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New Modular Equations of Signature Three in the Spirit of Ramanujan

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Dedicated to Prof. Chandrashekar Adiga on his 62^{nd} birthday.

Abstract. Srinivasa Ramanujan recorded many modular equations in his notebooks, which are useful in the computation of class invariants, continued fractions and the values of theta functions. In this paper, we prove some new modular equations of signature three by using theta function identities of composite degrees.

1. Introduction

Ramanujan [12] begins his study on modular equations in Chapter 15 by defining

$$ {}_1F_0\left(\frac{1}{2};x\right) := \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} x^n = (1-x)^{-1/2}, \qquad |x| < 1. $$

He then states a trivial identity

$$ {}_1F_0\left(\frac{2t}{1+t}\right) = (1+t)\ {}_1F_0(t^2). \tag{1} $$

After setting $\alpha = 2t/(1+t)$ and $\beta = 2t^2$, Ramanujan offers a modular equation of degree two

$$ \beta(2-\alpha)^2 = \alpha^2 $$

and the factor $(1+t)$ in (1) is called the multiplier. Further Ramanujan developed theory of elliptic functions in which q is replaced by one or the other functions for $n = 3, 4$ and 6.

$$ q_n := q_n(x) := exp\left(-\pi csc(\pi/n) \frac{{}_2F_1(1-x)}{{}_2F_1(x)}\right) \qquad 0 < x < 1, $$

where ${}_2F_1(x) = {}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)$ and ${}_2F_1$ represent the classical hypergeometric function defined as follows:

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} z^m, \qquad |z| < 1 $$

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and

$$(\alpha)_m = \alpha(\alpha+1)\dots(\alpha+m-1).$$

For $n = 3$ and 4, the theories are known as cubic and quartic theories respectively. Let us now take up a modular equation as given in the literature. A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1(1-\alpha)}{{}_2F_1(\alpha)} = \frac{{}_2F_1(1-\beta)}{{}_2F_1(\beta)}$$

and

$$m = \frac{z_1}{z_n}$$

is the multiplier connecting α and β, where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$. But a modular equation of degree n in the theory of signature three, is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)}.$$

Then, we say that β is of degree n over α and call the ratio

$$m = \frac{z_1}{z_n}$$

is the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)$. All through this paper, we have assumed $|q| < 1$. The standard q-shifted factorial is defined as

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{i=1}^{n}(1-aq^{i-1}) \qquad \text{and} \qquad (a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i).$$

In classical theory, for $q = q_2$, Ramanujan [12] has defined theta function [6] as follows:

$$\varphi(q) := \mathfrak{f}(q,q) = 1 + \sum_{i=1}^{\infty} q^{i^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{i=0}^{\infty} q^{i(i+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{i=0}^{\infty}(-1)^i q^{i(3i-1)/2} + \sum_{i=1}^{\infty}(-1)^i q^{i(3i+1)/2} = (q;q)_\infty.$$

For convenience, we write $f(-q^n)$ by f_n and one can easily deduce the following:

$$\varphi(-q) = \frac{f_1^2}{f_2} \qquad \text{and} \qquad \psi(q) = \frac{f_2^2}{f_1}. \tag{2}$$

Ramanujan [12, vol II] in his notebooks documented some cubic modular equations. Further these are proved by B. C. Berndt [7], he used parameterization and modular forms. Also C. Adiga et. al [1–3], M. S. M. Naika [9], M. S. M. Naika and S. Chandankumar [10] and N. Saikia and J. Chetry [14] obtained some modular equations of signature three. H. M. Srivastava and M. P. Chaudhary [15] established a set of four new results which depict the interrelationships between q-identities, continued fraction identities and combinatorial partition identities. Also in [16] H. M. Srivastava et. al. deduced some q-identities involving theta functions. These q-identities have relationship among three of the theta-type functions which arise from Jacobi's triple product identity. In section 2 of this paper, we list some $P-Q$ type theta function identities which will be utilized to demonstrate modular equations of signature three. In section 3, we prove composite degrees of cubic modular equations.

2. Preliminaries: $P-Q$ type theta function identities

Lemma 2.1. *[20] If*

$$P := \frac{\varphi(-q)}{\varphi(-q^3)} \qquad \text{and} \qquad Q := \frac{\varphi(-q^2)}{\varphi(-q^6)}$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 = \frac{3}{Q^2} - Q^2.$$

Lemma 2.2. *[13] If*

$$P := \frac{\varphi(q)}{\varphi(q^3)} \qquad \text{and} \qquad Q := \frac{\varphi(q^5)}{\varphi(q^{15})}$$

then

$$(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 + 5\left(\frac{Q}{P}\right)^2 + 5\left(\frac{Q}{P}\right) - 5\left(\frac{P}{Q}\right) + 5\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3.$$

Lemma 2.3. *[21] If*

$$P := \frac{\varphi(-q)}{\varphi(-q^3)} \qquad \text{and} \qquad Q := \frac{\varphi(-q^{11})}{\varphi(-q^{33})}$$

then

$$(PQ)^5 + \frac{3^5}{(PQ)^5} - 11\left[(PQ)^3 + \frac{3^3}{(PQ)^3}\right] + 308\left[PQ + \frac{3}{PQ}\right] = \left(\frac{P}{Q}\right)^6 + \left(\frac{Q}{P}\right)^6 + 22\left[\left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4\right]$$
$$\times\left[3 - \left(PQ + \frac{3}{PQ}\right)\right] + 11\left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right] \times \left[(PQ)^3 + \frac{3^3}{(PQ)^3} - 15\left(PQ + \frac{3}{PQ}\right) + 45\right] + 924.$$

Lemma 2.4. *[21] If*

$$P := \frac{\varphi(-q)}{\varphi(-q^3)} \qquad \text{and} \qquad Q := \frac{\varphi(-q^{13})}{\varphi(-q^{39})}$$

then

$$\left(\frac{P}{Q}\right)^7+\left(\frac{Q}{P}\right)^7+13\left[\left(\frac{P}{Q}\right)^6+\left(\frac{Q}{P}\right)^6\right]-26\left[\left(\frac{P}{Q}\right)^5+\left(\frac{Q}{P}\right)^5\right]-13\left[3\,(PQ)^2+\frac{27}{(PQ)^2}+10\right]\left[\left(\frac{P}{Q}\right)^4+\left(\frac{Q}{P}\right)^4\right]$$
$$+13\left[10\,(PQ)^2+\frac{90}{(PQ)^2}+68\right]\left[\left(\frac{P}{Q}\right)^3+\left(\frac{Q}{P}\right)^3\right]+13\left[(PQ)^4+\frac{81}{(PQ)^4}-20\left[(PQ)^2+\frac{9}{(PQ)^2}\right]-115\right]$$
$$\times\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]-13\left[(PQ)^4+\frac{81}{(PQ)^4}-10\left[(PQ)^2+\frac{9}{(PQ)^2}\right]-131\right]\left[\frac{P}{Q}+\frac{Q}{P}\right]$$
$$=(PQ)^6+\frac{729}{(PQ)^6}-26\left[(PQ)^4+\frac{81}{(PQ)^4}\right]+169\left[(PQ)^2+\frac{9}{(PQ)^2}\right]+832.$$

Lemma 2.5. *[11] If*

$$P:=\frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \textit{and} \qquad P:=\frac{\psi(q^3)}{q^{3/4}\psi(q^9)}$$

then

$$(PQ)^2+\frac{9}{(PQ)^2}=3+6\frac{Q^2}{P^2}+\frac{Q^4}{P^4}.$$

Lemma 2.6. *[17] If*

$$P:=\frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \qquad \textit{and} \qquad Q:=\frac{\psi(-q^{11})}{q^{11/4}\psi(-q^{33})}$$

then

$$(PQ)^5+\frac{3^5}{(PQ)^5}-11\left[(PQ)^3+\frac{3^3}{(PQ)^3}\right]+308\left[PQ+\frac{3}{PQ}\right]=\left(\frac{P}{Q}\right)^6+\left(\frac{Q}{P}\right)^6+22\left[\left(\frac{P}{Q}\right)^4+\left(\frac{Q}{P}\right)^4\right]$$
$$\times\left[3-\left(PQ+\frac{3}{PQ}\right)\right]+11\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]\left[(PQ)^3+\frac{3^3}{(PQ)^3}-15\left(PQ+\frac{3}{PQ}\right)+45\right]+924.$$

Lemma 2.7. *[5] If*

$$P:=\frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \textit{and} \qquad Q:=\frac{\psi(q^7)}{q^{7/4}\psi(q^{21})}$$

then

$$K_1\,(PQ)^3+K_2PQ=K_3\,(PQ)^2+K_4\left(\frac{P}{Q}\right)^2-K_5,$$

where

$$K_1=\left(\frac{P}{Q}\right)^8-1, \qquad K_2=14P^4\left(\left(\frac{P}{Q}\right)^4-1\right), \qquad K_3=P^4\left(7-P^4\right),$$
$$K_4=7P^4\left(P^4-3\right), \qquad K_5=27\left(\frac{P}{Q}\right)^4-7P^4\left(3+3\left(\frac{P}{Q}\right)^4-P^4\right).$$

Lemma 2.8. *[17] If*

$$P := \frac{\psi(-q)}{q^{1/4}\psi(-q^3)} \qquad \text{and} \qquad Q := \frac{\psi(-q^{13})}{q^{13/4}\psi(-q^{39})}$$

then

$$\begin{aligned}
&\left(\frac{P}{Q}\right)^7 + \left(\frac{Q}{P}\right)^7 + 13\left[\left(\frac{P}{Q}\right)^6 + \left(\frac{Q}{P}\right)^6\right] - 26\left[\left(\frac{P}{Q}\right)^5 + \left(\frac{Q}{P}\right)^5\right] - 13\left[3(PQ)^2 + \frac{27}{(PQ)^2} + 10\right]\left[\left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4\right]\\
&+ 13\left[10(PQ)^2 + \frac{90}{(PQ)^2} + 68\right]\left[\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3\right] + 13\left[(PQ)^4 + \frac{81}{(PQ)^4} - 20\left[(PQ)^2 + \frac{9}{(PQ)^2}\right] - 115\right]\\
&\times\left[\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2\right] - 13\left[(PQ)^4 + \frac{81}{(PQ)^4} - 10\left[(PQ)^2 + \frac{9}{(PQ)^2}\right] - 131\right]\left[\frac{P}{Q} + \frac{Q}{P}\right]\\
&= (PQ)^6 + \frac{729}{(PQ)^6} - 26\left[(PQ)^4 + \frac{81}{(PQ)^4}\right] + 169\left[(PQ)^2 + \frac{9}{(PQ)^2}\right] + 832.
\end{aligned}$$

Lemma 2.9. *[8] If*

$$P := q^{1/6}\frac{f_1 f_9}{f_3^2} \qquad \text{and} \qquad Q := q^{1/3}\frac{f_2 f_{18}}{f_6^2}$$

then

$$\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3 = \frac{1}{PQ} - 3PQ.$$

Lemma 2.10. *[18] If*

$$P := \frac{f_2 f_3}{q^{1/12} f_1 f_6} \qquad \text{and} \qquad Q := \frac{f_{10} f_{15}}{q^{5/12} f_5 f_{30}}$$

then

$$(PQ)^2 + \frac{1}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 + 5.$$

Lemma 2.11. *[18] If*

$$P := \frac{f_1 f_2}{q^{1/4} f_3 f_6} \qquad \text{and} \qquad Q := \frac{f_5 f_{10}}{q^{5/4} f_{15} f_{30}}$$

then

$$(PQ)^2 + \frac{81}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 - 5\left(\frac{Q}{P}\right)^2 - 5\left(\frac{P}{Q}\right)^2 - 5\left(\frac{Q}{P}\right) - 5\left(\frac{P}{Q}\right) + 20.$$

Lemma 2.12. *[19] If*

$$P := q^{1/12}\frac{f_1 f_6}{f_2 f_3} \qquad \text{and} \qquad Q := q^{1/4}\frac{f_3 f_{18}}{f_6 f_9}$$

then

$$\left[\left(\frac{P}{Q}\right)^6 + \left(\frac{Q}{P}\right)^6\right]\left[(PQ)^3 + \frac{1}{(PQ)^3} + 1\right] = (PQ)^6 + \frac{1}{(PQ)^6} + 10\left[(PQ)^3 + \frac{1}{(PQ)^3}\right] + 20.$$

Lemma 2.13. *[4] If*

$$P := \frac{f_1 f_2}{q^{1/4} f_3 f_6} \qquad \text{and} \qquad Q := \frac{f_2 f_4}{q^{1/2} f_6 f_{12}}$$

then

$$\left(\frac{Q}{P}\right)^8 + \left(\frac{P}{Q}\right)^8 - 7\left[\left(\frac{Q}{P}\right)^4 + \left(\frac{P}{Q}\right)^4\right] = \left[\left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2\right]\left[(PQ)^2 + \frac{81}{(PQ)^2}\right] + 24.$$

Lemma 2.14. *[11] If*

$$P := \frac{\varphi^2(-q)}{\varphi^2(-q^3)} \qquad \text{and} \qquad Q := \frac{\varphi^2(-q^3)}{\varphi^2(-q^9)}$$

then

$$PQ + \frac{9}{PQ} = 3 + 6\frac{Q}{P} + \frac{Q^2}{P^2}.$$

3. Main Results: New modular equations of composite degree in signature three

Theorem 3.1. *If β, γ and δ be the second, forth and eighth degrees respectively with respect to α and if*

$$u = \left(\frac{(1-\alpha)^2\beta^3(1-\gamma)}{\alpha^2(1-\beta)^3\gamma}\right)^{1/12} \qquad \text{and} \qquad v = \left(\frac{(1-\beta)^2\gamma^3(1-\delta)}{\beta^2(1-\gamma)^3\delta}\right)^{1/12}$$

then

$$u^4v^2 - u^3 - 2u^2v^2 + v^4u^3 + uv^4 - u + v^2 = 0.$$

Proof. Let

$$A = \frac{\varphi(-q)}{\varphi(-q^3)}, \qquad B = \frac{\varphi(-q^2)}{\varphi(-q^6)} \qquad \text{and} \qquad C = \frac{\varphi(-q^4)}{\varphi(-q^{12})}. \tag{3}$$

On employing (3) in Lemma 2.1, it is observed that

$$\frac{3}{B^2} - B^2 = \left(\frac{A}{B}\right)^2 + \left(\frac{B}{A}\right)^2 \tag{4}$$

and

$$\frac{3}{C^2} - C^2 = \left(\frac{B}{C}\right)^2 + \left(\frac{C}{B}\right)^2. \tag{5}$$

From [14], we have

$$\frac{f_1}{q^{1/12} f_3} = 3^{1/4}\left(\frac{1}{\alpha} - 1\right)^{1/12}. \tag{6}$$

Using (2) and (6) in (3), we observe that

$$\frac{A}{B} = \left(\frac{(1-\alpha)^2\beta^3(1-\gamma)}{\alpha^2(1-\beta)^3\gamma}\right)^{1/12} = u \tag{7}$$

and

$$\frac{B}{C} = \left(\frac{(1-\beta)^2\gamma^3(1-\delta)}{\beta^2(1-\gamma)^3\delta}\right)^{1/12} = v. \tag{8}$$

On using (7) and (8) in (4) and (5) , we obtain

$$u^2B^4 + (u^4+1)B^2 - 3u^2 = 0 \tag{9}$$

and

$$B^4 + (v^4+1)B^2 - 3v^4 = 0, \tag{10}$$

respectively. From (9) and (10), we deduce that

$$\frac{B^4}{3u^2(v^4+1) - 3v^4(u^4+1)} = \frac{B^2}{3u^2v^4 - 3u^2} = \frac{1}{u^2(v^4+1) - u^4 - 1}.$$

Which implies

$$B^4 = \frac{3u^2(v^4+1) - 3v^4(u^4+1)}{u^2(v^4+1) - u^4 - 1} \tag{11}$$

and

$$B^2 = \frac{3u^2v^4 - 3u^2}{u^2(v^4+1) - u^4 - 1}. \tag{12}$$

On combining (11) and (12) and then factorizing, we obtain

$$M(u,v)N(u,v) = 0,$$

where

$$M(u,v) = u^4v^2 - u^3 - u - 2v^2u^2 + v^2 + u^3v^4 + uv^4$$

and

$$N(u,v) = u^4v^2 - u^3 + u - 2v^2u^2 + v^2 - u^3v^4 - uv^4.$$

The series expansion of u and v are

$$u = 1 - 2q + 2q^2 - 2q^3 + 2q^4 - 4q^5 + 6q^6 - 8q^7 + 4q^8 + \dots$$

and

$$v = 1 - 2q^2 + 2q^4 - 2q^6 + 2q^8 - 4q^{10} + 8q^{12} + \dots.$$

Using these in $M(u,v)$ and $N(u,v)$, we obtain

$$M(u,v) = 96q^9 - 384q^{10} + 416q^{11} + 240q^{12} - 192q^{13} + \dots$$

and

$$N(u,v) = 2 - 12q + 4q^2 + 52q^3 - 60q^4 - 88q^5 + \dots.$$

One can see that $q^{-1}M(u,v) \to$ to 0 as $q \to 0$, where as $q^{-1}N(u,v) \nrightarrow 0$ as $q \to 0$. By analytic continuation, we obtain

$$u^4v^2 - u^3 - 2u^2v^2 + u^3v^4 + uv^4 - u + v^2 = 0.$$

□

Theorem 3.2. *If β, γ, δ, η and ζ be the second, fifth, tenth, twenty fifth and fiftieth degrees respectively with respect to α and if*

$$u=\left(\frac{(1-\alpha)^2\beta\gamma^2(1-\delta)}{\alpha^2(1-\beta)(1-\gamma)^2\delta}\right)^{1/6} \qquad \textit{and} \qquad v=\left(\frac{(1-\gamma)^2\delta\eta(1-\zeta)}{\gamma^2(1-\delta)(1-\eta)^2\zeta}\right)^{1/6}$$

then

$$(bu^2-av^4)(av^2-bu^4)=9u^4v^4(1-u^2v^2)^2,$$

where

$$a=1+5u+5u^2-5u^4+5u^5-u^6 \qquad \textit{and} \qquad b=1+5v+5v^2-5v^4+5v^5-v^6.$$

Proof. Let

$$A=\frac{\varphi(-q)}{\varphi(-q^3)}, \qquad B=\frac{\varphi(-q^5)}{\varphi(-q^{15})} \qquad \text{and} \qquad C=\frac{\varphi(-q^{25})}{\varphi(-q^{75})}. \tag{13}$$

On employing (13) in Lemma 2.2, it is observed that

$$(AB)^2+\frac{9}{(AB)^2}=\left(\frac{B}{A}\right)^3+5\left(\frac{B}{A}\right)^2+5\left(\frac{B}{A}\right)-5\left(\frac{A}{B}\right)+5\left(\frac{A}{B}\right)^2-\left(\frac{A}{B}\right)^3 \tag{14}$$

and

$$(BC)^2+\frac{9}{(BC)^2}=\left(\frac{C}{B}\right)^3+5\left(\frac{C}{B}\right)^2+5\left(\frac{C}{B}\right)-5\left(\frac{B}{C}\right)+5\left(\frac{B}{C}\right)^2-\left(\frac{B}{C}\right)^3. \tag{15}$$

Using (2) and (6) in (13), we deduce

$$\left(\frac{A}{B}\right)^2=\left(\frac{(1-\alpha)^2\beta\gamma^2(1-\delta)}{\alpha^2(1-\beta)(1-\gamma)^2\delta}\right)^{1/6}=u \tag{16}$$

and

$$\left(\frac{B}{C}\right)^2=\left(\frac{(1-\gamma)^2\delta\eta(1-\zeta)}{\gamma^2(1-\delta)(1-\eta)^2\zeta}\right)^{1/6}=v. \tag{17}$$

On using (16) and (17) in (14) and (15), we obtain

$$B^8u^4-aB^4+9u^2=0 \tag{18}$$

and

$$B^8v^2-bB^4+9v^4=0, \tag{19}$$

respectively, where a and b are as defined as in Theorem 3.2. From (18) and (19), we deduce that

$$\frac{B^8}{9bu^2-9av^4}=\frac{B^4}{9u^2v^2-9u^4v^4}=\frac{1}{av^2-bu^4}.$$

Which implies

$$B^8=\frac{9(bu^2-av^4)}{av^2-bu^4} \tag{20}$$

and

$$B^4=\frac{9u^2v^2(1-u^2v^2)}{av^2-bu^4}. \tag{21}$$

On combining and simplifying (20) and (21), we deduce the result. □

Theorem 3.3. *If β, γ, δ, η and ζ be the second, third, sixth, ninth and eighteenth degrees respectively with respect to α and if*

$$u = \sqrt{3}\left(\frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2}\right)^{1/12} \qquad \textit{and} \qquad v = \sqrt{3}\left(\frac{\gamma(1-\delta)^2\eta(1-\zeta)^2}{(1-\gamma)\delta^2(1-\eta)\zeta^2}\right)^{1/12}$$

then

$$6v^4(v^2+6a)(v^4-u^2b) = u^2a^2b^2,$$

where

$$a = u^4 - 3u^2 + 9 \qquad \textit{and} \qquad b = v^4 - 3v^2 + 9.$$

Proof. Let

$$A = \frac{\psi(q)}{q^{1/4}\psi(q^3)}, \qquad B = \frac{\psi(q^3)}{q^{3/4}\psi(q^9)} \qquad \text{and} \qquad C = \frac{\psi(q^9)}{q^{9/4}\psi(q^{27})} \tag{22}$$

On employing (22) in Lemma 2.5, it is observed that

$$(AB)^2 + \frac{9}{(AB)^2} = 3 + 6\left(\frac{B}{A}\right)^2 + \left(\frac{B}{A}\right)^4 \tag{23}$$

and

$$(BC)^2 + \frac{9}{(BC)^2} = 3 + 6\left(\frac{C}{B}\right)^2 + \left(\frac{C}{B}\right)^4. \tag{24}$$

Using (2) and (6) in (22), we deduce

$$AB = \sqrt{3}\left(\frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2}\right)^{1/12} = u \tag{25}$$

and

$$BC = \sqrt{3}\left(\frac{\gamma(1-\delta)^2\eta(1-\zeta)^2}{(1-\gamma)\delta^2(1-\eta)\zeta^2}\right)^{1/12} = v. \tag{26}$$

On using (25) and (26) in (23) and (24), we obtain

$$B^8 + 6u^2B^4 - au^2 = 0 \tag{27}$$

and

$$bB^8 - 6v^4B^4 - v^6 = 0, \tag{28}$$

respectively, where a and b are as defined in Theorem 3.3. From (27) and (28), we deduce that

$$\frac{-B^8}{u^2v^6 + 6u^2v^4a} = \frac{B^4}{v^6 - u^2ab} = \frac{-1}{6v^4 - 6u^2b}.$$

Which implies

$$B^8 = \frac{u^2v^6 + 6u^2v^4a}{6v^4 - 6u^2b} \tag{29}$$

and

$$B^4 = \frac{u^2ab - v^6}{6v^4 - 6u^2b}. \tag{30}$$

On combining and simplifying (29) and (30), we obtain the required result.

□

Theorem 3.4. *If β, γ, δ, η and ζ be the second, third, forth, sixth and twelfth degrees respectively with respect to α and if*

$$u=\left(\frac{\gamma\delta(1-\alpha)(1-\beta)}{\alpha\beta(1-\gamma)(1-\delta)}\right)^{1/12} \qquad \text{and} \qquad v=\left(\frac{\delta\zeta(1-\beta)(1-\eta)}{\beta\eta(1-\delta)(1-\zeta)}\right)^{1/12}$$

then

$$u^8-9u^6v^4+5u^6v^2+12u^4v^4-u^4v^2-u^2v^4+5u^2v^6-9u^4v^6+v^8=0.$$

Proof. Let

$$A=q^{1/6}\frac{f_1f_9}{f_3^2}, \qquad B=q^{1/3}\frac{f_2f_{18}}{f_6^2} \qquad \text{and} \qquad C=q^{2/3}\frac{f_4f_{36}}{f_{12}^2}. \tag{31}$$

On employing (31) in Lemma 2.9, it is observed that

$$\frac{1}{AB}-3AB=\left(\frac{A}{B}\right)^3+\left(\frac{B}{A}\right)^3 \tag{32}$$

and

$$\frac{1}{BC}-3BC=\left(\frac{B}{C}\right)^3+\left(\frac{C}{B}\right)^3. \tag{33}$$

Using (6) in (31), we deduce

$$AB=\left(\frac{(1-\alpha)(1-\beta)\gamma\delta}{\alpha\beta(1-\gamma)(1-\delta)}\right)^{1/12}=u \tag{34}$$

and

$$BC=\left(\frac{(1-\beta)\delta(1-\eta)\zeta}{\beta(1-\delta)\eta(1-\zeta)}\right)^{1/12}=v. \tag{35}$$

On using (34) and (35) in (32) and (33) respectively, we obtain

$$B^{12}-u^2(1-3u^2)B^6+u^6=0 \tag{36}$$

and

$$B^{12}-v^2(1-3v^2)B^6+v^6=0, \tag{37}$$

respectively. From (36) and (37), we deduce that

$$\frac{B^{12}}{u^6v^2(1-3v^2)-u^2v^6(1-3u^2)}=\frac{B^6}{u^6-v^6}=\frac{1}{u^2(1-3u^2)-v^2(1-3v^2)}.$$

Which implies

$$B^{12}=\frac{u^6v^2(1-3v^2)-u^2v^6(1-3u^2)}{u^2(1-3u^2)-v^2(1-3v^2)} \tag{38}$$

and

$$B^6=\frac{u^6-v^6}{u^2(1-3u^2)-v^2(1-3v^2)}. \tag{39}$$

On combining (38) and (39), we obtain

$$(u-v)^2(u+v)^2(u^8-9u^6v^4+5u^6v^2+12u^4v^4-u^4v^2-u^2v^4+5u^2v^6-9u^4v^6+v^8)=0.$$

Since $u \neq v$ and $u \neq -v$, we obtain

$$u^8-9u^6v^4+5u^6v^2+12u^4v^4-u^4v^2-u^2v^4+5u^2v^6-9u^4v^6+v^8=0.$$

□

Theorem 3.5. *If β, γ, δ, η and ζ be the second, fifth, tenth, twenty fifth and fiftieth degrees respectively with respect to α and if*

$$u=\left(\frac{\alpha(1-\beta)\gamma(1-\delta)}{(1-\alpha)\beta(1-\gamma)\delta}\right)^{1/12} \qquad \text{and} \qquad v=\left(\frac{\gamma(1-\delta)\eta(1-\zeta)}{(1-\gamma)\delta(1-\eta)\zeta}\right)^{1/12}$$

then

$$(u^6vb-uv^6a)(ua-vb)=(u^6-v^6)^2,$$

where

$$a=u^4-5u^2+1 \qquad \text{and} \qquad b=v^4-5v^2+1.$$

Proof. Let

$$A=\frac{f_2f_3}{q^{1/12}f_1f_6}, \qquad B=\frac{f_{10}f_{15}}{q^{5/12}f_5f_{30}} \qquad \text{and} \qquad C=\frac{f_{50}f_{75}}{q^{25/12}f_{25}f_{150}}. \tag{40}$$

On employing (40) in Lemma 2.10, it is observed that

$$(AB)^2+\frac{1}{(AB)^2}=\left(\frac{B}{A}\right)^3+\left(\frac{A}{B}\right)^3+5 \tag{41}$$

and

$$(BC)^2+\frac{1}{(BC)^2}=\left(\frac{C}{B}\right)^3+\left(\frac{B}{C}\right)^3+5. \tag{42}$$

Employing (6) in (40), we deduce

$$AB=\left(\frac{\alpha(1-\beta)\gamma(1-\delta)}{(1-\alpha)\beta(1-\gamma)\delta}\right)^{1/12}=u \tag{43}$$

and

$$BC=\left(\frac{\gamma(1-\delta)\eta(1-\zeta)}{(1-\gamma)\delta(1-\eta)\zeta}\right)^{1/12}=v. \tag{44}$$

On using (43) and (44) in (41) and (42) respectively, we obtain

$$B^{12}-auB^6+u^6=0 \tag{45}$$

and

$$B^{12}-bvB^6+v^6=0, \tag{46}$$

respectively, where a and b are as defined as in Theorem 3.5. From (45) and (46), we deduce that

$$\frac{B^{12}}{bvu^6 - auv^6} = \frac{B^6}{u^6 - v^6} = \frac{1}{au - bv}.$$

Which implies

$$B^{12} = \frac{bu^6v - auv^6}{au - bv} \tag{47}$$

and

$$B^6 = \frac{u^6 - v^6}{au - bv}. \tag{48}$$

On combining and simplifying (47) and (48), we deduce the result. □

Theorem 3.6. *If β, γ, δ, η and ζ be the second, fifth, tenth, twenty fifth and fiftieth degrees respectively with respect to α and if*

$$u = \left(\frac{\gamma\delta(1-\alpha)(1-\beta)}{\alpha\beta(1-\gamma)(1-\delta)}\right)^{1/12} \qquad \textit{and} \qquad v = \left(\frac{\eta\zeta(1-\gamma)(1-\delta)}{\gamma\delta(1-\eta)(1-\zeta)}\right)^{1/12}$$

then

$$(bu - av^5)(av - bu^5) = 81uv(1 - u^4v^4)^2,$$

where

$$a = 1 - 5u - 5u^2 + 20u^3 - 5u^4 - 5u^5 + u^6 \qquad \textit{and} \qquad b = 1 - 5v - 5v^2 + 20v^3 - 5v^4 - 5v^5 + v^6.$$

Proof. Let

$$A = \frac{f_1 f_2}{q^{1/4} f_3 f_6}, \qquad B = \frac{f_5 f_{10}}{q^{5/4} f_{15} f_{30}} \qquad \text{and} \qquad C = \frac{f_{25} f_{50}}{q^{25/4} f_{75} f_{150}}. \tag{49}$$

On employing (49) in Lemma 2.11, we have

$$(AB)^2 + \frac{81}{(AB)^2} = \left(\frac{B}{A}\right)^3 + \left(\frac{A}{B}\right)^3 - 5\left(\frac{B}{A}\right)^2 - 5\left(\frac{A}{B}\right)^2 - 5\left(\frac{B}{A}\right) - 5\left(\frac{A}{B}\right) + 20 \tag{50}$$

and

$$(BC)^2 + \frac{81}{(BC)^2} = \left(\frac{C}{B}\right)^3 + \left(\frac{B}{C}\right)^3 - 5\left(\frac{C}{B}\right)^2 - 5\left(\frac{B}{C}\right)^2 - 5\left(\frac{C}{B}\right) - 5\left(\frac{B}{C}\right) + 20. \tag{51}$$

Using (6) in (49), it is observed that

$$\frac{A}{B} = \left(\frac{\gamma\delta(1-\alpha)(1-\beta)}{\alpha\beta(1-\gamma)(1-\delta)}\right)^{1/12} = u \tag{52}$$

and

$$\frac{B}{C} = \left(\frac{\eta\zeta(1-\gamma)(1-\delta)}{\gamma\delta(1-\eta)(1-\zeta)}\right)^{1/12} = v. \tag{53}$$

On employing (52) and (53) in (50) and (51) respectively, we obtain

$$u^5B^8 - aB^4 + 81u = 0 \tag{54}$$

and

$$vB^8 - bB^4 + 81v^5 = 0, \tag{55}$$

respectively, where a and b are as defined as in Theorem 3.6. From (54) and (55), we deduce

$$\frac{B^8}{81(bu - av^5)} = \frac{B^4}{81uv(1 - u^4v^4)} = \frac{1}{av - bu^5}.$$

Which implies

$$B^8 = \frac{81(bu - av^5)}{av - bu^5} \tag{56}$$

and

$$B^4 = \frac{81uv(1 - u^4v^4)}{av - bu^5}. \tag{57}$$

On combining and simplifying (56) and (57), we deduce the result. □

Theorem 3.7. *If β, γ and δ be the second, forth and eighth degrees respectively with respect to α and if*

$$u = \left(\frac{(1-\alpha)\gamma}{\alpha(1-\gamma)}\right)^{1/12} \qquad \textit{and} \qquad v = \left(\frac{(1-\beta)\delta}{\beta(1-\delta)}\right)^{1/12},$$

then

$$u^2(bu^2v^4(8 + 11a^2 - a^4) - av^2(8 + 11b^2 - b^4))(au^2v^2(8 + 11b^2 - b^4) - b(8 + 11a^2 - a^4)) = 81a^2b^2(1 - u^4v^4)^2,$$

where

$$a = u^2 + \frac{1}{u^2} \qquad \textit{and} \qquad b = v^2 + \frac{1}{v^2}.$$

Proof. Let

$$A = \frac{f_1f_2}{q^{1/4}f_3f_6}, \qquad B = \frac{f_2f_4}{q^{1/2}f_6f_{12}} \qquad \text{and} \qquad C = \frac{f_4f_8}{qf_{12}f_{24}}. \tag{58}$$

On employing Lemma 2.13, we obtain

$$\left(\frac{A}{B}\right)^8 + \left(\frac{B}{A}\right)^8 - 7\left[\left(\frac{A}{B}\right)^4 + \left(\frac{B}{A}\right)^4\right] = \left[\left(\frac{A}{B}\right)^2 + \left(\frac{B}{A}\right)^2\right]\left[(AB)^2 + \frac{81}{(AB)^2}\right] + 24 \tag{59}$$

and

$$\left(\frac{B}{C}\right)^8 + \left(\frac{C}{B}\right)^8 - 7\left[\left(\frac{B}{C}\right)^4 + \left(\frac{C}{B}\right)^4\right] = \left[\left(\frac{B}{C}\right)^2 + \left(\frac{C}{B}\right)^2\right]\left[(BC)^2 + \frac{81}{(BC)^2}\right] + 24. \tag{60}$$

Employing (6) in (58), it is observed that

$$\frac{A}{B} = \left(\frac{(1-\alpha)\gamma}{\alpha(1-\gamma)}\right)^{1/12} = u \qquad \text{and} \qquad \frac{B}{C} = \left(\frac{(1-\beta)\delta}{\beta(1-\delta)}\right)^{1/12} = v. \tag{61}$$

Using (61) in (59) and (60), we obtain

$$au^4B^8 + (8 + 11a^2 - a^4)u^2B^4 + 81a = 0 \tag{62}$$

and

$$bB^8 + (8 + 11b^2 - b^4)v^2B^4 + 81bv^4 = 0, \tag{63}$$

respectively, where a and b are as defined as in Theorem 3.7. From (62) and (63), we deduce

$$\frac{B^8}{81bu^2v^4(8 + 11a^2 - a^4) - 81av^2(8 + 11b^2 - b^4)} = \frac{B^4}{81ab(1 - u^4v^4)}$$
$$= \frac{1}{au^4v^2(8 + 11b^2 - b^4) - bu^2(8 + 11a^2 - a^4)}.$$

Which implies

$$B^8 = \frac{81(bu^2v^4(8 + 11a^2 - a^4) - av^2(8 + 11b^2 - b^4))}{u^2(au^2v^2(8 + 11b^2 - b^4) - b(8 + 11a^2 - a^4))} \tag{64}$$

and

$$B^4 = \frac{81ab(1 - u^4v^4)}{u^2(au^2v^2(8 + 11b^2 - b^4) - b(8 + 11a^2 - a^4))}. \tag{65}$$

On combining and simplifying (64) and (65), we deduce the result. □

Theorem 3.8. *If β, γ, δ, η and ζ be the second, third, sixth, ninth and eighteenth degrees respectively with respect to α and if*

$$u = \left(\frac{(1-\alpha)^2\beta\gamma^2(1-\delta)}{\alpha^2(1-\beta)(1-\gamma)^2\delta}\right)^{1/6} \quad \text{and} \quad v = \left(\frac{(1-\gamma)^2\delta\eta^2(1-\zeta)}{\gamma^2(1-\delta)(1-\eta)^2\zeta}\right)^{1/6}$$

then

$$(u(3v^2 + 6v + 1) - v^3(3u^2 + 6u + 1))(v(3u^2 + 6u + 1) - u^3(3v^2 + 6v + 1)) = 9(uv - u^3v^3)^2.$$

Proof. Let

$$A = \frac{\varphi^2(-q)}{\varphi^2(-q^3)}, \qquad B = \frac{\varphi^2(-q^3)}{\varphi^2(-q^9)} \qquad \text{and} \qquad C = \frac{\varphi^2(-q^9)}{\varphi^2(-q^{27})}. \tag{66}$$

On employing (66) in Lemma 2.14, it is observed that

$$AB + \frac{9}{AB} = 3 + 6\frac{B}{A} + \frac{B^2}{A^2} \tag{67}$$

and

$$BC + \frac{9}{BC} = 3 + 6\frac{C}{B} + \frac{C^2}{B^2}. \tag{68}$$

Using (2) and (6) in (66), we deduce

$$\frac{A}{B} = \left(\frac{(1-\alpha)^2\beta\gamma^2(1-\delta)}{\alpha^2(1-\beta)(1-\gamma)^2\delta}\right)^{1/6} = u \tag{69}$$

and

$$\frac{B}{C}=\left(\frac{(1-\gamma)^2\delta\eta^2(1-\zeta)}{\gamma^2(1-\delta)(1-\eta)^2\zeta}\right)^{1/6}=v. \tag{70}$$

Using (69) and (70) in (67) and (68), we obtain

$$u^3B^4-(3u^2+6u+1)B^2+9u=0 \tag{71}$$

and

$$vB^4-(3v^2+6v+1)B^2+9v^3=0, \tag{72}$$

respectively. From (71) and (72), we deduce that

$$\frac{B^4}{9(u(3v^2+6v+1)-v^3(3u^2+6u+1))}=\frac{B^2}{9(uv-u^3v^3)}=\frac{1}{v(3u^2+6u+1)-u^3(3v^2+6v+1)}.$$

Which implies

$$B^4=\frac{9(u(3v^2+6v+1)-v^3(3u^2+6u+1))}{v(3u^2+6u+1)-u^3(3v^2+6v+1)} \tag{73}$$

and

$$B^2=\frac{9(uv-u^3v^3)}{v(3u^2+6u+1)-u^3(3v^2+6v+1)}. \tag{74}$$

On combining and simplifying (73) and (74), we deduce the result. □

Theorem 3.9. *If β and γ be the second and forth degrees respectively with respect to α, then we have*

$$\sqrt{3}\left(\frac{\beta^2(1-\gamma)}{(1-\beta)^2\gamma}\right)^{1/6}-\sqrt{3}\left(\frac{(1-\beta)^2\gamma}{\beta^2(1-\gamma)}\right)^{1/6}=\left(\frac{\alpha^2(1-\beta)^3\gamma}{(1-\alpha)^2\beta^3(1-\gamma)}\right)^{1/6}+\left(\frac{(1-\alpha)^2\beta^3(1-\gamma)}{\alpha^2(1-\beta)^3\gamma}\right)^{1/6}.$$

Proof. Let

$$A=\frac{\varphi(-q)}{\varphi(-q^3)} \qquad \text{and} \qquad B=\frac{\varphi(-q^2)}{\varphi(-q^6)}. \tag{75}$$

Using (2) and (6) in (75), it is observed that

$$A=3^{1/4}\left(\frac{\beta(1-\alpha)^2}{\alpha^2(1-\beta)}\right)^{1/12} \qquad \text{and} \qquad B=3^{1/4}\left(\frac{\gamma(1-\beta)^2}{\beta^2(1-\gamma)}\right)^{1/12}.$$

From the above, we deduce

$$\frac{A}{B}=\left(\frac{(1-\alpha)^2\beta^3(1-\gamma)}{\alpha^2(1-\beta)^3\gamma}\right)^{1/12}. \tag{76}$$

By employing (76) in Lemma 2.1, we deduce the required identity. □

Theorem 3.10. *If β, γ and δ be the second, eleventh and twenty second degrees respectively with respect to α, then we have*

$$9\sqrt{3}\left(a^{5/12}+\frac{1}{a^{5/12}}\right)-33\sqrt{3}\left(a^{1/4}+\frac{1}{a^{1/4}}\right)+308\sqrt{3}\left(a^{1/12}+\frac{1}{a^{1/12}}\right)=b^{1/2}+\frac{1}{b^{1/2}}+22\sqrt{3}\left(b^{1/3}+\frac{1}{b^{1/3}}\right)$$
$$\left(\sqrt{3}-\left(a^{1/12}+\frac{1}{a^{12}}\right)\right)+33\sqrt{3}\left(b^{1/6}+\frac{1}{b^{1/6}}\right)\left(a^{1/4}+\frac{1}{a^{1/4}}-5\left(a^{1/12}+\frac{1}{a^{1/12}}\right)+5\sqrt{3}\right)+924,$$

where

$$a=\frac{(1-\alpha)^2\beta(1-\gamma)^2\delta}{\alpha^2(1-\beta)\gamma^2(1-\delta)} \qquad \textit{and} \qquad b=\frac{(1-\alpha)^2\beta\gamma^2(1-\delta)}{\alpha^2(1-\beta)(1-\gamma)^2\delta}.$$

Proof. Let

$$A=\frac{\varphi(-q)}{\varphi(-q^3)}\quad\text{and}\quad B=\frac{\varphi(-q^{11})}{\varphi(-q^{33})}. \tag{77}$$

Using (2) and (6) in (77), it is observed that

$$A=3^{1/4}\left(\frac{\beta(1-\alpha)^2}{\alpha^2(1-\beta)}\right)\quad\text{and}\quad B=3^{1/4}\left(\frac{\delta(1-\gamma)^2}{\gamma^2(1-\delta)}\right)^{1/12}.$$

From the above, we have

$$AB=\sqrt{3}\left(\frac{(1-\alpha)^2\beta(1-\gamma)^2\delta}{\alpha^2(1-\beta)\gamma^2(1-\delta)}\right)^{1/12}\quad\text{and}\quad \frac{A}{B}=\left(\frac{(1-\alpha)^2\beta\gamma^2(1-\delta)}{\alpha^2(1-\beta)(1-\gamma)^2\delta}\right)^{1/12}. \tag{78}$$

Employ (78) in Lemma 2.3, we deduce the required result. □

Theorem 3.11. *If β, γ and δ be the second, thirteenth and twenty sixth degrees respectively with respect to α, then we have*

$$\begin{aligned}&b^{7/12}+\frac{1}{b^{7/12}}+13\left(b^{1/2}+\frac{1}{b^{1/2}}\right)-26\left(b^{5/2}+\frac{1}{b^{5/2}}\right)-13\left(9a^{1/6}+\frac{9}{a^{1/6}}+10\right)\left(b^{1/3}+\frac{1}{b^{1/3}}\right)\\&+13\left(30a^{1/6}+\frac{30}{a^{1/6}}+68\right)\left(b^{1/4}+\frac{1}{b^{1/4}}\right)+13\left(9a^{1/3}+\frac{9}{a^{1/3}}-60\left(a^{1/6}+\frac{1}{a^{1/6}}\right)-115\right)\left(b^{1/6}+\frac{1}{b^{1/6}}\right)\\&-13\left(9a^{1/3}+\frac{9}{a^{1/3}}-30\left(a^{1/6}+\frac{1}{a^{1/6}}\right)-131\right)\left(b^{1/12}+\frac{1}{b^{1/12}}\right)=27a^{1/2}+\frac{27}{a^{1/2}}-234\left(a^{1/3}+\frac{1}{a^{1/3}}\right)\\&+507\left(a^{1/6}+\frac{1}{a^{1/6}}\right)+832,\end{aligned}$$

where

$$a=\frac{(1-\alpha)^2\beta(1-\gamma)^2\delta}{\alpha^2(1-\beta)\gamma^2(1-\delta)}\quad\textit{and}\quad b=\frac{(1-\alpha)^2\beta\gamma^2(1-\delta)}{\alpha^2(1-\beta)(1-\gamma)^2\delta}.$$

Proof. Let

$$A=\frac{\varphi(-q)}{\varphi(-q^3)}\quad\text{and}\quad B=\frac{\varphi(-q^{13})}{\varphi(-q^{39})}. \tag{79}$$

Using (2) and (6) in (79), it is observed that

$$A=3^{1/4}\left(\frac{\beta(1-\alpha)^2}{\alpha^2(1-\beta)}\right)^{1/12}\quad\text{and}\quad B=3^{1/4}\left(\frac{\delta(1-\gamma)^2}{\gamma^2(1-\delta)}\right)^{1/12}.$$

From the above, we have

$$AB=\sqrt{3}\left(\frac{\beta\delta(1-\alpha)^2(1-\gamma)^2}{\alpha^2\gamma^2(1-\beta)(1-\delta)}\right)^{1/12}\quad\text{and}\quad \frac{A}{B}=\left(\frac{\beta\gamma^2(1-\alpha)^2(1-\delta)}{\alpha^2\delta(1-\beta)(1-\gamma)^2}\right)^{1/12}. \tag{80}$$

By employing (80) in Lemma 2.4, we deduce the required result. □

Theorem 3.12. *If β, γ and δ be the second, third and sixth degrees respectively with respect to α, then we have*

$$\begin{aligned}3\left(\frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2}\right)^{1/6}+3\left(\frac{(1-\alpha)\beta^2(1-\gamma)\delta^2}{\alpha(1-\beta)^2\gamma(1-\delta)^2}\right)^{1/6}&=3+6\left(\frac{(1-\alpha)\beta^2\gamma(1-\delta)^2}{\alpha(1-\beta)^2(1-\gamma)\delta^2}\right)^{1/6}\\&+\left(\frac{(1-\alpha)\beta^2\gamma(1-\delta)^2}{\alpha(1-\beta)^2(1-\gamma)\delta^2}\right)^{1/3}.\end{aligned}$$

Proof. Let

$$A = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad B = \frac{\psi(q^3)}{q^{3/4}\psi(q^9)}. \tag{81}$$

Using (2) and (6) in (81), it is observed that

$$A = 3^{1/4}\left(\frac{\alpha(1-\beta)^2}{\beta^2(1-\alpha)}\right)^{1/12} \quad \textit{and} \quad B = 3^{1/4}\left(\frac{\gamma(1-\delta)^2}{\delta^2(1-\gamma)}\right)^{1/12}.$$

From the above, we deduce

$$AB = \sqrt{3}\left(\frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2}\right)^{1/12} \quad \text{and} \quad \frac{A}{B} = \left(\frac{\alpha(1-\beta)^2(1-\gamma)\delta^2}{(1-\alpha)\beta^2\gamma(1-\delta)^2}\right)^{1/12}. \tag{82}$$

Employ (82) in Lemma 2.5, to complete the proof. □

Theorem 3.13. *If β, γ and δ be the second, eleventh and twenty second degrees respectively with respect to α, then we have*

$$\begin{aligned}
&9\sqrt{3}\left(a^{5/12} + \frac{1}{a^{5/12}}\right) - 33\sqrt{3}\left(a^{1/4} + \frac{1}{a^{1/4}}\right) + 308\sqrt{3}\left(a^{1/12} + \frac{1}{a^{1/12}}\right) = b^{1/2} + \frac{1}{b^{1/2}} + 22\sqrt{3}\left(b^{1/3} + \frac{1}{b^{1/3}}\right)\\
&\times\left(\sqrt{3} - a^{1/12} - \frac{1}{a^{1/12}}\right) + 33\sqrt{3}\left(b^{1/6} + \frac{1}{b^{1/6}}\right)\left(a^{1/4} + \frac{1}{a^{1/4}} - 5\sqrt{3}\left(a^{1/12} + \frac{1}{a^{1/12}}\right) + 15\sqrt{3}\right) + 924,
\end{aligned}$$

where

$$a = \frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2} \quad \textit{and} \quad b = \frac{\alpha(1-\beta)^2(1-\gamma)\delta^2}{(1-\alpha)\beta^2\gamma(1-\delta)^2}.$$

Proof. Let

$$A = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \quad \text{and} \quad B = \frac{\psi(q^{11})}{q^{11/4}\psi(q^{33})}. \tag{83}$$

Using (2) and (6) in (83), it is observed that

$$A = 3^{1/4}\left(\frac{\alpha(1-\beta)^2}{\beta^2(1-\alpha)}\right)^{1/12} \quad \text{and} \quad B = 3^{1/4}\left(\frac{\gamma(1-\delta)^2}{\delta^2(1-\gamma)}\right)^{1/12}.$$

From the above, it is easy to observed that

$$AB = \sqrt{3}\left(\frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2}\right)^{1/12} \quad \text{and} \quad \frac{A}{B} = \left(\frac{\alpha(1-\beta)^2(1-\gamma)\delta^2}{(1-\alpha)\beta^2\gamma(1-\delta)^2}\right)^{1/12}. \tag{84}$$

Employ (84) in Lemma 2.6, to complete the proof. □

Theorem 3.14. *If β, γ and δ be the second, seventh and fourteenth degrees respectively with respect to α, then we have*

$$\begin{aligned}
&a^{1/4}\left(b^{2/3} - 1\right) + 42a^{1/12}c^{1/3}(b^{1/3} - 1) = 3\sqrt{3}a^{1/6}c^{1/3}(7 - 3c^{1/3}) + 21\sqrt{3}b^{1/6}c^{1/3}(c^{1/3} - 1)\\
&- 9\sqrt{3}b^{1/3} + 21\sqrt{3}c^{1/3}(1 + b^{1/3} - c^{1/3}),
\end{aligned}$$

where

$$a = \frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2}, \quad b = \frac{\alpha(1-\beta)^2(1-\gamma)\delta^2}{(1-\alpha)\beta^2\gamma(1-\delta)^2} \quad \textit{and} \quad c = \frac{\alpha(1-\beta)^2}{(1-\alpha)\beta^2}.$$

Proof. Let

$$A = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \text{and} \qquad B = \frac{\psi(q^7)}{q^{7/4}\psi(q^{21})}. \tag{85}$$

Using (2) and (6) in (85), it is observed that

$$A = 3^{1/4}\left(\frac{\alpha(1-\beta)^2}{\beta^2(1-\alpha)}\right)^{1/12} \qquad \text{and} \qquad B = 3^{1/4}\left(\frac{\gamma(1-\delta)^2}{\delta^2(1-\gamma)}\right)^{1/12}.$$

From the above, we have

$$AB = \sqrt{3}\left(\frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2}\right)^{1/12} \qquad \text{and} \qquad \frac{A}{B} = \left(\frac{\alpha(1-\beta)^2(1-\gamma)\delta^2}{(1-\alpha)\beta^2\gamma(1-\delta)^2}\right)^{1/12}. \tag{86}$$

Employ (86) in Lemma 2.7, to complete the proof. □

Theorem 3.15. *If β, γ and δ be the second, thirteenth and twenty sixth degrees respectively with respect to α, then we have*

$$\begin{aligned} &b^{7/12} + \frac{1}{b^{7/12}} + 13\left(b^{1/2} + \frac{1}{b^{1/2}}\right) - 26\left(b^{5/12} + \frac{1}{b^{5/12}}\right) - 13\left(9a^{1/6} + \frac{9}{a^{1/6}} + 10\right)\left(b^{1/3} + \frac{1}{b^{1/3}}\right) \\ &+ 26\left(15a^{1/6} + \frac{15}{a^{1/6}} + 34\right)\left(b^{1/4} + \frac{1}{b^{1/4}}\right) + 13\left(9a^{1/3} + \frac{9}{a^{1/3}} - 60\left(a^{1/6} + \frac{1}{a^{1/6}}\right) - 115\right)\left(b^{1/6} + \frac{1}{b^{1/6}}\right) \\ &- 13\left(9a^{1/3} + \frac{9}{a^{1/3}} - 30\left(a^{1/6} + \frac{1}{a^{1/6}}\right) - 131\right)\left(b^{1/12} + \frac{1}{b^{1/12}}\right) = 27a^{1/2} + \frac{27}{a^{1/2}} - 234\left(a^{1/3} + \frac{1}{a^{1/3}}\right) \\ &+ 507\left(a^{1/6} + \frac{1}{a^{1/6}}\right) + 832, \end{aligned}$$

where

$$a = \frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2} \qquad \textit{and} \qquad b = \frac{\alpha(1-\beta)^2(1-\gamma)\delta^2}{(1-\alpha)\beta^2\gamma(1-\delta)^2}.$$

Proof. Let

$$A = \frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \text{and} \qquad B = \frac{\psi(q^{13})}{q^{13/4}\psi(q^{39})}. \tag{87}$$

Using (2) and (6) in (87), it is observed that

$$A = 3^{1/4}\left(\frac{\alpha(1-\beta)^2}{\beta^2(1-\alpha)}\right)^{1/12} \qquad \text{and} \qquad B = 3^{1/4}\left(\frac{\gamma(1-\delta)^2}{\delta^2(1-\gamma)}\right)^{1/12}.$$

From the above, we have

$$AB = \sqrt{3}\left(\frac{\alpha(1-\beta)^2\gamma(1-\delta)^2}{(1-\alpha)\beta^2(1-\gamma)\delta^2}\right)^{1/12} \qquad \text{and} \qquad \frac{A}{B} = \left(\frac{\alpha(1-\beta)^2(1-\gamma)\delta^2}{(1-\alpha)\beta^2\gamma(1-\delta)^2}\right)^{1/12}. \tag{88}$$

By employing (88) in Lemma 2.8, we deduce the required identity . □

Theorem 3.16. *If β, γ and δ be the second, third and sixth degrees respectively with respect to α, then we have*

$$\left(\frac{\alpha\beta(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)\gamma\delta}\right)^{1/12} - 3\left(\frac{(1-\alpha)(1-\beta)\gamma\delta}{\alpha\beta(1-\gamma)(1-\delta)}\right)^{1/12} = \left(\frac{(1-\alpha)\beta\gamma(1-\delta)}{\alpha(1-\beta)(1-\gamma)\delta}\right)^{1/4} + \left(\frac{\alpha(1-\beta)(1-\gamma)\delta}{(1-\alpha)\beta\gamma(1-\delta)}\right)^{1/4}.$$

Proof. Let

$$A = q^{1/6}\frac{f_1 f_9}{f_3^2} \qquad \text{and} \qquad B = q^{1/3}\frac{f_2 f_{18}}{f_6^2}. \tag{89}$$

Using (6) in (89), it is observed that

$$A = \left(\frac{\gamma(1-\alpha)}{\alpha(1-\gamma)}\right)^{1/12} \qquad \text{and} \qquad B = \left(\frac{\delta(1-\beta)}{\beta(1-\delta)}\right)^{1/12}.$$

From the above, we find that

$$AB = \left(\frac{(1-\alpha)(1-\beta)\gamma\delta}{\alpha\beta(1-\gamma)(1-\delta)}\right)^{1/12} \qquad \text{and} \qquad \frac{A}{B} = \left(\frac{(1-\alpha)\beta\gamma(1-\delta)}{\alpha(1-\beta)(1-\gamma)\delta}\right)^{1/12}. \tag{90}$$

Employ (90) in Lemma 2.9, to complete the proof. □

Theorem 3.17. *If β, γ and δ be the second, fifth and tenth degrees respectively with respect to α, then we have*

$$\left(\frac{\alpha(1-\beta)\gamma(1-\delta)}{(1-\alpha)\beta(1-\gamma)\delta}\right)^{1/6} + \left(\frac{(1-\alpha)\beta(1-\gamma)\delta}{\alpha(1-\beta)\gamma(1-\delta)}\right)^{1/6} = \left(\frac{(1-\alpha)\beta\gamma(1-\delta)}{\alpha(1-\beta)(1-\gamma)\delta}\right)^{1/4} + \left(\frac{\alpha(1-\beta)(1-\gamma)\delta}{(1-\alpha)\beta\gamma(1-\delta)}\right)^{1/4} + 5.$$

Proof. Let

$$A = \frac{f_2 f_3}{q^{1/12} f_1 f_6} \qquad \text{and} \qquad B = \frac{f_{10} f_{15}}{q^{5/12} f_5 f_{30}}. \tag{91}$$

Using (6) in (91), it is observed that

$$A = \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1/12} \qquad \text{and} \qquad B = \left(\frac{\gamma(1-\delta)}{\delta(1-\gamma)}\right)^{1/12}.$$

From the above, we observe that

$$AB = \left(\frac{\alpha(1-\beta)\gamma(1-\delta)}{(1-\alpha)\beta(1-\gamma)\delta}\right)^{1/12} \qquad \text{and} \qquad \frac{A}{B} = \left(\frac{\alpha(1-\beta)(1-\gamma)\delta}{(1-\alpha)\beta\gamma(1-\delta)}\right)^{1/12}. \tag{92}$$

Employ (92) in Lemma 2.10, to complete the proof. □

Theorem 3.18. *If β, γ and δ be the second, fifth and tenth degrees respectively with respect to α, then we have*

$$9a^{1/6} + \frac{9}{a^{1/6}} = b^{1/4} + \frac{1}{b^{1/4}} - 5b^{1/6} - \frac{5}{b^{1/6}} - 5b^{1/12} - \frac{5}{b^{1/12}} + 20,$$

where

$$a = \frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}{\alpha\beta\gamma\delta} \qquad \text{and} \qquad b = \frac{\gamma\delta(1-\alpha)(1-\beta)}{\alpha\beta(1-\gamma)(1-\delta)}.$$

Proof. Let

$$A = \frac{f_1 f_2}{q^{1/4} f_3 f_6} \qquad \text{and} \qquad B = \frac{f_5 f_{10}}{q^{5/4} f_{15} f_{30}}. \tag{93}$$

Employing (6) in (93), it is observed that

$$A = \sqrt{3}\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/12} \quad \text{and} \quad B = \sqrt{3}\left(\frac{(1-\gamma)(1-\delta)}{\gamma\delta}\right)^{1/12}.$$

From the above, we find that

$$AB = 3\left(\frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}{\alpha\beta\gamma\delta}\right)^{1/12} \quad \text{and} \quad \frac{A}{B} = \left(\frac{\gamma\delta(1-\alpha)(1-\beta)}{\alpha\beta(1-\gamma)(1-\delta)}\right)^{1/12}. \tag{94}$$

By employing (94) in Lemma 2.11, we deduce the required identity. □

Theorem 3.19. *If β, γ and δ be the second, third and sixth degrees respectively with respect to α, then we have*

$$\left(b^{1/2} + \frac{1}{b^{1/2}}\right)\left(a^{1/4} + \frac{1}{a^{1/4}} + 1\right) = a^{1/2} + \frac{1}{a^{1/2}} + 10\left(a^{1/4} + \frac{1}{a^{1/4}}\right) + 20,$$

where

$$a = \frac{(1-\alpha)\beta(1-\gamma)\delta}{\alpha(1-\beta)\gamma(1-\delta)} \quad \text{and} \quad b = \frac{(1-\alpha)\beta\gamma(1-\delta)}{\alpha(1-\beta)(1-\gamma)\delta}.$$

Proof. Let

$$A = q^{1/12}\frac{f_1 f_6}{f_2 f_3} \quad \text{and} \quad B = q^{1/4}\frac{f_3 f_{18}}{f_6 f_9}. \tag{95}$$

Using (6) in (95), it is observed that

$$A = \left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right)^{1/12} \quad \text{and} \quad B = \left(\frac{\delta(1-\gamma)}{\gamma(1-\delta)}\right)^{1/12}.$$

From the above, we find that

$$AB = \left(\frac{(1-\alpha)\beta(1-\gamma)\delta}{\alpha(1-\beta)\gamma(1-\delta)}\right)^{1/12} \quad \text{and} \quad \frac{A}{B} = \left(\frac{(1-\alpha)\beta\gamma(1-\delta)}{\alpha(1-\beta)(1-\gamma)\delta}\right)^{1/12}. \tag{96}$$

By employing (96) in Lemma 2.12, we deduce the required identity. □

Theorem 3.20. *If β and γ be the second and fourth degrees respectively with respect to α, then we have*

$$\left(\frac{\alpha(1-\gamma)}{(1-\alpha)\gamma}\right)^{2/3} + \left(\frac{(1-\alpha)\gamma}{\alpha(1-\gamma)}\right)^{2/3} - 7\left[\left(\frac{\alpha(1-\gamma)}{(1-\alpha)\gamma}\right)^{1/3} + \left(\frac{(1-\alpha)\gamma}{\alpha(1-\gamma)}\right)^{1/3}\right] = 9\left[\left(\frac{\alpha(1-\gamma)}{(1-\alpha)\gamma}\right)^{1/6} + \left(\frac{(1-\alpha)\gamma}{\alpha(1-\gamma)}\right)^{1/6}\right]$$
$$\times\left[\left(\frac{(1-\alpha)(1-\beta)^2(1-\gamma)}{\alpha\beta^2\gamma}\right)^{1/6} + \left(\frac{\alpha\beta^2\gamma}{(1-\alpha)(1-\beta)^2(1-\gamma)}\right)^{1/6}\right] + 24.$$

Proof. Let

$$A = \frac{f_1 f_2}{q^{1/4} f_3 f_6} \quad \text{and} \quad B = \frac{f_2 f_4}{q^{1/4} f_6 f_{12}}. \tag{97}$$

Using (6) in (97), it is observed that

$$A = \sqrt{3}\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/12} \quad \text{and} \quad B = \sqrt{3}\left(\frac{(1-\beta)(1-\gamma)}{\beta\gamma}\right)^{1/12}. \tag{98}$$

From the above, we find that

$$AB = 3\left(\frac{(1-\alpha)(1-\beta)^2(1-\gamma)}{\alpha\beta^2\gamma}\right)^{1/12} \quad \text{and} \quad \frac{A}{B} = \left(\frac{(1-\alpha)\gamma}{\alpha(1-\gamma)}\right)^{1/12}. \tag{99}$$

By employing (99) in Lemma 2.13, we deduce the required identity. □

Theorem 3.21. *If β and γ be the third and ninth degrees respectively with respect to α, then we have*

$$3\left(\frac{(1-\alpha)^2\beta(1-\gamma)^2\delta}{\alpha^2(1-\beta)\gamma^2(1-\delta)}\right)^{1/6} + 3\left(\frac{\alpha^2(1-\beta)\gamma^2(1-\delta)}{(1-\alpha)^2\beta(1-\gamma)^2\delta}\right)^{1/6} = 3 + 6\left(\frac{\alpha^2(1-\beta)(1-\gamma)^2\delta}{(1-\alpha)\beta\gamma^2(1-\delta)}\right)^{1/6} + \left(\frac{\alpha^2(1-\beta)(1-\gamma)^2\delta}{(1-\alpha)\beta\gamma^2(1-\delta)}\right)^{1/3}.$$

Proof. Let

$$A = \frac{\varphi^2(-q)}{\varphi^2(-q^3)} \quad \text{and} \quad B = \frac{\varphi^2(-q^3)}{\varphi^2(-q^9)}. \tag{100}$$

Using (2) and (6) in (100), it is observed that

$$A = \sqrt{3}\left(\frac{\beta(1-\alpha)^2}{\alpha^2(1-\beta)}\right)^{1/6} \quad \text{and} \quad B = \sqrt{3}\left(\frac{\delta(1-\gamma)^2}{\gamma^2(1-\delta)}\right)^{1/6}.$$

From the above, we have

$$AB = 3\left(\frac{(1-\alpha)^2\beta(1-\gamma)^2\delta}{\alpha^2(1-\beta)\gamma^2(1-\delta)}\right)^{1/6} \quad \text{and} \quad \frac{A}{B} = \left(\frac{(1-\alpha)\beta\gamma^2(1-\delta)}{\alpha^2(1-\beta)(1-\gamma)^2\delta}\right)^{1/6}. \tag{101}$$

By employing (101) in Lemma 2.14, we deduce the required identity. □

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