Filomat 34:9 (2020), 2829–2846
https://doi.org/10.2298/FIL2009829S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/filomat`

Determining Crossing Numbers of the Join Products of Two Specific Graphs of Order Six With the Discrete Graph

Michal Staš[a]

[a]*Department of Mathematics and Theoretical Informatics, FEEI TUKE, 042 00 Košice, Slovak Republic.*

Abstract. The main aim of the paper is to give the crossing number of the join product $G^* + D_n$ for the connected graph G^* of order six consisting of $P_4 + D_1$ and of one leaf incident with some inner vertex of the path P_4 on four vertices, and where D_n consists of n isolated vertices. In the proofs, it will be extend the idea of the minimum numbers of crossings between two different subgraphs from the set of subgraphs which do not cross the edges of the graph G^* onto the set of subgraphs by which the edges of G^* are crossed exactly once. Due to the mentioned algebraic topological approach, we are able to extend known results concerning crossing numbers for join products of new graphs. The proofs are done with the help of software that generates all cyclic permutations for a given number k, and creates a new graph COG for calculating the distances between all $(k-1)!$ vertices of the graph. Finally, by adding one edge to the graph G^*, we are able to obtain the crossing number of the join product of one other graph with the discrete graph D_n.

1. Introduction

The problem of reducing the number of crossings on the edges in the drawings of graphs was studied in many areas, and the most prominent area is VLSI technology. Introduction of the VLSI technology revolutionized circuit design and had a strong impact on parallel computing. A lot of research aiming at efficient use of the new technologies has been done and further investigations are in progress. As a crossing of two edges of the communication graph requires unit area in its VLSI-layout, the crossing number together with the number of vertices of the graph immediately provide a lower bound for the area of the VLSI-layout of the communication graph. The crossing numbers have been also studied to improve the readability of hierarchical structures and automated graph drawings. The visualized graph should be easy to read and understand. For the understandability of graph drawings, the reducing of crossings is by far the most important.

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane. (For the definition of a *drawing* see [11].) It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let D ($D(G)$) be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number

2010 *Mathematics Subject Classification*. Primary 05C10, Secondary 05C38.

Keywords. graph, drawing, crossing number, join product, rotation.

Received: 10 September 2019; Revised: 27 September 2020; Accepted: 13 October 2020

Communicated by Paola Bonacini

The research was supported by the internal faculty research project no. FEI-2017-39.

Email address: `michal.stas@tuke.sk` (Michal Staš)

of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$

$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

In the paper, some proofs will be also based on the Kleitman's result on crossing numbers of the complete bipartite graphs [8]. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \Big\lfloor \frac{m}{2} \Big\rfloor \Big\lfloor \frac{m-1}{2} \Big\rfloor \Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor, \quad \text{if} \quad \min\{m,n\} \leq 6.$$

Using Kleitman's result [8], the crossing numbers for the join product of two paths, the join product of two cycles, and also for the join product of a path and a cycle were studied by Klešč [9]. Moreover, the exact values for crossing numbers of $G + D_n$ and of $G + P_n$ for all graphs G of order at most four are given by Klešč and Schrötter [13]. It is also important to note that the crossing numbers of the graphs $G + D_n$ are known for few graphs G of order five and six in [1], [3], [5], [7], [10], [11], [12], [14], and [16]. In all these cases, the graph G is connected and contains at least one cycle. Obviously, with the growing number of edges in graphs, it is much more difficult to determine their crossing numbers, and so the purpose of this article is to extend the known results concerning this topic to new graphs G with $|V(G)| < |E(G)|$. The crossing numbers of $G + D_n$ are also known only for some disconnected graphs G, see [4], [15], and [17].

The methods presented in the paper are based on multiple combinatorial properties of the cyclic permutations. The similar methods were partially used earlier by Hernández-Vélez *et al.* [6]. The properties of cyclic permutations have been already verified with the help of software by Berežný and Staš in [3] and [4]. Also in this article, some parts of proofs can be simplified by utilizing the work of the software COGA that generates all cyclic permutations by Berežný and Buša [2]. C++ version of the program is located on the website `http://web.tuke.sk/fei-km/coga/`. The list with the short names of $6!/6 = 120$ cyclic permutations of six elements are collected in Table 1 of [3]. Note that we were unable to determine the crossing number of the join product $G^* + D_n$ using the methods used in [11], [13], and [14].

2. Cyclic Permutations and Configurations

Let G^* be the connected graph of order six consisting of $P_4 + D_1$ and of one leaf incident with some inner vertex of the path P_4 on four vertices. We consider the join product of G^* with the discrete graph on n vertices denoted by D_n. The graph $G^* + D_n$ consists of one copy of the graph G^* and of n vertices $t_1, t_2, \dots, t_n$, where any vertex t_j, $j = 1, 2, \dots, n$, is adjacent to every vertex of G^*. Let T^j, $j = 1, \dots, n$, denote the subgraph induced by the six edges incident with the vertex t_j. This means that the graph $T^1 \cup \dots \cup T^n$ is isomorphic with the complete bipartite graph $K_{6,n}$ and therefore, we can write

$$G^* + D_n = G^* \cup K_{6,n} = G^* \cup \left(\bigcup_{j=1}^{n} T^j \right). \tag{1}$$

Let D be a good drawing of the graph $G^* + D_n$. The *rotation* $\mathrm{rot}_D(t_j)$ of a vertex t_j in the drawing D is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_j, see [6]. We use the notation (123456) if the counter-clockwise order the edges incident with the vertex t_j is t_jv_1, t_jv_2, t_jv_3, t_jv_4, t_jv_5, and t_jv_6. We emphasize that a rotation is a cyclic permutation; that is, (123456), (234561), (345612), (456123), (561234), and (612345) denote the same rotation. Thus, $6!/6 = 120$ different $\mathrm{rot}_D(t_j)$ can appear in a drawing of the graph $G^* + D_n$. By $\overline{\mathrm{rot}}_D(t_j)$ we understand the inverse rotation of $\mathrm{rot}_D(t_j)$. In the given drawing D, we separate all subgraphs T^j, $j = 1, \dots, n$, of the graph $G^* + D_n$ into three mutually disjoint subsets depending on how many times the considered T^j crosses the edges of G^* in D. For $j = 1, \dots, n$, let

$R_D = \{T^j : \mathrm{cr}_D(G^*, T^j) = 0\}$ and $S_D = \{T^j : \mathrm{cr}_D(G^*, T^j) = 1\}$. Every other subgraph T^j crosses the edges of G^* at least twice in D. For $T^j \in R_D \cup S_D$, let F^j denote the subgraph $G^* \cup T^j$, $j \in \{1, 2, \ldots, n\}$, of $G^* + D_n$ and let $D(F^j)$ be its subdrawing induced by D. Due to arguments in the proof of Theorem 3.4, at least one of the sets R_D and S_D must be nonempty in a good drawing D of $G^* + D_n$ with the smallest number of crossings. Thus, we will deal with only drawings of the graph G^* with the possibility of an existence of a subgraph T^j that crosses the edges of G^* at most once. This assumption confirms that there are five non isomorphic planar drawings of G^* given in Fig. 1 in which the vertex notation of the graph G^* will be justified later.

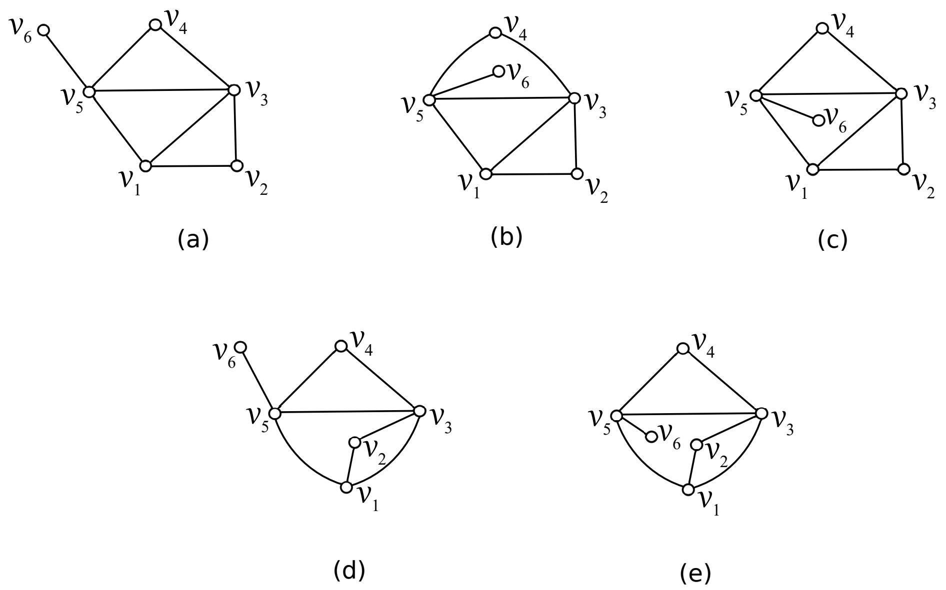

Figure 1: Five non isomorphic planar drawings of the graph G^*.

Let us first assume the drawing of G^* with the corresponding vertex notation in such a way as shown in Fig. 1(a). Our aim is to list all possible rotations $\mathrm{rot}_D(t_j)$ which can appear in D if the edges of T^j do not cross the edges of G^*. Since there is only one subdrawing of $F^j \setminus v_5$ represented by the rotation (16432), there are two ways for how to obtain the subdrawing of F^j depending on in which region the edge t_jv_5 is placed. We denote these two possibilities under our consideration by $\mathcal{R}_1$ and $\mathcal{R}_2$. As for our considerations does not play role which of the regions is unbounded, assume the drawings shown in Fig. 2.

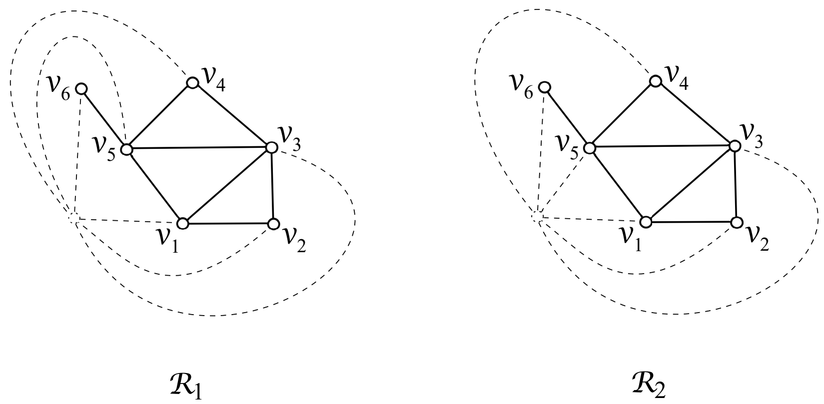

Figure 2: Drawings of two possible configurations from $\mathcal{M}$ of the subgraph F^j.

In the rest of the paper, we represent a cyclic permutation by the permutation with 1 in the first position. Thus, the configurations $\mathcal{R}_1$ and $\mathcal{R}_2$ are represented by the cyclic permutations (165432) and (156432), respectively. Of course, in a fixed drawing of the graph $G^* + D_n$, some configurations from $\mathcal{M} = \{\mathcal{R}_1, \mathcal{R}_2\}$ need not appear. So, we denote by $\mathcal{M}_D$ the set of all configurations of $\mathcal{M}$ that appear in D. Now, we deal with the minimum numbers of crossings between two different subgraphs T^i and T^j depending on the configurations of subgraphs F^i and F^j. Let D be a good drawing of the graph $G^* + D_n$, and let $\mathcal{X}$, $\mathcal{Y}$ be configurations from $\mathcal{M}_D$. We shortly denote by $\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations $\mathcal{X}$, $\mathcal{Y}$, respectively. Finally, let $\mathrm{cr}(\mathcal{X}, \mathcal{Y}) = \min\{\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})\}$ over all pairs $\mathcal{X}$ and $\mathcal{Y}$ from $\mathcal{M}$ among all good drawings of the graph $G^* + D_n$. Our aim is to establish $\mathrm{cr}(\mathcal{X}, \mathcal{Y})$ for all pairs $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$.

Let $\overline{P_j}$ denotes the inverse cyclic permutation to the permutation P_j, for $j = 1, \dots, 120$, where the list with the short names of $6!/6 = 120$ cyclic permutations of six elements was collected in Table 1 of [3]. Woodall [18] has been defined the cyclic-ordered graph COG with the set of vertices $V = \{P_1, P_2, \dots, P_{120}\}$, and with the set of edges E, where two vertices are joined by the edge if the vertices correspond to the permutations P_i and P_j, which are formed by the exchange of exactly two adjacent elements of the 6-tuple (i. e. an ordered set with 6 elements). Hence, if $d_{COG}(''\mathrm{rot}_D(t_i)'', ''\overline{\mathrm{rot}_D(t_j)}'')$ denotes the distance between two vertices which correspond to the cyclic permutations $\mathrm{rot}_D(t_i)$ and $\overline{\mathrm{rot}_D(t_j)}$ in the graph COG, then

$$\mathrm{cr}_D(T^i, T^j) \geq Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) = d_{COG}(''\mathrm{rot}_D(t_i)'', ''\overline{\mathrm{rot}_D(t_j)}'') \tag{2}$$

holds for any two different subgraphs T^i and T^j, where $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ was defined in [3] as the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$ or, equivalently, from $\mathrm{rot}_D(t_j)$ to the inverse of $\mathrm{rot}_D(t_i)$. In particular, the configurations $\mathcal{R}_1$ and $\mathcal{R}_2$ are represented by the cyclic permutations $P_{120} = (165432)$ and $P_{119} = (156432)$, respectively. Since $\overline{P_{119}} = (123465) = P_{25}$, we have $\mathrm{cr}(\mathcal{R}_1, \mathcal{R}_2) \geq 5$ using of $d_{COG}(''P_{25}'', ''P_{120}'') = 5$. Clearly, also $\mathrm{cr}(\mathcal{R}_p, \mathcal{R}_p) \geq 6$ for each $p = 1, 2$. Details have been worked out by Woodall [18]. For easier and more accurate labeling in the proofs of assertions, let us define notation of regions in some subdrawings of $G^* + D_n$. For $T^j \in R_D$, the unique drawing of F^j contains six regions with the vertex t_j on its boundary. For example, if F^j has the configuration $\mathcal{R}_1$, then let us denote these six regions by $\omega_{1,2}$, $\omega_{2,3}$, $\omega_{3,4}$, $\omega_{4,5}$, $\omega_{5,6}$, and $\omega_{1,5,6}$ depending on which of vertices are located on the boundary of the corresponding region. A similar designation may also be used for the case of $T^j \in S_D$.

In the case of $R_D = \emptyset$, our aim shall be to list all possible rotations $\mathrm{rot}_D(t_j)$ which can appear in D if the edges of T^j cross the edges of G^* exactly once. Since the edge v_2v_3 can be crossed by t_jv_1 and the edge t_jv_3 can cross one of the edges v_1v_2, v_1v_5, and v_4v_5, we obtain $4 \times 2 = 8$ possibilities depending on in which region the edge t_jv_5 is placed. Further, if any of the edges t_jv_1 and t_jv_4 crosses the edge v_5v_6, then there is only one possibility for a placement of the edge t_jv_5. Clarity of edge placing of t_jv_5 gives the last way if the edge v_3v_4 is crossed by t_jv_5. We denote these eleven possibilities under our consideration by $\mathcal{A}_p$, for $p = 1, \dots, 11$. Again, as for our considerations, it does not play a role in which of the regions is unbounded; assume the drawings shown in Fig. 3. Thus, the configurations $\mathcal{A}_p$ are represented by the cyclic permutations given in Table 1.

$\mathrm{conf}(F^j)$	$\mathrm{rot}_D(t_j)$	$\mathrm{conf}(F^j)$	$\mathrm{rot}_D(t_j)$
$\mathcal{A}_1$	(135642)	$\mathcal{A}_7$	(136542)
$\mathcal{A}_2$	(126543)	$\mathcal{A}_8$	(125643)
$\mathcal{A}_3$	(156423)	$\mathcal{A}_9$	(165423)
$\mathcal{A}_4$	(164532)	$\mathcal{A}_{10}$	(154326)
$\mathcal{A}_5$	(154632)	$\mathcal{A}_{11}$	(156342)
$\mathcal{A}_6$	(165342)		

Table 1: The corresponding rotations of t_j for $F^j = G^* \cup T^j$, where $T^j \in S_D$.

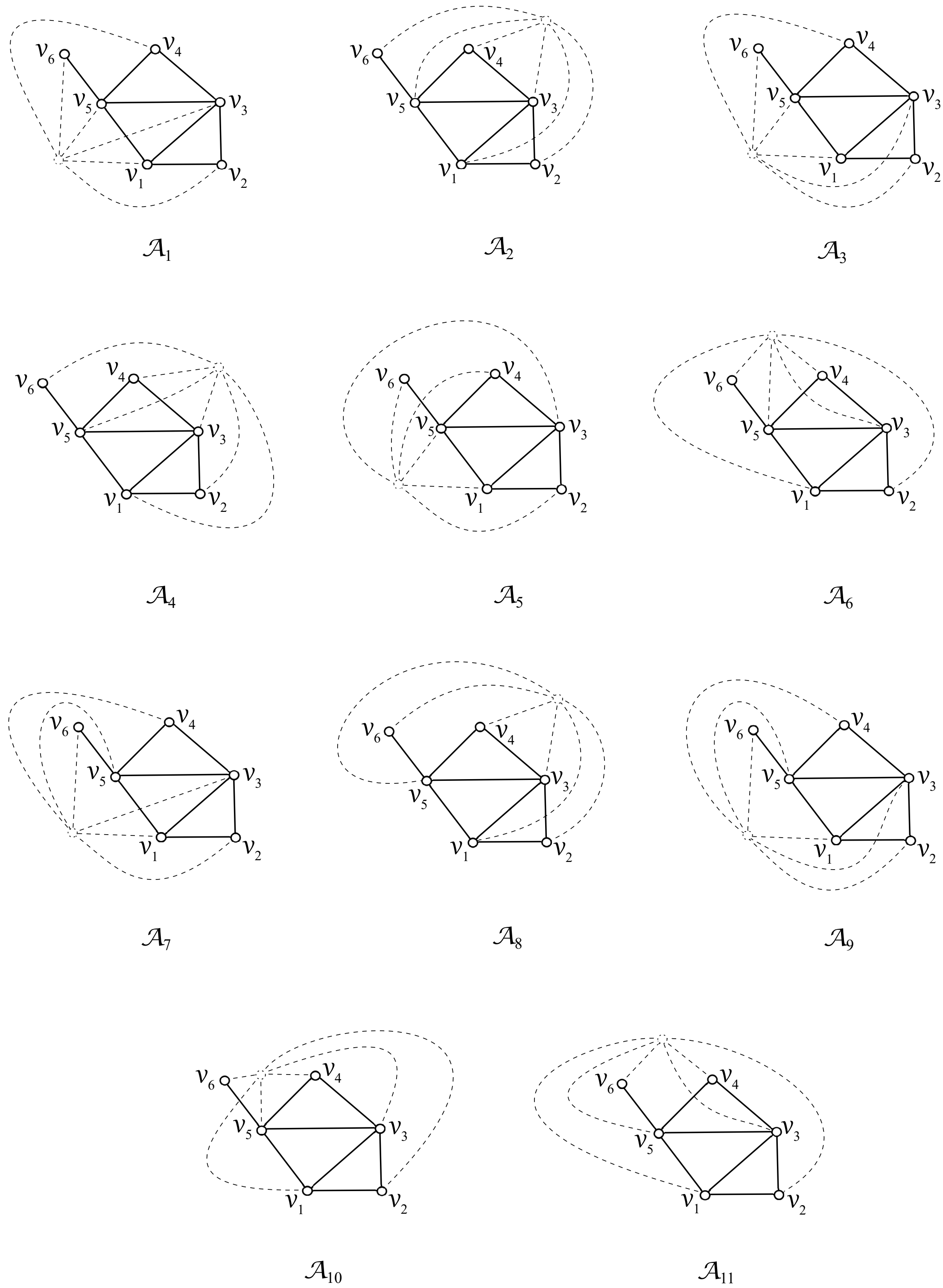

Figure 3: Drawings of eleven possible configurations from $\mathcal{N}$ of the subgraph F^j.

Of course, in a fixed drawing of the graph $G^* + D_n$, some configurations from $\mathcal{N} = \{\mathcal{A}_1, \dots, \mathcal{A}_{11}\}$ need not appear. So, we denote by $\mathcal{N}_D$ the subset of $\mathcal{N}$ consisting of all configurations that exist in the drawing D. Due to the properties of the cyclic rotations, one can easily verify that $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_2) \geq 2$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_3) \geq 5$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_4) \geq 2$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_5) \geq 3$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_6) \geq 3$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_7) \geq 5$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_8) \geq 3$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_9) \geq 4$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_{10}) \geq 4$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_{11}) \geq 4$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_3) \geq 3$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_4) \geq 4$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_5) \geq 3$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_6) \geq 4$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_7) \geq 3$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_8) \geq 5$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_9) \geq 4$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_{10}) \geq 4$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_{11}) \geq 4$, $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_4) \geq 3$, $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_5) \geq 4$, $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_6) \geq 3$, $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_7) \geq 4$, $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_8) \geq 4$, $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_9) \geq 5$, $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_{10}) \geq 3$, $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_{11}) \geq 4$, $\mathrm{cr}(\mathcal{A}_4, \mathcal{A}_5) \geq 3$, $\mathrm{cr}(\mathcal{A}_4, \mathcal{A}_6) \geq 4$, $\mathrm{cr}(\mathcal{A}_4, \mathcal{A}_7) \geq 3$, $\mathrm{cr}(\mathcal{A}_4, \mathcal{A}_8) \geq 3$, $\mathrm{cr}(\mathcal{A}_4, \mathcal{A}_9) \geq 4$, $\mathrm{cr}(\mathcal{A}_4, \mathcal{A}_{10}) \geq 4$, $\mathrm{cr}(\mathcal{A}_4, \mathcal{A}_{11}) \geq 4$, $\mathrm{cr}(\mathcal{A}_5, \mathcal{A}_6) \geq 4$, $\mathrm{cr}(\mathcal{A}_5, \mathcal{A}_7) \geq 2$, $\mathrm{cr}(\mathcal{A}_5, \mathcal{A}_8) \geq 4$, $\mathrm{cr}(\mathcal{A}_5, \mathcal{A}_9) \geq 3$, $\mathrm{cr}(\mathcal{A}_5, \mathcal{A}_{10}) \geq 4$, $\mathrm{cr}(\mathcal{A}_5, \mathcal{A}_{11}) \geq 4$, $\mathrm{cr}(\mathcal{A}_6, \mathcal{A}_7) \geq 4$, $\mathrm{cr}(\mathcal{A}_6, \mathcal{A}_8) \geq 4$, $\mathrm{cr}(\mathcal{A}_6, \mathcal{A}_9) \geq 4$, $\mathrm{cr}(\mathcal{A}_6, \mathcal{A}_{10}) \geq 4$, $\mathrm{cr}(\mathcal{A}_6, \mathcal{A}_{11}) \geq 5$, $\mathrm{cr}(\mathcal{A}_7, \mathcal{A}_8) \geq 2$, $\mathrm{cr}(\mathcal{A}_7, \mathcal{A}_9) \geq 5$, $\mathrm{cr}(\mathcal{A}_7, \mathcal{A}_{10}) \geq 4$, $\mathrm{cr}(\mathcal{A}_7, \mathcal{A}_{11}) \geq 3$, $\mathrm{cr}(\mathcal{A}_8, \mathcal{A}_9) \geq 3$, $\mathrm{cr}(\mathcal{A}_8, \mathcal{A}_{10}) \geq 4$, $\mathrm{cr}(\mathcal{A}_8, \mathcal{A}_{11}) \geq 4$, $\mathrm{cr}(\mathcal{A}_9, \mathcal{A}_{10}) \geq 4$, $\mathrm{cr}(\mathcal{A}_9, \mathcal{A}_{11}) \geq 3$, and $\mathrm{cr}(\mathcal{A}_{10}, \mathcal{A}_{11}) \geq 3$. Moreover, by a discussion of possible subdrawings, we can verify that $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_5) \geq 4$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_4) \geq 6$, $\mathrm{cr}(\mathcal{A}_4, \mathcal{A}_8) \geq 4$, $\mathrm{cr}(\mathcal{A}_5, \mathcal{A}_7) \geq 4$, $\mathrm{cr}(\mathcal{A}_5, \mathcal{A}_9) \geq 4$, $\mathrm{cr}(\mathcal{A}_7, \mathcal{A}_8) \geq 4$, $\mathrm{cr}(\mathcal{A}_7, \mathcal{A}_{11}) \geq 4$, $\mathrm{cr}(\mathcal{A}_8, \mathcal{A}_9) \geq 4$, $\mathrm{cr}(\mathcal{A}_9, \mathcal{A}_{11}) \geq 4$, and $\mathrm{cr}(\mathcal{A}_{10}, \mathcal{A}_{11}) \geq 4$. Clearly, also $\mathrm{cr}(\mathcal{A}_p, \mathcal{A}_p) \geq 6$ for any $p = 1, \dots, 11$. The resulting lower bounds for the number of crossings of configurations from $\mathcal{N}$ are summarized in the symmetric Table 2 (here, $\mathcal{A}_p$ and $\mathcal{A}_q$ are configurations of the subgraphs F^i and F^j, where $p, q \in \{1, \dots, 11\}$).

–	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9$	$\mathcal{A}_{10}$	$\mathcal{A}_{11}$
$\mathcal{A}_1$	6	2	5	2	4	3	5	3	4	4	4
$\mathcal{A}_2$	2	6	3	6	3	4	3	5	4	4	4
$\mathcal{A}_3$	5	3	6	3	4	3	4	4	5	3	4
$\mathcal{A}_4$	2	6	3	6	3	4	3	4	4	4	4
$\mathcal{A}_5$	4	3	4	3	6	4	4	4	4	4	4
$\mathcal{A}_6$	3	4	3	4	4	6	4	4	4	4	5
$\mathcal{A}_7$	5	3	4	3	4	4	6	4	5	4	4
$\mathcal{A}_8$	3	5	4	4	4	4	4	6	4	4	4
$\mathcal{A}_9$	4	4	5	4	4	4	5	4	6	4	4
$\mathcal{A}_{10}$	4	4	3	4	4	4	4	4	4	6	4
$\mathcal{A}_{11}$	4	4	4	4	4	5	4	4	4	4	6

Table 2: The necessary number of crossings between two different subgraphs T^i and T^j for the configurations $\mathcal{A}_p$ and $\mathcal{A}_q$.

Now, let us suppose the drawing of G^* with the considered vertex notations in such a way as shown in Fig. 1(d). In this case, the set R_D is empty, and our aim is to list again all possible rotations $\mathrm{rot}_D(t_j)$ which can appear in D if T^j crosses the edges of G^* exactly once. Of course, the vertex t_j must be placed in the pentagonal region of $D(G^*)$ and the edge t_jv_2 have to cross one edge of G^*. Since there is only one subdrawing of $F^j \setminus \{v_2, v_5\}$ represented by the rotation (1643), there are four ways for how to obtain the subdrawing of F^j depending on in which region the edge t_jv_5 is placed and which of the edges of G^* is crossed by t_jv_2. These four possibilities under our consideration are denoted by $\mathcal{D}_p$, for $p = 1, 2, 3, 4$. Again, as for our considerations, it does not play a role in which of the regions is unbounded; assume the drawings shown in Fig. 4. Thus, the configurations $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ are represented by the cyclic permutations (126543), (156432), (165432), and (125643), respectively. Of course, in a fixed drawing of the graph $G^* + D_n$, some configurations from $\mathcal{O} = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4\}$ need not appear. We denote by $\mathcal{O}_D$ the subset of $\mathcal{O}$ consisting of all configurations that exist in the drawing D. The verification of the lower bounds for number of crossings of two configurations from $\mathcal{O}$ proceeds in the same way like above, and so they can be summarized in the symmetric Table 3 (here, $\mathcal{D}_p$ and $\mathcal{D}_q$ are configurations of the subgraphs F^i and F^j, where $p, q \in \{1, 2, 3, 4\}$).

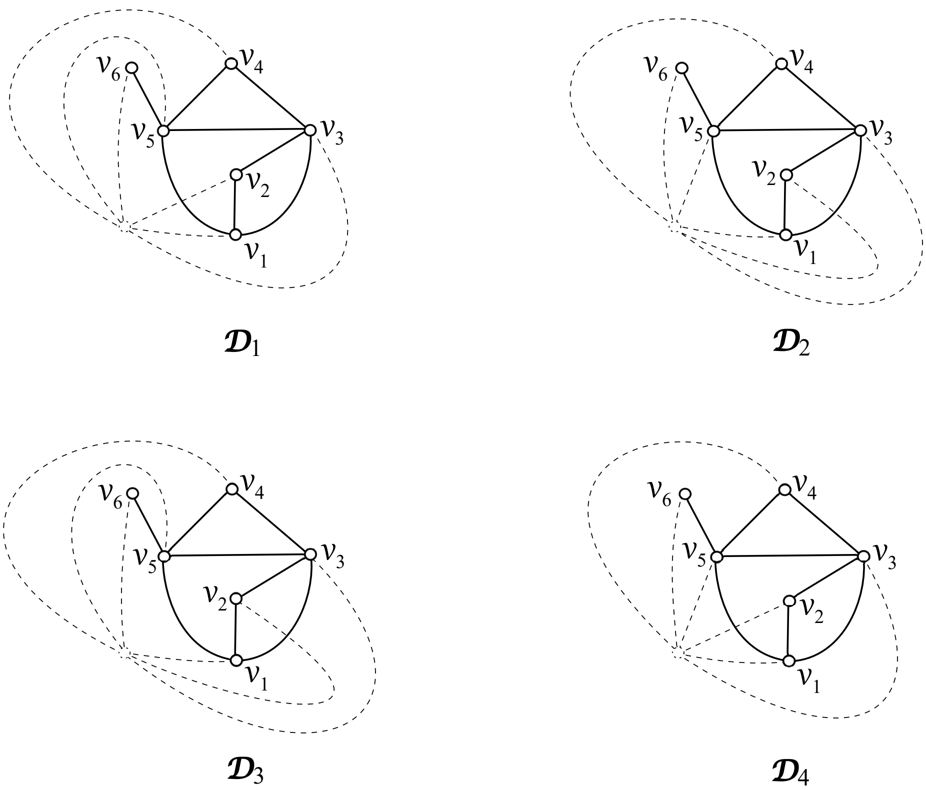

Figure 4: Drawings of four possible configurations from $\mathcal{O}$ of the subgraph F^j.

−	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$
$\mathcal{D}_1$	6	5	5	5
$\mathcal{D}_2$	5	6	5	5
$\mathcal{D}_3$	5	5	6	4
$\mathcal{D}_4$	5	5	4	6

Table 3: The necessary number of crossings between two different subgraphs T^i and T^j for the configurations $\mathcal{D}_p$ and $\mathcal{D}_q$.

Finally, without loss of generality, we consider the drawing with vertex notations of the graph G^* in such a way as shown in Fig. 1(e). In this case, the set R_D is also empty, and our aim is to list again all possible rotations $\mathrm{rot}_D(t_j)$ which can appear in D if $T^j \in S_D$. Of course, the vertex t_j must be placed in the pentagonal region of $D(G^*)$ and the edge t_jv_4 have to cross one edge of G^*. Since there is only one subdrawing of $F^j \setminus \{v_4, v_5\}$ represented by the rotation (1236), there are four ways for how to obtain the subdrawing of F^j depending on in which region the edge t_jv_5 is placed and which of the edges of G^* is crossed by t_jv_4. These four possibilities under our consideration are denoted by $\mathcal{E}_p$, for $p = 1, 2, 3, 4$. Again, as for our considerations, it does not play a role in which of the regions is unbounded; assume the drawings shown in Fig. 5. Thus, the configurations $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, and $\mathcal{E}_4$ are represented by the cyclic permutations (123465), (123456), (123564), and (123654), respectively. Similarly, we denote by $\mathcal{P}_D$ the subset of $\mathcal{P} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ consisting of all configurations that exist in the drawing D. Further, due to the properties of the cyclic rotations, all lower bounds of number of crossings of two configurations from $\mathcal{P}$ can be summarized in the symmetric Table 4 (here, $\mathcal{E}_p$ and $\mathcal{E}_q$ are configurations of the subgraphs F^i and F^j, where $p, q \in \{1, 2, 3, 4\}$).

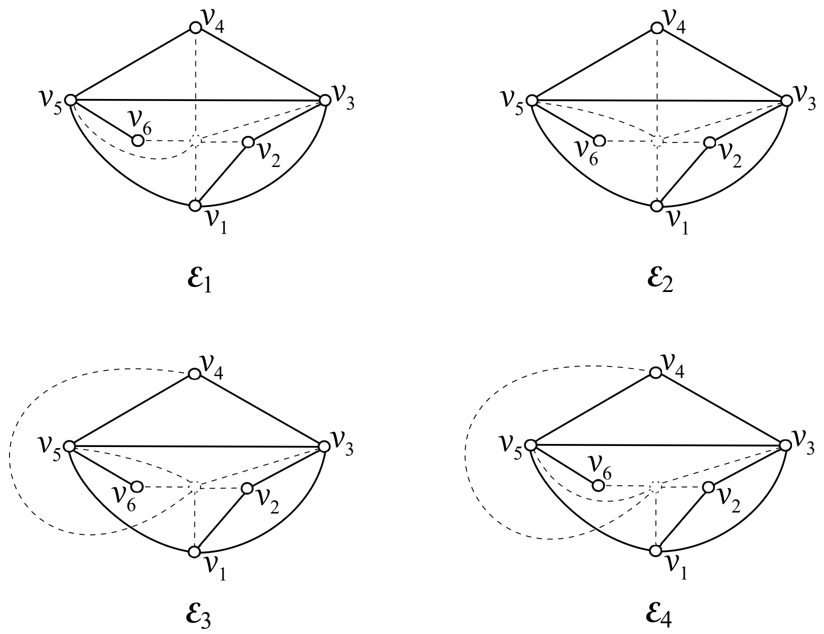

Figure 5: Drawings of four possible configurations from $\mathcal{P}$ of the subgraph F^j.

$-$	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{E}_1$	6	5	4	4
$\mathcal{E}_2$	5	6	4	3
$\mathcal{E}_3$	4	4	6	5
$\mathcal{E}_4$	4	3	5	6

Table 4: The necessary number of crossings between two different subgraphs T^i and T^j for the configurations $\mathcal{E}_p$ and $\mathcal{E}_q$.

3. The Crossing Number of $G^* + D_n$

Two vertices t_i and t_j of the graph $G^* + D_n$ are *antipodal* in a drawing of $G^* + D_n$ if the subgraphs T^i and T^j do not cross. A drawing is *antipode-free* if it has no antipodal vertices. In the proof of Theorem 3.4, the following statements related to some restricted subdrawings of the graph G^*+D_n are needful. Let us first note that if D is a good and antipode-free drawing of G^*+D_n with the vertex notation of the graph G^* in such a way as shown in Fig. 1(a), and $T^j \in S_D$ such that F^j has configuration $\mathcal{A}_p \in \mathcal{N}_D$, then $\mathrm{cr}_D(G^* \cup T^j, T^l) \geq 3$ holds for any T^l, $l \neq j$, see Fig. 3. Further, there are possibilities of obtaining a subgraph $T^l \notin R_D \cup S_D$ with $\mathrm{cr}_D(G^* \cup T^j, T^l) = 3$ only for the cases of the configurations $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_4$ of F^j.

Lemma 3.1. *Let D be a good and antipode-free drawing of $G^* + D_n$, for $n > 2$, with the vertex notation of the graph G^* in such a way as shown in Fig. 1(a). If T^i, $T^j \in S_D$ are different subgraphs such that F^i, F^j have different configurations from any of the sets $\{\mathcal{A}_1, \mathcal{A}_2\}$ and $\{\mathcal{A}_1, \mathcal{A}_4\}$, then*

$$\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) \geq 7 \qquad \textit{for any } T^l \notin R_D \cup S_D.$$

Proof. Let us assume the configurations $\mathcal{A}_1$ of F^i and $\mathcal{A}_2$ of F^j, and remark that they are represented by the cyclic permutations $P_{109} = (135642)$ and $P_{87} = (126543)$, respectively. Let T^l be any subgraph with $l \neq i, j$. We are able to use the property of crossings among edges of its subgraph $K_{6,2}$ with the help of Woodall's results in [18], that is, $\mathrm{cr}_D(T^i \cup T^j, T^l) \geq Q\left(\mathrm{rot}_D(t_i), \overline{\mathrm{rot}_D(t_j)}\right)$ in the subdrawing of $T^i \cup T^j \cup T^l$ induced by

D for any $l \neq i, j$. As $d_{COG}("P_{109}", "P_{87}") = 4$, this enforces $\mathrm{cr}_D(T^i \cup T^j, T^l) \geq 4$. It is obvious that the case $\mathrm{cr}_D(G^*, T^l) \geq 3$ implies $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) \geq 3 + 4 = 7$. In addition, we will only deal with a subgraph T^l that crosses the edges of G^* exactly twice.

Moreover, if we still assume a T^l with $\mathrm{cr}_D(T^i, T^l) = 1$, then the vertex t_l must be placed in the quadrangular region of $D(F^i)$ with three vertices v_2, v_3, and v_4 of G^* on its boundary, i.e., $t_l \in \omega_{2,3,4}$. This enforces that the edge v_2v_3 and v_3v_4 of the graph G^* must be crossed by the edge t_lv_1 and t_lv_5, respectively, and $\mathrm{cr}_D(T^i, T^l) = 1$ only for T^l with $\mathrm{rot}_D(t_l) = (126453) = P_{81}$. Using $\overline{P_{87}} = (134562) = P_{97}$, and $d_{COG}("P_{81}", "P_{97}") = 5$ we obtain $\mathrm{cr}_D(T^j, T^l) \geq 5$. Hence, $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) \geq 2 + 1 + 5 = 8$. Since we can apply the same idea for the case of $\mathrm{cr}_D(T^j, T^l) = 1$, in addition, let us suppose that $\mathrm{cr}_D(T^i, T^l) \geq 2$ and $\mathrm{cr}_D(T^j, T^l) \geq 2$ for any such T^l with $\mathrm{cr}_D(G^*, T^l) = 2$. Of course, if $\mathrm{cr}_D(T^i, T^l) > 2$ or $\mathrm{cr}_D(T^j, T^l) > 2$, we obtain the considered result $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) \geq 2 + 3 + 2 = 7$.

Finally, let us assume a T^l with $\mathrm{cr}_D(G^*, T^l) = 2$, $\mathrm{cr}_D(T^i, T^l) = 2$, and $\mathrm{cr}_D(T^j, T^l) = 2$. The vertex t_l must be placed in one quadrangular region of $D(F^i)$ with three vertices of G^* on its boundary, i.e., $t_l \in \omega_{2,3,4} \cup \omega_{4,5,6}$. We can easy to verify if $t_l \in \omega_{2,3,4}$ then t_lv_3 does not cross any edge of G^*, and the edge v_4v_5 of G^* must be crossed by t_lv_3 in the case of $t_l \in \omega_{4,5,6}$. As in both cases the edge t_lv_4 cannot cross any edge of G^*, likewise, it must be true for the subdrawing $D(F^j)$. The assumptions $\mathrm{cr}_D(G^*, T^l) = 2$ and $\mathrm{cr}_D(T^j, T^l) = 2$ imply that the vertex t_l must be placed in the pentagonal region of $D(F^j)$ with four vertices of G^* on its boundary, i.e., $t_l \in \omega_{1,2,5,6}$. Since the edge t_lv_4 cannot cross any edge of G^*, then t_lv_4 have to cross exactly two edges of the subgraph T^i. This enforces that no edge of t_lv_1, t_lv_2, t_lv_5, and t_lv_6 is crossed in the subdrawing $D(F^i \cup F^j)$. Since the edge t_lv_3 cannot cross two edges of G^*, we obtain a contradiction.

The similar arguments can be applied for the pair $\{\mathcal{A}_1, \mathcal{A}_4\}$, and the proof is done. □

Lemma 3.2. *Let D be a good and antipode-free drawing of $G^* + D_n$, for $n > 2$, with the vertex notation of the graph G^* in such a way as shown in Fig. 1(a). If T^i, $T^j \in S_D$ are different subgraphs such that F^i, F^j have different configurations from any of the sets $\{\mathcal{A}_1, \mathcal{A}_8\}$, $\{\mathcal{A}_2, \mathcal{A}_3\}$, $\{\mathcal{A}_2, \mathcal{A}_5\}$, $\{\mathcal{A}_2, \mathcal{A}_7\}$, $\{\mathcal{A}_3, \mathcal{A}_4\}$, $\{\mathcal{A}_3, \mathcal{A}_6\}$, $\{\mathcal{A}_3, \mathcal{A}_{10}\}$, $\{\mathcal{A}_4, \mathcal{A}_5\}$, and $\{\mathcal{A}_4, \mathcal{A}_7\}$, then*

$$\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) \geq 7 \qquad \textit{for any } T^l \textit{ with } \mathrm{cr}_D(G^*, T^l) = 2.$$

Proof. Let us assume the configurations $\mathcal{A}_1$ of F^i and $\mathcal{A}_8$ of F^j, and note that they are represented by the cyclic permutations $P_{109} = (135642)$ and $P_{85} = (125643)$, respectively, and let also $T^l \notin R_D \cup S_D$ be a subgraph that crosses the edges of G^* exactly twice. If $\mathrm{cr}_D(T^i, T^l) = 1$, then the subdrawing $D(F^l)$ can be represented only by the cyclic permutation $P_{81} = (126453)$ due to the arguments in the proof of Lemma 3.1. Using $\overline{P_{85}} = (134652) = P_{103}$, and $d_{COG}("P_{81}", "P_{103}") = 4$ we obtain $\mathrm{cr}_D(T^j, T^l) \geq 4$. Thus, $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) \geq 2+1+4 = 7$. We can apply the same idea for the case of $\mathrm{cr}_D(T^j, T^l) = 2$. Let us assume that $\mathrm{cr}_D(T^i, T^l) \geq 2$ and $\mathrm{cr}_D(T^j, T^l) \geq 3$ for any such subgraph T^l, which yields that $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) \geq 2+2+3 = 7$ clearly holds for any $T^l \notin R_D \cup S_D$ with $\mathrm{cr}_D(G^*, T^l) = 2$. The similar arguments can be used for the remaining pairs of configurations, and this completes the proof. □

We have to emphasize that, in Lemma 3.2, the assumption $\mathrm{cr}_D(G^*, T^l) = 2$ is inevitable. For $T^l \notin R_D \cup S_D$ with $\mathrm{cr}_D(G^*, T^l) = 3$, the reader can easily find a subdrawing of $G^* \cup T^i \cup T^j \cup T^l$ in which $\mathrm{cr}_D(T^i \cup T^j, T^l) = 3$, i.e., $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) = 6$. Further, we cannot generalize Lemma 3.2 for all pairs of different configurations from $\mathcal{N}$. If we consider the configurations $\mathcal{A}_1$ of F^i and $\mathcal{A}_6$ of F^j, then the reader also can easily find a subdrawing of $G^* \cup T^i \cup T^j \cup T^l$ in which $\mathrm{cr}_D(G^* \cup T^i \cup T^j, T^l) = 6$ with $\mathrm{cr}_D(G^*, T^l) = 2$.

Lemma 3.3. *Let D be a good and antipode-free drawing of $G^* + D_n$, $n > 2$, with the vertex notation of the graph G^* in such a way as shown in Fig. 1(a). Let $T^j \in R_D$ be a subgraph such that F^j has configuration $\mathcal{R}_1 \in \mathcal{M}_D$. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^j, T^k) = 3$, then*

a) $\mathrm{cr}_D(G^* \cup T^k \cup T^j, T^l) \geq 8$ *for any subgraph $T^l \in S_D$, $l \neq k$;*
b) $\mathrm{cr}_D(G^* \cup T^k \cup T^j, T^l) \geq 7$ *for any subgraph $T^l \notin R_D \cup S_D$ with* $\mathrm{cr}_D(G^*, T^l) = 2$.

Proof. Let us assume the configuration $\mathcal{R}_1$ of F^j, and remark that it is represented by the cyclic permutation $P_{120} = (165432)$. The unique drawing of F^j contains six regions with the vertex t_j on their boundaries, see Fig. 2. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^j, T^k) = 3$, then the vertex t_k must be placed in the quadrangular region of $D(F^j)$ with three vertices of G^* on its boundary, i.e., $t_k \in \omega_{1,5,6}$. This enforces that the edge v_1v_5 of the graph G^* must be crossed by the edge t_kv_3 and $\mathrm{cr}_D(T^j, T^k) = 3$ only for T^k with $\mathrm{rot}_D(t_k) = (135642) = P_{109}$.

a) As $T^k \in S_D$ and $\mathrm{rot}_D(t_k) = (135642) = P_{109}$, the considered subdrawing $D(F^k)$ can be described as the configuration $\mathcal{A}_1$, for more see Fig. 3. Now, for each $T^l \in S_D$ with $l \neq k$, we are able to determine the minimum numbers of crossings of T^l with the subgraphs T^k and T^j in the first two columns of Table 5. The values in the first column of Table 5 are given by the lower bounds from the first column of Table 2. Since $\overline{P_{120}} = (123456) = P_1$, the values in the second column can be determined by $d_{COG}(''P_1'', ''P_i'')$, where P_i are the corresponding cyclic permutations for all possible configurations $\mathcal{A}_p$, $p = 1, \ldots, 11$ of the subgraph F^l. The smallest value in the last column of Table 5 gives the required minimum number of crossings.

$\mathrm{conf}(F^l)$	$\mathrm{cr}_D(T^k, T^l)$	$\mathrm{cr}_D(T^j, T^l)$	$\mathrm{cr}_D(T^k \cup T^j, T^l)$	$\mathrm{cr}_D(G^* \cup T^k \cup T^j, T^l)$
$\mathcal{A}_1$	6	3	9	10
$\mathcal{A}_2$	2	5	7	8
$\mathcal{A}_3$	5	4	9	10
$\mathcal{A}_4$	2	5	7	8
$\mathcal{A}_5$	4	4	8	9
$\mathcal{A}_6$	3	5	8	9
$\mathcal{A}_7$	5	4	9	10
$\mathcal{A}_8$	3	4	7	8
$\mathcal{A}_9$	4	5	9	10
$\mathcal{A}_{10}$	4	5	9	10
$\mathcal{A}_{11}$	4	4	8	9

Table 5: All possibilities of the subgraph F^l for $T^l \in S_D$ with $\mathrm{cr}_D(T^j, T^k) = 3$, and $T^k \in S_D$.

b) Let $T^l \notin R_D \cup S_D$ be a subgraph with $\mathrm{cr}_D(G^*, T^l) = 2$, that is, the vertex t_l cannot be placed inside the triangular region of $D(G^*)$. If $\mathrm{cr}_D(T^j, T^l) = 2$, then t_l must be placed in the quadrangular region of $D(F^j)$ with three vertices of G^* on its boundary, i.e., $t_l \in \omega_{1,5,6}$. This enforces that the edge v_1v_5 and v_5v_6 of G^* must be crossed by the edge t_lv_3 and t_lv_4, respectively, and $\mathrm{cr}_D(T^j, T^l) = 2$ only for T^l with $\mathrm{rot}_D(t_l) = (135462) = P_{99}$. Using $\overline{P_{99}} = (126453) = P_{81}$, and $d_{COG}(''P_{81}'', ''P_{109}'') = 5$ we obtain $\mathrm{cr}_D(T^k, T^l) \geq 5$. Thus, $\mathrm{cr}_D(G^* \cup T^k \cup T^j, T^l) \geq 2 + 5 + 2 = 9$. We can apply the similar idea in the case of $\mathrm{cr}_D(T^k, T^l) = 1$, i.e., $\mathrm{rot}_D(t_l) = (126453) = P_{81}$, and the distance $d_{COG}(''P_{99}'', ''P_{120}'') = 4$ implies $\mathrm{cr}_D(G^* \cup T^k \cup T^j, T^l) \geq 2 + 1 + 4 = 7$. It remains to consider the case where $\mathrm{cr}_D(T^j, T^l) \geq 3$ and $\mathrm{cr}_D(T^k, T^l) \geq 2$, which yields that $\mathrm{cr}_D(G^* \cup T^k \cup T^j, T^l) \geq 2+2+3 = 7$ trivially holds for each $T^l \notin R_D \cup S_D$ with $\mathrm{cr}_D(G^*, T^l) = 2$.

□

Theorem 3.4. $\mathrm{cr}(G^* + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$.

Proof. Fig. 6 shows the drawing of $G^* + D_n$ with exactly $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 3\lfloor \frac{n}{2} \rfloor$ crossings. Thus,

$$\mathrm{cr}(G^* + D_n) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.$$

We prove the reverse inequality by induction on n. The graph $G^* + D_1$ is planar; hence, $\mathrm{cr}(G^* + D_1) = 0$. The graph $G^* + D_2$ contains a subgraph that is a subdivision of the graph $P_4 + C_3$. It was proved by Klešč [9]

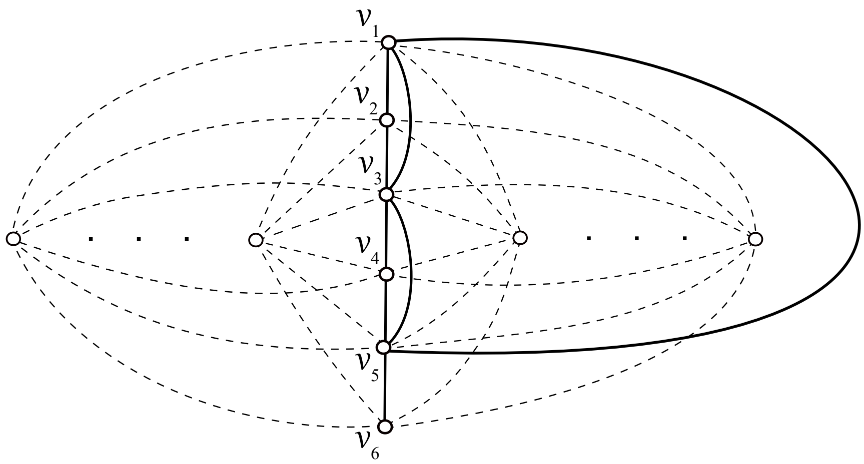

Figure 6: The drawing of $G^* + D_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 3\lfloor \frac{n}{2} \rfloor$ crossings.

that $\mathrm{cr}(P_4 + C_3) = 3$. So, the result is true for $n = 1$ and $n = 2$. Suppose now that, for some $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G^* + D_n) < 6\Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor + 3\Big\lfloor \frac{n}{2} \Big\rfloor, \tag{3}$$

and that

$$\mathrm{cr}(G^* + D_m) = 6\Big\lfloor \frac{m}{2} \Big\rfloor \Big\lfloor \frac{m-1}{2} \Big\rfloor + 3\Big\lfloor \frac{m}{2} \Big\rfloor \qquad \text{for any positiver integer } m < n. \tag{4}$$

We claim that the considered drawing D must be antipode-free. For a contradiction suppose, without loss of generality, that $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. Using positive values in Tables 2, 3 and 4, one can easily verify that both subgraphs T^{n-1} and T^n cannot be from the set S_D. If at least one of T^{n-1} and T^n, say T^n, does not cross G^*, it is not difficult to verify in Fig. 2 that T^{n-1} must cross $G^* \cup T^n$ at least trice, that is, $\mathrm{cr}_D(G^*, T^{n-1} \cup T^n) \geq 3$. By Kleitman [8], we already know that $\mathrm{cr}(K_{6,3}) = 6$, which yields that each T^k, $k = 1, 2, \dots, n-2$, crosses the edges of the subgraph $T^{n-1} \cup T^n$ at least six times. So, for the number of crossings in D we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &= \mathrm{cr}_D\,(G^* + D_{n-2}) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{6,n-2}, T^{n-1} \cup T^n) + \mathrm{cr}_D(G^*, T^{n-1} \cup T^n) \\
&\geq 6\Big\lfloor \frac{n-2}{2} \Big\rfloor \Big\lfloor \frac{n-3}{2} \Big\rfloor + 3\Big\lfloor \frac{n-2}{2} \Big\rfloor + 0 + 6(n-2) + 3 \\
&= 6\Big\lfloor \frac{n}{2} \Big\rfloor \Big\lfloor \frac{n-1}{2} \Big\rfloor + 3\Big\lfloor \frac{n}{2} \Big\rfloor.
\end{aligned}$$

This contradiction with the assumption (3) confirms that D is antipode-free. Moreover, if $r = |R_D|$ and $s = |S_D|$, the assumption (4) together with the well-known fact $\mathrm{cr}(K_{6,n}) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ imply that, in D, if $r = 0$ then there are at least $\lceil \frac{n}{2} \rceil + 1$ subgraphs T^j by which the edges of the graph G^* are crossed exactly once. More precisely:

$$\mathrm{cr}_D(G^*) + \mathrm{cr}_D(G^*, K_{6,n}) < 3\left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$\mathrm{cr}_D(G^*) + 0r + 1s + 2(n - r - s) < 3\left\lfloor \frac{n}{2} \right\rfloor. \tag{5}$$

This forces that $2r + s \geq 2n - 3\lfloor \frac{n}{2} \rfloor + 1$, and if $r = 0$ then $s \geq 2n - 3\lfloor \frac{n}{2} \rfloor + 1 \geq \lceil \frac{n}{2} \rceil + 1$. Now, for $T^j \in R_D \cup S_D$, we will discuss about the existence of possible configurations of subgraph $F^j = G^* \cup T^j$ in the drawing D and we will show that in all cases a contradiction with the assumption (3) is obtained.

Case 1: $\mathrm{cr}_D(G^*) = 0$ and there is the possibility of obtaining a subdrawing of $G^* \cup T^j$ in D for some $T^j \in R_D$. Without loss of generality, we can choose the vertex notation of the graph G^* in such a way as shown in Fig. 1(a). As the set R_D can be empty, two subcases may occur:

a) Let R_D be the nonempty set, i.e., there is a subgraph $T^j \in R_D$. Let us first note that if we denote by t the number of subgraphs T^k whose edges cross the graph G^* exactly twice then the modified inequality (5), for $1s + 2t + 3(n - r - s - t) < 3\lfloor \frac{n}{2} \rfloor$, forces that $r + s + t \geq \lceil \frac{n}{2} \rceil$ and $3r + 2s + t > 3n - 3\lfloor \frac{n}{2} \rfloor$. As we deal with the configurations belonging to the nonempty set $\mathcal{M}_D$, we consider two possibilities. In the case of $\mathcal{R}_2 \in \mathcal{M}_D$, let us assume that $T^j \in R_D$ with the configuration $\mathcal{R}_2$ of F^j. By fixing the subgraph $G^* \cup T^j$ and using a discussion in all possible regions of $D(F^j)$ for $\mathcal{R}_2$ in Fig. 2, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &= \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G^* \cup T^j) + \mathrm{cr}_D(G^* \cup T^j) \\
&\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5(r-1) + 5s + 5t + 4(n - r - s - t) + 0 \\
&= 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n + (r + s + t) - 5 \\
&\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n + \left\lceil \frac{n}{2} \right\rceil - 5 \\
&\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

This contradicts the assumption of D, and therefore, in the next part, let $\mathcal{R}_2 \notin \mathcal{M}_D$, that is, $\mathcal{M}_D = \{\mathcal{R}_1\}$. Without lost of generality, we can assume the configuration $\mathcal{R}_1$ of F^n. It is not difficult to verify that the edges of T^n are crossed by each subgraph $T^k \in S_D$ at least thrice. So, let us denote $S_D(T^n) = \{T^k \in S_D : \mathrm{cr}_D(T^n, T^k) = 3\}$. If T^k is a subgraph from the nonempty set $S_D(T^n)$ then $\mathrm{cr}_D(G^* \cup T^n \cup T^k, T^l) \geq 6 + 3 = 9$ is fulfilling for any $T^l \in R_D$, $l \neq n$ provided by $\mathrm{rot}_D(t_n) = \mathrm{rot}_D(t_l)$. As $\mathrm{cr}_D(G^* \cup T^n \cup T^k) = 4$, by fixing the subgraph $G^* \cup T^n \cup T^k$ and using Lemma 3.3, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 9(r-1) + 8(s-1) + 7t + 6(n - r - s - t) + 4 \\
&= 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6n + (3r + 2s + t) - 13 \\
&\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6n + \left(3n - 3\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - 13 \\
&\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

If the set $S_D(T^n)$ is empty then, by fixing the subgraph $G^* \cup T^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 6(r-1) + 5s + 4(n - r - s) + 0 \\
&= 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n + (2r + s) - 6 \\
&\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n + \left(2n - 3\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - 6 \\
&\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

Both subcases confirm a contradiction with the assumption in D.

b) Let R_D be the empty set, that is, each subgraph T^j crosses the edges of G^* at least once in D. Thus, we deal with the configurations belonging to the nonempty set $\mathcal{N}_D$. Let us first assume that $\{\mathcal{A}_1, \mathcal{A}_p\} \subseteq \mathcal{N}_D$ for some $p \in \{2, 4\}$. Without lost of generality, let us consider two different subgraphs T^{n-1}, $T^n \in S_D$ such that F^{n-1} and F^n have different configurations from $\{\mathcal{A}_1, \mathcal{A}_2\}$. Then, $\mathrm{cr}_D(T^{n-1} \cup T^n, T^l) \geq 7$ is true for any $T^l \in S_D$ with $l \neq n-1, n$ by summing the values in all columns in the first two rows of Table 2. Moreover, $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^l) \geq 7$ is fulfilling for any subgraph $T^l \notin S_D$ by Lemma 3.1. As $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n) \geq 1 + 1 + 2 = 4$, by fixing the subgraph $G^* \cup T^{n-1} \cup T^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(s-2) + 7(n-s) + 4 \\
&= 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7n + s - 12 \\
&\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7n + \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) - 12 \\
&\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

This also contradicts the assumption of D and the same arguments can be used for the case of different configurations from $\{\mathcal{A}_1, \mathcal{A}_4\}$ due to their symmetry. In addition, let us suppose that $\{\mathcal{A}_1, \mathcal{A}_p\} \not\subseteq \mathcal{N}_D$ for $p = 2, 4$. Now, let us assume that some of the sets $\{\mathcal{A}_1, \mathcal{A}_8\}$, $\{\mathcal{A}_2, \mathcal{A}_3\}$, $\{\mathcal{A}_2, \mathcal{A}_5\}$, $\{\mathcal{A}_2, \mathcal{A}_7\}$, $\{\mathcal{A}_3, \mathcal{A}_4\}$, $\{\mathcal{A}_3, \mathcal{A}_6\}$, $\{\mathcal{A}_3, \mathcal{A}_{10}\}$, $\{\mathcal{A}_4, \mathcal{A}_5\}$, and $\{\mathcal{A}_4, \mathcal{A}_7\}$ is a subset of $\mathcal{N}_D$. Without lost of generality, let us consider two different subgraphs T^{n-1}, $T^n \in S_D$ such that F^{n-1} and F^n have different configurations $\mathcal{A}_1$ and $\mathcal{A}_8$, respectively. Then, $\mathrm{cr}_D(T^{n-1} \cup T^n, T^l) \geq 7$ is also true for any $T^l \in S_D$, $l \neq n-1, n$ by summing of two corresponding values of Table 2. Moreover, $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n, T^l) \geq 7$ is fulfilling for any subgraph $T^l \notin S_D$ with $\mathrm{cr}_D(G^*, T^l) = 2$ by Lemma 3.2. Again, if we denote by t the number of subgraphs T^k by which the edges of G^* are crossed exactly twice then the modified inequality (5), for $1s + 2t + 3(n-s-t) < 3\lfloor \frac{n}{2} \rfloor$, confirms that $2s + t > 3n - 3\lfloor \frac{n}{2} \rfloor$. As $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n) \geq 1 + 1 + 3 = 5$, by fixing the subgraph $G^* \cup T^{n-1} \cup T^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 8(s-2) + 7t + 6(n-s-t) + 5 \\
&= 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6n + (2s+t) - 11 \\
&\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6n + \left(3n - 3\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - 11 \\
&\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

All these pairs of configurations confirm a contradiction with the assumption in D, and so in the next, suppose that this case does not occur. Further, at this point, if we consider $\{\mathcal{A}_1, \mathcal{A}_6\} \subseteq \mathcal{N}_D$ then, by fixing the subgraph $G^* \cup T^{n-1} \cup T^n$ with $\mathcal{A}_1$ of F^{n-1} and $\mathcal{A}_6$ of F^n, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 9(s-2) + 6(n-s) + 5 \\
&= 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6n + 3s - 13 \\
&\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6n + 3\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) - 13 \\
&\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

In addition, let us also suppose that $\{\mathcal{A}_1, \mathcal{A}_6\} \not\subseteq \mathcal{N}_D$. Therewith, the minimal numbers of crossings between the edges of two different subgraphs from the set S_D are at least four in the following two subcases:

If we assume $\mathcal{A}_p \in \mathcal{N}_D$ for some $p \in \{3, 5, 6, 7, 8, 9, 10, 11\}$ then, for $T^j \in S_D$ with $\mathcal{A}_p \in \mathcal{N}_D$ of F^j, one can easily verify that $\mathrm{cr}_D(G^* \cup T^j, T^l) \geq 4$ holds for any subgraph $T^l \notin S_D$ using the subdrawing of F^j induced by D, see Fig. 3. Hence, by fixing the subgraph $G^* \cup T^j$, we have

$$\begin{aligned}\mathrm{cr}_D(G^* + D_n) &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 5(s-1) + 4(n-s) + 1\\ &= 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4n + s - 4\\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4n + \left(\left\lceil\frac{n}{2}\right\rceil + 1\right) - 4\\ &\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 3\left\lfloor\frac{n}{2}\right\rfloor.\end{aligned}$$

Finally, in the case either $\mathcal{N}_D = \{\mathcal{A}_p\}$ for only one $p \in \{1, 2, 4\}$ or $\mathcal{N}_D = \{\mathcal{A}_2, \mathcal{A}_4\}$, without lost of generality, let us assume that $T^n \in S_D$ with the configuration $\mathcal{A}_p$ of F^n. Then, $\mathrm{cr}_D(T^n, T^l) \geq 6$ holds for any $T^l \in S_D$, $l \neq n$ by the remaining values of Table 2. Thus, by fixing the subgraph $G^* \cup T^n$, we have

$$\begin{aligned}\mathrm{cr}_D(G^* + D_n) &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 7(s-1) + 3(n-s) + 1\\ &= 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + 4s - 6\\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + 4\left(\left\lceil\frac{n}{2}\right\rceil + 1\right) - 6\\ &\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 3\left\lfloor\frac{n}{2}\right\rfloor.\end{aligned}$$

Case 2: $\mathrm{cr}_D(G^*) = 0$ and there is no possibility of an existence of subgraph $T^j \in R_D$. Since the set R_D is empty, we only need to consider the four subdrawings of G^* in D shown in Fig. 1(b)-(e). In all considered cases, the inequality (5) enforces that there are at least $\lceil\frac{n}{2}\rceil + 1$ subgraphs T^j by which the edges of the graph G^* are crossed exactly once.

b) $\mathrm{cr}_D(G^*) = 0$ and we consider the drawing of G^* with the vertex notation as shown in Fig. 1(b). For $T^j \in S_D$, our aim is to list again all possible rotations $\mathrm{rot}_D(t_j)$ which can appear in D. Since there is only one subdrawing of $F^j \setminus v_6$ represented by the rotation (15432), there are two ways for how to obtain the subdrawing of F^j depending on which edge of G^* is crossed by the edge t_jv_6. These two possibilities under our consideration are denoted by $\mathcal{B}_1$ and $\mathcal{B}_2$, and they are represented by the cyclic permutations (154632) and (156432), respectively. Further, due to the properties of the cyclic permutations, we can easily verify that $\mathrm{cr}(\mathcal{B}_1, \mathcal{B}_2) \geq 5$ (let us note that this idea has been used for an establishing the values in Table 2). As there is a $T^j \in S_D$, by fixing the subgraph $G^* \cup T^j$, we have

$$\begin{aligned}\mathrm{cr}_D(G^* + D_n) &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 6(s-1) + 3(n-s) + 1\\ &= 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + 3s - 5\\ &\geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3n + 3\left(\left\lceil\frac{n}{2}\right\rceil + 1\right) - 5\\ &\geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 3\left\lfloor\frac{n}{2}\right\rfloor.\end{aligned}$$

c) $\mathrm{cr}_D(G^*) = 0$ and we choose the drawing with the vertex notation of G^* as shown in Fig. 1(c). In this case, for a $T^j \in S_D$, the reader can easily verify that the subgraph $F^j = G^* \cup T^j$ is uniquely represented by $\mathrm{rot}_D(t_j) = (165432)$ and $\mathrm{cr}_D(T^j, T^l) \geq 6$ holds for any $T^l \in S_D$, $l \neq j$ provided by $\mathrm{rot}_D(t_j) = \mathrm{rot}_D(t_l)$. Thus, we can apply the same idea as in the previous subcase.

d) $\mathrm{cr}_D(G^*) = 0$ and we consider the drawing with the vertex notation of G^* as shown in Fig. 1(d). In this case, we deal with the configurations belonging to the nonempty set $\mathcal{O}_D$. Note that the lower bounds for the number of crossings of two configurations from $\mathcal{O}$ have been already established in Table 3. Since there is the possibility to find a subdrawing of $G^* \cup T^j \cup T^l$ in which $\mathrm{cr}_D(G^* \cup T^j, T^l) = 3$ with $T^j \in S_D$ and $T^l \notin S_D$, we discuss two following subcases. If we consider a subgraph $T^j \in S_D$ with the configuration $\mathcal{D}_p \in \mathcal{O}_D$ of F^j, where $p \in \{3, 4\}$, then $\mathrm{cr}_D(G^* \cup T^j, T^l) \geq 1 + 4 = 5$ holds for any $T^l \in S_D$, $l \neq j$ using the smallest values of Table 3. Moreover, it is not difficult to verify in possible regions of $D(G^* \cup T^j)$ that $\mathrm{cr}_D(G^* \cup T^j, T^l) \geq 4$ is fulfilling for any subgraph $T^l \notin S_D$. Hence, by fixing the subgraph $G^* \cup T^j$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &\geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 5(s-1) + 4(n-s) + 1 \\
&= 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 4n + s - 4 \\
&\geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 4n + \left(\Big\lceil\frac{n}{2}\Big\rceil + 1\right) - 4 \\
&\geq 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + 3\Big\lfloor\frac{n}{2}\Big\rfloor.
\end{aligned}$$

This subcase contradicts the assumption of D, and therefore, in the next part, suppose that $\mathcal{D}_p \in \mathcal{M}_D$ only for some $p \in \{1, 2\}$. By fixing the subgraph $G^* \cup T^j$ for $T^j \in S_D$ with the configuration either $\mathcal{D}_1$ or $\mathcal{D}_2$ of F^j, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &\geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 6(s-1) + 3(n-s) + 1 \\
&= 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 3n + 3s - 5 \\
&\geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 3n + 3\left(\Big\lceil\frac{n}{2}\Big\rceil + 1\right) - 5 \\
&\geq 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + 3\Big\lfloor\frac{n}{2}\Big\rfloor.
\end{aligned}$$

e) $\mathrm{cr}_D(G^*) = 0$ and we consider the drawing of G^* with the vertex notation as shown in Fig. 1(e). Now, we deal with the configurations belonging to the nonempty set $\mathcal{P}_D$. The lower bounds of number of crossings of two configurations from $\mathcal{P}$ have been also already established in Table 4. Again, since there is the possibility to find a subdrawing of $G^* \cup T^j \cup T^l$ in which $\mathrm{cr}_D(G^* \cup T^j, T^l) = 3$ with $T^j \in S_D$ and $T^l \notin S_D$, we discuss three following subcases:

First, let us assume that $\{\mathcal{E}_2, \mathcal{E}_4\} \subseteq \mathcal{P}_D$ and, in the rest of paper, let us consider two different subgraphs T^{n-1}, $T^n \in S_D$ such that F^{n-1} and F^n have different configurations $\mathcal{E}_2$ and $\mathcal{E}_4$, respectively. Then, $\mathrm{cr}_D(T^{n-1} \cup T^n, T^l) \geq 9$ holds for any $T^l \in S_D$, $l \neq n-1, n$ by summing of two corresponding values in all columns of Table 4. As $\mathrm{cr}_D(G^* \cup T^{n-1} \cup T^n) \geq 1 + 1 + 3 = 5$, by fixing the subgraph $G^* \cup T^{n-1} \cup T^n$, we have

$$\begin{aligned}
\mathrm{cr}_D(G^* + D_n) &\geq 6\Big\lfloor\frac{n-2}{2}\Big\rfloor\Big\lfloor\frac{n-3}{2}\Big\rfloor + 10(s-2) + 5(n-s) + 5 \\
&= 6\Big\lfloor\frac{n-2}{2}\Big\rfloor\Big\lfloor\frac{n-3}{2}\Big\rfloor + 5n + 5s - 15
\end{aligned}$$

$$\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 5n + 5\left(\left\lceil \frac{n}{2} \right\rceil + 1 \right) - 15$$
$$\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.$$

For the subcase $\{\mathcal{E}_2, \mathcal{E}_4\} \not\subseteq \mathcal{P}_D$, if we consider a subgraph $T^j \in S_D$ with the configuration $\mathcal{E}_p \in \mathcal{P}_D$ of F^j, where $p \in \{3, 4\}$, then $\mathrm{cr}_D(G^* \cup T^j, T^l) \geq 1 + 4 = 5$ holds for any $T^l \in S_D$, $l \neq j$ by the remaining values in Table 4. Moreover, we can easy to verify in possible regions of $D(G^* \cup T^j)$ that $\mathrm{cr}_D(G^* \cup T^j, T^l) \geq 4$ is also true for any subgraph $T^l \notin S_D$. Hence, by fixing the subgraph $G^* \cup T^j$, we have

$$\mathrm{cr}_D(G^* + D_n) \geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5(s-1) + 4(n-s) + 1$$
$$= 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n + s - 4$$
$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n + \left(\left\lceil \frac{n}{2} \right\rceil + 1 \right) - 4$$
$$\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.$$

Both subcases contradict the assumption of D, and therefore, in the next part, suppose that $\mathcal{E}_p \in \mathcal{P}_D$ only for some $p \in \{1, 2\}$. Finally, by fixing the subgraph $G^* \cup T^j$ for $T^j \in S_D$ with the configuration either $\mathcal{E}_1$ or $\mathcal{E}_2$ of F^j, we have

$$\mathrm{cr}_D(G^* + D_n) \geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 6(s-1) + 3(n-s) + 1$$
$$= 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3n + 3s - 5$$
$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3n + 3\left(\left\lceil \frac{n}{2} \right\rceil + 1 \right) - 5$$
$$\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor.$$

Case 3: $\mathrm{cr}_D(G^*) \geq 1$. For all possible subdrawings of the graph G^* in D with at least one crossing among edges of G^* and also with a possibility of obtaining a subgraph T^j that crosses the edges of G^* at most once, we are able to apply one of the ideas of the previous subcases.

Let us turn to the possible subdrawings of G^* with $\mathrm{cr}_D(G^*) = 1$. Since the graph G^* contains $P_4 + D_1$ as a subgraph, we only need to consider possibilities either of crossings between planar subdrawings of $P_4 + D_1$ and the bridge of G^*, or of subdrawings of $P_4 + D_1$ with exactly one crossing and with the placement of the bridge of G^* without any new crossing. Thus, assuming a subgraph $T^j \in R_D \cup S_D$, we obtain twelve possible non isomorphic drawings of G^* with one crossing among its edges shown in Fig. 7. If we consider the drawings with the vertex notation of G^* as shown in Fig. 7(a)-(c), the proof can proceed in the similar way as in Case 1. For the subdrawings of G^* in D as shown in Fig. 7(d)-(l), we can use one of the ideas of Case 2. Obviously, with the growing number of crossings in the induced subdrawing $D(G^*)$ for $T^j \in R_D \cup S_D$, the edges of the subgraph $G^* \cup T^j$ will be crossed by T^l, $l \neq j$ more often, and therefore, the considered subcases will be easier to discuss than in Cases 1 and 2.

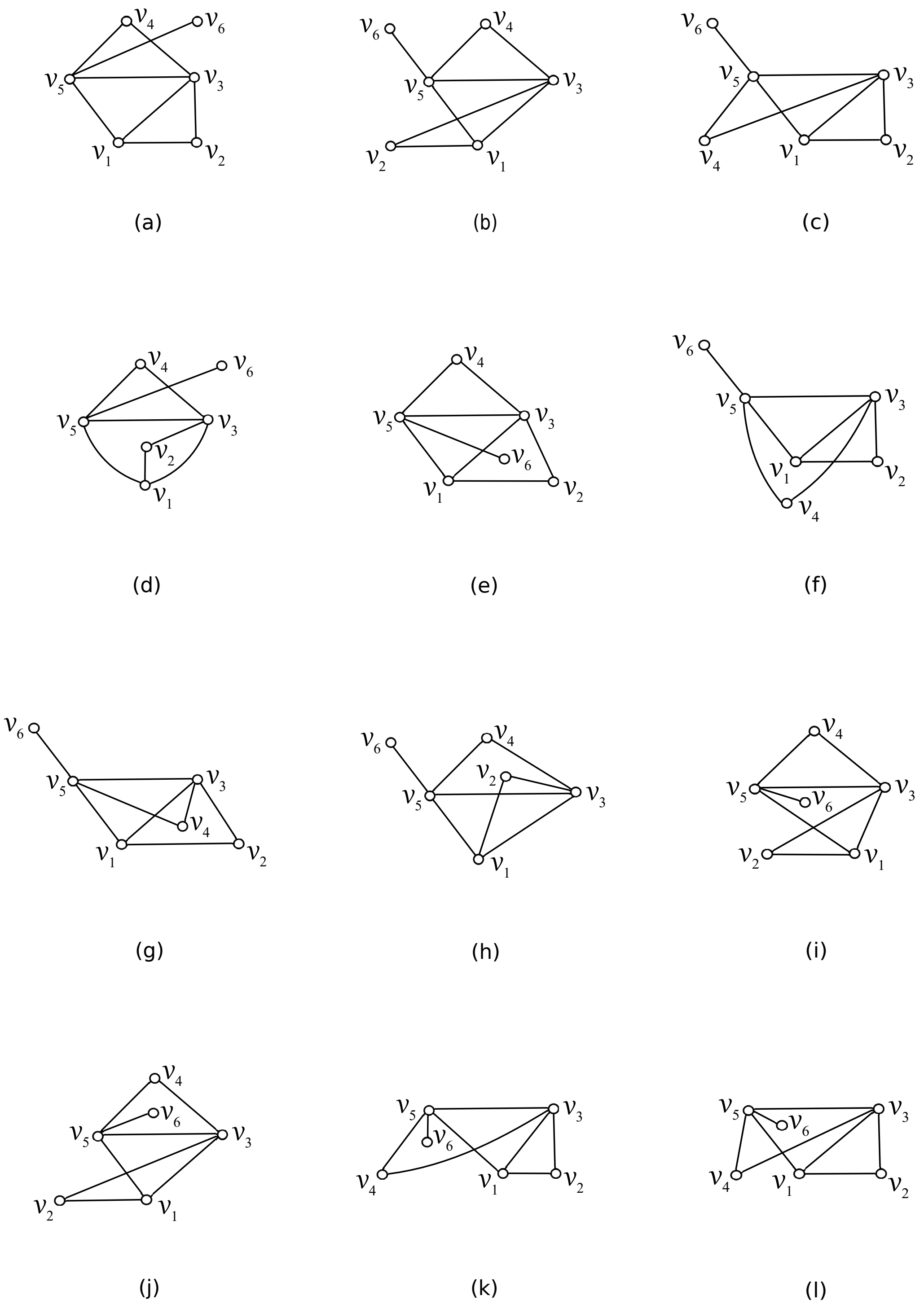

Figure 7: Twelve non isomorphic drawings of the graph G^* with $\mathrm{cr}_D(G^*) = 1$.

Thus, it was shown in all mentioned cases that there is no good drawing D of the graph $G^* + D_n$ with fewer than $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 3\lfloor\frac{n}{2}\rfloor$ crossings, and the proof of the theorem is complete. $\square$

4. One Other Graph

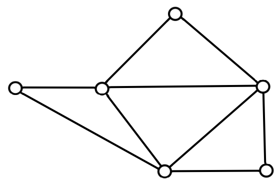

Figure 8: One graph G_1 by adding one edge to the graph G^*.

Finally, into the drawing in Fig. 6, we are able to add the edge v_1v_6 to the graph G^* without additional crossings, and we obtain new graph G_1 represented in Fig. 8. Therefore, the drawing of the graph $G_1 + D_n$ with $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor$ crossings is obtained. On the other hand, $G^* + D_n$ is a subgraph of $G_1 + D_n$, and therefore, $\mathrm{cr}(G_1 + D_n) \geq \mathrm{cr}(G^* + D_n)$. Thus, the next result is obvious.

Corollary 4.1. $\mathrm{cr}(G_1 + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 3\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$.

Acknowledgment

The author thanks the referees for very careful readings and many constructive comments to improve the quality of the paper.

References

[1] K. Asano, *The crossing number of $K_{1,3,n}$ and $K_{2,3,n}$*, J. Graph Theory 10 (1986) 1–8.
[2] Š. Berežný, J. Buša Jr., *Algorithm of the Cyclic-Order Graph Program* (*Implementation and Usage*), Math. Model. Geom. 7(3) (2019) 1–8.
[3] Š. Berežný, M. Staš, *Cyclic permutations and crossing numbers of join products of symmetric graph of order six*, Carpathian J. Math. 34(2) (2018) 143–155.
[4] Š. Berežný, M. Staš, *Cyclic permutations and crossing numbers of join products of two symmetric graphs of order six*, Carpathian J. Math. 35(2) (2019) 137–146.
[5] Š. Berežný, M. Staš, *On the crossing number of join of the wheel on six vertices with the discrete graph*, Carpathian J. Math. 36(3) (2020) 381–390.
[6] C. Hernández-Vélez, C. Medina, G. Salazar, *The optimal drawing of $K_{5,n}$*, Electron. J. Combin. 21 (2014) ♯ P4.1, 29 pp.
[7] P. T. Ho, *The crossing number of $K_{1,1,3,n}$*, Ars Combinatoria 99 (2011) 461–471.
[8] D. J. Kleitman, *The crossing number of $K_{5,n}$*, J. Combinatorial Theory 9 (1970) 315–323.
[9] M. Klešč, *The join of graphs and crossing numbers*, Electron. Notes in Discrete Math. 28 (2007) 349–355.
[10] M. Klešč, *On the Crossing Numbers of Cartesian Products of Stars and Graphs on Five Vertices*, Combinatorial Algorithms, Springer, LNCS, 5874 (2009) 324–333.
[11] M. Klešč, *The crossing number of join of the special graph on six vertices with path and cycle*, Discrete Math. 310(9) (2010) 1475–1481.
[12] M. Klešč, E. Draženská, *The crossing numbers of products of the graph $K_{2,2,2}$ with stars*, Carpathian J. Math. 24(3) (2008) 327–331.
[13] M. Klešč, Š. Schrötter, *The crossing numbers of join products of paths with graphs of order four*, Discuss. Math. Graph Theory 31(2) (2011) 321–331.
[14] M. Klešč, Š. Schrötter, em The crossing numbers of join of paths and cycles with two graphs of order five, Combinatorial Algorithms, Springer, LNCS, 7125 (2012) 160–167.
[15] M. Staš, *Determining crossing number of join of the discrete graph with two symmetric graphs of order five*, Symmetry 11(2) (2019) 1–9.
[16] M. Staš, *On the crossing number of join of the wheel on five vertices with the discrete graph*, Bull. Aust. Math. Soc. 101(3) (2020), 353–361.
[17] M. Staš, *On the crossing number of join product of the discrete graph with special graphs of order five*, Electron. J. Graph Theory Appl. 8(2) (2020) 339–351.
[18] D. R. Woodall, *Cyclic-order graphs and Zarankiewicz's crossing number conjecture*, J. Graph Theory 17(6) (1993) 657–671.