergence rate of the greedy rank-one algorithm [2, 17, 18, 24, 25].

5. Conclusions

In this paper, we established a Brauer-type Z-eigenvalue inclusion set for even-order real tensors by Z-identity tensor and proposed some sufficient conditions for the positive definiteness of multivariate homogeneous forms. Note that the suitable parameter α has a great influence on the numerical effects and positive definiteness of $f_{\mathcal{A}}(x)$. Therefore, how to select the suitable parameter α is our further research.

Competing Interests

The authors declare that they have no competing interests.

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