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Asymptotic Normality of Coefficients of Some Polynomials Related to Dowling Lattices

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Abstract. Recently, we introduced two sequences of polynomials $(B_n(x,y,z))$ and $(F_n(x,y,z))$, which unify many familiar polynomials related to Dowling lattices, such as the Bell polynomials, the Dowling polynomials, the ordered Bell polynomials, the r-Bell polynomials and the r-Dowling polynomials. In this paper, we show the asymptotic normality of coefficients of $B_n(x,y,z)$ and $F_n(x,y,z)$. As applications, we obtain the asymptotic normality of coefficients of some polynomials related to Dowling lattices in a unified approach.

1. Introduction

Let $a(n,k)$ be a double-indexed sequence of nonnegative numbers and let

$$p(n,k) = \frac{a(n,k)}{\sum_{j=0}^{n} a(n,j)} \tag{1}$$

denote the normalized probabilities. Following Bender [2], we say that the sequence $a(n,k)$ is *asymptotically normal by a central limit theorem*, if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{k\le\mu_n+x\sigma_n} p(n,k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0, \tag{2}$$

where μ_n and σ_n^2 are the mean and the variance of (1) respectively. We say $a(n,k)$ is *asymptotically normal by a local limit theorem* on $\mathbb{R}$, if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sigma_n p(n, \lfloor \mu_n + x\sigma_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0. \tag{3}$$

In this case,

$$a(n,k) \sim \frac{e^{-x^2/2} \sum_{j=0}^{n} a(n,j)}{\sigma_n \sqrt{2\pi}}, \text{ as } n \to \infty,$$

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where $k = \mu_n + x\sigma_n$ and $x = O(1)$. Clearly, the validity of (2) implies that of (3).

Let $\left\{ {n \atop k} \right\}$ be the Stirling number of the second kind, which counts the number of distinct partitions of an n-set. The Bell polynomials [1]

$$B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k, \tag{4}$$

defined as the associated generating function, have only real zeros [22]. Using this fact, Harper [16] showed that $\left\{ {n \atop k} \right\}$ is approximately normally distributed. The number $k!\left\{ {n \atop k} \right\}$, which is closely related to the Stirling number of the second kind, counts the number of distinct ordered partitions of an n-set. The ordered Bell polynomials

$$F_n(x) = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} x^k \tag{5}$$

also have only real zeros [4, 22].

In 1973, Dowling [15] introduced a class of geometric lattices over a finite group G of order $m \geq 1$, called Dowling lattices. Let $Q_n(G)$ be Dowling lattices of rank n associated to G. When $m = 1$, that is, G is the trivial group, $Q_n(G)$ is isomorphic to the lattice Π_{n+1} of partitions of an $(n+1)$-set. So Dowling lattices can be viewed as a group-theoretic analog of partition lattices. We denote the Whitney numbers of the second kind by $W_m(n, k)$. The Dowling polynomials $D_{n,m}(x)$ are defined by

$$D_{n,m}(x) = \sum_{k=0}^{n} W_m(n, k) x^k, \tag{6}$$

generalized the Bell polynomials $B_n(x)$, i.e., $B_n(x) = D_{n,1}(x)$ (see [4]). Benoumhani [4] also introduced two generalized Dowling polynomials

$$F_{n,m,1}(x) = \sum_{k=0}^{n} k! W_m(n, k) x^k, \quad F_{n,m,2}(x) = \sum_{k=0}^{n} k! W_m(n, k) m^k x^k, \tag{7}$$

which generalized $F_n(x)$, i.e., $F_n(x) = F_{n,1,1}(x) = F_{n,1,2}(x)$. There has been an amount of results concerned with the Dowling polynomials (see [3–5, 12, 13, 22] for instance). For example, Benoumhani [3–5] gave the recurrence relations, the exponential generating functions and the reality of zeros of these Dowling polynomials.

Based on the r-Stirling numbers given by Broder [10], Mezö defined the r-Bell polynomials by

$$B_{n,r}(x) = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^k, \tag{8}$$

where $\left\{ {n \atop k} \right\}_r$ is the r-Stirling numbers, enumerating the number of partitions of the set $[n]$ having k non-empty disjoint subsets, such that the numbers $1, 2, \ldots, r$ are in distinct subsets [10]. In particular, when $r = 0$, we have $B_{n,0}(x) = B_n(x)$. The strong x-log-convexity of the r-Bell polynomials has been obtained by Liu and Li [20].

Choen and Jung [12] defined the r-Dowling polynomials by

$$D_{n,m,r}(x) = \sum_{k=0}^{n} W_{m,r}(n, k) x^k, \tag{9}$$

where $W_{m,r}(n,k)$ is the r-Whitney numbers of the second kind.

Recently, the first author and Ma [21] introduced two polynomials sequences $(B_n(x,y,z))$ and $(F_n(x,y,z))$, whose generating functions are

$$\sum_{n\geq 0} B_n(x,y,z)\frac{t^n}{n!} = \exp\left(zt + \frac{x}{y}(e^{yt}-1)\right) \tag{10}$$

and

$$\sum_{n\geq 0} F_n(x,y,z)\frac{t^n}{n!} = \frac{e^{zt}}{1-\frac{x}{y}(e^{yt}-1)} \tag{11}$$

respectively. These two polynomials sequences unify many polynomials related to Dowling lattices, such as

(1) the Bell polynomials $B_n(x) = B_n(x,1,0)$;

(2) the Dowling polynomials $D_{n,m}(x) = B_n(x,m,1)$;

(3) the r-Bell polynomials $B_{n,r}(x) = B_n(x,1,r)$;

(4) the r-Dowling polynomials $D_{n,m,r}(x) = B_n(x,m,r)$;

(5) the ordered Bell polynomials $F_n(x) = F_n(x,1,0)$;

(6) the generalized Dowling polynomials $F_{n,m,1}(x) = F_n(mx,m,1)$;

(7) the generalized Dowling polynomials $F_{n,m,2}(x) = F_n(x,m,1)$.

Let

$$B_n(x,y,z) = \sum_{k=0}^{n} S_{y,z}(n,k)x^k \tag{12}$$

and

$$F_n(x,y,z) = \sum_{k=0}^{n} F_{y,z}(n,k)x^k. \tag{13}$$

Denote by $B_n(y,z) = \sum_{k=0}^{n} S_{y,z}(n,k)$ and $F_n(y,z) = \sum_{k=0}^{n} F_{y,z}(n,k)$. In this paper, we first present the asymptotic formulas of $B_n(y,z)$ and $F_n(y,z)$. Then we give the asymptotic normality of $S_{y,z}(n,k)$ and $F_{y,z}(n,k)$. More precisely, we have the following.

Theorem 1.1. *For nonnegative integer n and nonnegative numbers y,z, we have*

$$B_n(y,z) \sim \frac{n!}{R_1^n\sqrt{2\pi(n+yR_1^2e^{yR_1})}}\exp\left(zR_1 + \frac{e^{yR_1}-1}{y}\right) \tag{14}$$

and

$$F_n(y,z) \sim \frac{yn!e^{zR_2}}{R_2^n\sqrt{2\pi(n(y+1-e^{yR_2})^2 + y^2(y+1)R_2^2e^{yR_2})}}, \tag{15}$$

where R_1 is the unique positive solution of $R(z+e^{yR}) = n$ and R_2 is the solution of $zR + \frac{yRe^{yR}}{y+1-e^{yR}} = n$ satisfying $0 < R_2 < 1$.

Theorem 1.2. (1) *The coefficients $S_{y,z}(n,k)$ are asymptotically normal for $y,z \geq 0$.*

(2) *The coefficients $F_{y,z}(n,k)$ are asymptotically normal for $y \geq z \geq 0$.*

2. Proof of Theorems 1.1 and 1.2

In this section, we present the proof of Theorems 1.1 and 1.2 respectively.

Proof. [Proof of Theorem 1.1] We first prove the asymptotic formula of $B_n(y,z)$. By (10), we have the exponential generating function of $B_n(y,z)$ is

$$\sum_{k=0}^{n} B_n(y,z)\frac{t^n}{n!} = \exp\left(zt + \frac{e^{yt}-1}{y}\right). \tag{16}$$

Following Moser and Wyman [23], the sequence $B_n(y,z)$ can be expressed as follows by Cauchy's formula.

$$B_n(y,z) = \frac{n!}{2\pi i}\oint_{|t|=R} \frac{\exp\left(zt + \frac{e^{yt}-1}{y}\right)}{t^{n+1}} dt. \tag{17}$$

Set $t = Re^{i\theta}$. Then

$$B_n(y,z) = \frac{n!}{2\pi R^n}\int_{-\pi}^{\pi} \exp\left(zRe^{i\theta} + \frac{e^{yRe^{i\theta}}-1}{y} - in\theta\right) d\theta.$$

We decompose this last integral into three parts

$$\left(\int_{-\pi}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\pi}\right) \exp\left(F(\theta)\right) d\theta,$$

with

$$F(\theta) = zRe^{i\theta} + \frac{e^{yRe^{i\theta}}-1}{y} - in\theta, \text{ and } \varepsilon = n^{-1/4}.$$

Next we prove that the integrals $\int_{-\pi}^{-\varepsilon}$ and $\int_{\varepsilon}^{\pi}$ are negligible, and then the greatest contribution comes from the medium part $\int_{-\varepsilon}^{\varepsilon}$. Since

$$F'(\theta) = iRe^{i\theta}(z + e^{yRe^{i\theta}}) - in$$

and

$$F''(\theta) = -Re^{i\theta}(z + e^{yRe^{i\theta}} + yRe^{i\theta}e^{yRe^{i\theta}}),$$

we have

$$\begin{aligned} F(0) &= zR + \frac{e^{yR}-1}{y}, \\ F'(0) &= iR(z + e^{yR}) - in, \\ F''(0) &= -R(z + e^{yR} + yRe^{yR}). \end{aligned}$$

Expanding the integral $\int_{\varepsilon}^{\pi}$ in a Taylor series about $\theta = 0$, we obtain

$$\begin{aligned} & \left|\int_{\varepsilon}^{\pi} \exp(F(\theta)) d\theta\right| \\ = & \left|\int_{\varepsilon}^{\pi} \exp\left(F(0) + F'(0)\theta + F''(0)\frac{\theta^2}{2} + o(\theta^2)\right) d\theta\right| \\ = & \exp(F(0))\left|\int_{\varepsilon}^{\pi} \exp\left(i\theta(R(z+e^{yR}) - n)\right)\exp\left(F''(0)\frac{\theta^2}{2} + o(\theta^2)\right) d\theta\right| \\ \leq & \exp\left(zR + \frac{e^{yR}-1}{y}\right)\int_{\varepsilon}^{\pi} \exp\left(-\frac{\theta^2 R}{2}(z + e^{yR} + yRe^{yR}) + o(\theta^2)\right) d\theta. \end{aligned}$$

Note that $F'(0) = 0$ is equivalent to $z + e^{yR} = \frac{n}{R}$. Let $f(R) = z + e^{yR}$ and $g(R) = \frac{n}{R}$. Clearly, $f(R)$ is increasing and $g(R)$ is decreasing in the interval $(0, \infty)$ respectively. It is easy to calculate that $f(0) = z$, $f(R) \to +\infty$, as $R \to +\infty$, and $g(R) \to +\infty$ as $R \to 0$, $g(R) \to 0$ as $R \to +\infty$. Thus there exists a point $R_1 \in (0, \infty)$ such that $f(R_1) = g(R_1)$. Hence the equation $F'(0) = 0$ has an unique (positive) solution R_1, i.e.,

$$R_1(z + e^{yR_1}) = n.$$

The integral in the last expression is

$$\int_{\varepsilon}^{\pi} \exp\left(-\frac{\theta^2}{2}(n + yR_1^2 e^{yR_1}) + o(\theta^2)\right) d\theta \le \int_{\varepsilon}^{\pi} \exp\left(-\frac{\varepsilon^2}{2}\right) d\theta \le \pi e^{-\frac{\sqrt{n}}{2}} \to 0, \text{ as } n \to \infty.$$

The same calculation are valid for $\int_{-\pi}^{-\varepsilon}$. So

$$B_n(y,z) \sim \frac{n!}{2\pi R_1^n} \exp\left(zR_1 + \frac{e^{yR_1} - 1}{y}\right) \int_{-\varepsilon}^{\varepsilon} \exp\left(-\frac{\theta^2}{2}(n + yR_1^2 e^{yR_1}) + o(\theta^2)\right) d\theta. \tag{18}$$

Putting

$$\psi = \sqrt{n + yR_1^2 e^{yR_1}}\,\theta$$

in (18) and observing that for n large enough, we integrate on the real axis

$$\begin{aligned} B_n(y,z) &\sim \frac{n!}{2\pi R_1^n \sqrt{n + yR_1^2 e^{yR_1}}} \exp\left(zR_1 + \frac{e^{yR_1} - 1}{y}\right) \int_{-\infty}^{+\infty} \exp\left(-\frac{\psi^2}{2}\right) d\psi \\ &= \frac{n!}{R_1^n \sqrt{2\pi(n + yR_1^2 e^{yR_1})}} \exp\left(zR_1 + \frac{e^{yR_1} - 1}{y}\right). \end{aligned}$$

Then we prove the asymptotic formula of $F_n(y,z)$. By (11), the sequence $F_n(y,z)$ can be expressed by

$$F_n(y,z) = \frac{n!}{2\pi i} \oint_{|t|=R} \frac{ye^{zt}}{t^{n+1}(y + 1 - e^{yt})} dt. \tag{19}$$

Also set $t = Re^{i\theta}$, we have

$$F_n(y,z) = \frac{yn!}{2\pi R^n} \int_{-\pi}^{\pi} \frac{\exp(zRe^{i\theta} - in\theta)}{y + 1 - \exp(yRe^{i\theta})} d\theta.$$

We decompose this last integral into three parts

$$\left(\int_{-\pi}^{-\varepsilon} + \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\pi}\right) \exp(F(\theta)) d\theta, \tag{20}$$

in this case

$$F(\theta) = \ln \frac{\exp(zRe^{i\theta} - in\theta)}{y + 1 - \exp(yRe^{i\theta})} = zRe^{i\theta} - in\theta - \ln(y + 1 - \exp(yRe^{i\theta})), \text{ and } \varepsilon = n^{-\frac{1}{4}}.$$

Since

$$F'(\theta) = izRe^{i\theta} - in + \frac{iyRe^{i\theta} \exp(yRe^{i\theta})}{y + 1 - \exp(yRe^{i\theta})}$$

and

$$F''(\theta) = -zRe^{i\theta} - \frac{yRe^{i\theta}\exp(yRe^{i\theta})(1+yRe^{i\theta})}{y+1-\exp(yRe^{i\theta})} - \frac{y^2R^2e^{2i\theta}\exp(2yRe^{i\theta})}{(y+1-\exp(yRe^{i\theta}))^2},$$

we have

$$\begin{aligned}
F(0) &= zR - \ln(y+1-e^{yR}),\\
F'(0) &= izR - in + \frac{iyRe^{yR}}{y+1-e^{yR}},\\
F''(0) &= -\left(zR + \frac{yRe^{yR}}{y+1-e^{yR}}\right) - \frac{y^2(y+1)R^2e^{yR}}{(y+1-e^{yR})^2}.
\end{aligned}$$

Note that $F'(0)=0$ is equivalent to $\frac{y^2+y}{y+1-e^{yR}} = \frac{n}{R} + y - z$. Let $f(R) = \frac{y^2+y}{y+1-e^{yR}}$ and $g(R) = \frac{n}{R} + y - z$. Clearly, $f(R)$ is increasing and $g(R)$ is decreasing in the interval $(0, \frac{\ln(y+1)}{y})$ respectively. It is easy to calculate that $f(0) = y+1, f(R) \to +\infty$, as $R \to \frac{\ln(y+1)}{y}$, and $g(R) \to +\infty$ as $R \to 0$, $g(\frac{\ln(y+1)}{y}) = \frac{ny}{\ln(y+1)} + y - z$. Thus there exists a point $R_2 \in (0, \frac{\ln(y+1)}{y})$ such that $f(R_2) = g(R_2)$. Since $\frac{\ln(y+1)}{y} < 1$, the equation $F'(0) = 0$ has an unique solution R_2, which is greater than zero and less than one. Now $F''(0) = -n - \frac{y^2(y+1)R_2^2e^{yR_2}}{(y+1-e^{yR_2})^2}$. Then expanding the integral in a Taylor series about $\theta = 0$, we have

$$\begin{aligned}
\left|\int_\varepsilon^\pi \exp(F(\theta))d\theta\right| &\le \int_\varepsilon^\pi \left|\exp\left(F(0) + \frac{\theta^2}{2}F''(0) + o(\theta^2)\right)\right| d\theta\\
&= \frac{e^{zR_2}}{y+1-e^{yR_2}} \int_\varepsilon^\pi \exp\left(-\frac{\theta^2}{2}\left(n + \frac{y^2(y+1)R_2^2e^{yR_2}}{(y+1-e^{yR_2})^2}\right) + o(\theta^2)\right) d\theta.
\end{aligned}$$

The integral in the last expression is

$$\int_\varepsilon^\pi \exp\left(-\frac{\theta^2}{2}\left(n + \frac{y^2(y+1)R_2^2e^{yR_2}}{(y+1-e^{yR_2})^2}\right) + o(\theta^2)\right) d\theta \le \pi e^{-\frac{\sqrt{n}}{2}} \to 0, \text{ as } n \to \infty.$$

The same calculation is valid for $\int_{-\pi}^{-\varepsilon}$. Finally, we obtain

$$F_n(y,z) \sim \frac{yn!e^{zR_2}}{2\pi R_2^n(y+1-e^{yR_2})} \int_{-\varepsilon}^{\varepsilon} \exp\left(-\frac{\theta^2}{2}\left(n + \frac{y^2(y+1)R_2^2e^{yR_2}}{(y+1-e^{yR_2})^2}\right)\right) d\theta. \tag{21}$$

Putting

$$\psi = \frac{\sqrt{n(y+1-e^{yR_2})^2 + y^2(y+1)R_2^2e^{yR_2}}}{y+1-e^{yR_2}}\theta.$$

Then

$$\begin{aligned}
F_n(y,z) &\sim \frac{yn!e^{yR_2}}{R_2^n\sqrt{2\pi(n(y+1-e^{yR_2})^2 + y^2(y+1)R_2^2e^{yR_2})}} \int_{-\infty}^{+\infty} \exp\left(-\frac{\psi^2}{2}\right) d\psi\\
&= \frac{yn!e^{yR_2}}{R_2^n\sqrt{2\pi(n(y+1-e^{yR_2})^2 + y^2(y+1)R_2^2e^{yR_2})}}
\end{aligned}$$

This completes our proof. □

A standard approach to demonstrating the asymptotic normality is the following criterion, which was used by Harper [16] to show the asymptotic normality of the Stirling numbers of the second kind. We refer the reader to the excellent surveys of the asymptotic normality by [2, 11, 14].

Theorem 2.1 ([26]). *Suppose that $A_n(x) = \sum_{k=0}^{n} a(n,k)x^k$ have only real zeros and $A_n(x) = \prod_{i=1}^{n}(x + r_i)$. Let*

$$\mu_n = \sum_{i=1}^{n} \frac{1}{1 + r_i}$$

and

$$\sigma_n^2 = \sum_{i=1}^{n} \frac{r_i}{(1 + r_i)^2}$$

Then if $\sigma_n \to +\infty$, the numbers $a(n,k)$ are asymptotically normal with the mean μ_n and the variance σ_n^2.

Remark 2.2 ([11]). *Suppose that $A_n(x) = \sum_{k=0}^{n} a(n,k)x^k$. Then the mean and the variance of $a(n,k)$ are given by the following expressions*

$$\begin{aligned} \mu_n &= \frac{A_n'(1)}{A_n(1)} = \frac{\sum_{k=0}^{n} ka(n,k)}{\sum_{k=0}^{n} a(n,k)}, \\ \sigma_n^2 &= \frac{A_n'(1)}{A_n(1)} + \frac{A_n''(1)}{A_n(1)} - \left(\frac{A_n'(1)}{A_n(1)}\right)^2 = \frac{\sum_{k=0}^{n} k^2 a(n,k)}{\sum_{k=0}^{n} a(n,k)} - \mu_n^2. \end{aligned}$$

From the exponential generating functions (10) and (11), it is easy to obtain that the recurrence relations of $B_n(x,y,z)$ and $F_n(x,y,z)$ are

$$B_n(x,y,z) = (x+z)B_{n-1}(x,y,z) + xyB_{n-1}'(x,y,z) \tag{22}$$

and

$$F_n(x,y,z) = (x+z)F_{n-1}(x,y,z) + x(x+y)F_{n-1}'(x,y,z) \tag{23}$$

respectively. So the coefficients $S_{y,z}(n,k)$ and $F_{y,z}(n,k)$ satisfy

$$S_{y,z}(n,k) = S_{y,z}(n-1,k-1) + (z+yk)S_{y,z}(n-1,k) \tag{24}$$

and

$$F_{y,z}(n,k) = kF_{y,z}(n-1,k-1) + (z+yk)F_{y,z}(n-1,k) \tag{25}$$

respectively.

Let f and g be two real polynomials with only real zeros and with positive leading coefficients. Denote their zeros by $r_1(f) \geq r_2(f) \geq \cdots \geq r_n(f)$ and $r_1(g) \geq r_2(g) \geq \cdots \geq r_m(g)$ respectively. For convenience, we set that $r_i(f) = +\infty$ for $i < 1$ and $r_i(f) = -\infty$ for $i > n$. We say that $f(x)$ *interlaces* $g(x)$, denoted by $f \preceq g$, if $n \leq m \leq n+1$ and $r_i(g) \geq r_i(f) \geq r_{i+1}(g)$ for all i. Obviously, if f has only real zeros then $f' \preceq f$. Wang and Yeh [25] gave the following criteria for the reality of zeros of polynomials.

Theorem 2.3 ([25]). *Suppose that f, g are polynomials with coefficients having the same sign and only have real zeros. If $g \preceq f$ and $ad \geq bc$, then the polynomial $(ax+b)f(x) + x(cx+d)g(x)$ also has only real zeros.*

Based on Theorem 2.3, we have the reality of zeros of $B_n(x,y,z)$ and $F_n(x,y,z)$ as polynomials of x.

Theorem 2.4. (1) *The polynomial $B_n(x,y,z)$ has only real zeros for $y \geq 0$.*

(2) *The polynomial $F_n(x,y,z)$ has only real zeros for $y \geq z \geq 0$.*

Now we are in the position to prove Theorem 1.2.

Proof. [Proof of Theorem 1.2] It suffices to prove the variances of $S_{y,z}(n,k)$ and $F_{y,z}(n,k)$ tending to ∞, as $n \to \infty$ by Theorem 2.4 respectively.

By the recurrence (24), we have

$$\sum_{k=0}^{n} kS_{y,z}(n,k) = \frac{B_{n+1}(y,z) - (z+1)B_n(y,z)}{y}$$

$$\sum_{k=0}^{n} k^2 S_{y,z}(n,k) = \frac{B_{n+2}(y,z) - (2+2z)B_{n+1}(y,z) + (z^2+2z+1-y)B_n(y,z)}{y^2}.$$

So the mean and the variance of $S_{y,z}(n,k)$ are

$$\mu_n = \frac{B_{n+1}(y,z)}{yB_n(y,z)} - \frac{z+1}{y}$$

and

$$\sigma_n^2 = \frac{B_{n+2}(y,z)}{y^2 B_n(y,z)} - \left(\frac{B_{n+1}(y,z)}{yB_n(y,z)}\right)^2 - \frac{1}{y}.$$

Using the asymptotic formula (14) of $B_n(y,z)$, we have

$$\begin{aligned} \sigma_n^2 &\sim \frac{(n+2)(n+1)}{R_1^2 y^2}\sqrt{1 - \frac{2}{n+2+yR_1 e^{yR}}} - \frac{(n+1)^2}{R_1^2 y^2}\left(1 - \frac{1}{n+1+yR_1 e^{yR}}\right) - \frac{1}{y} \\ &\sim \frac{n+1}{R_1^2 y^2} - \frac{1}{y}. \end{aligned}$$

Thus $\sigma_n^2 \to \infty$, as $n \to \infty$.

By the recurrence (25), we have

$$\sum_{k=0}^{n} kF_{y,z}(n,k) = \frac{F_{n+1}(y,z) - (1+z)F_n(y,z)}{(y+1)}$$

$$\sum_{k=0}^{n} k^2 F_{y,z}(n,k) = \frac{F_{n+2}(y,z) - (3+2z)F_{n+1}(y,z) - (y-3z-1-z^2)F_n(y,z)}{(y+1)^2}.$$

So the mean and the variance are

$$\mu_n = \frac{F_{n+1}(y,z)}{(y+1)F_n(y,z)} - \frac{z+1}{y+1}$$

and

$$\sigma_n^2 = \frac{F_{n+2}(y,z)}{(y+1)^2 F_n(y,z)} - \frac{F_{n+1}(y,z)}{(y+1)^2 F_n(y,z)} - \left(\frac{F_{n+1}(y,z)}{(y+1)F_n(y,z)}\right)^2 + \frac{z-y}{(y+1)^2}.$$

Using the asymptotic formula (15) of $F_n(y,z)$, we have

$$\begin{aligned}
\sigma_n^2 \quad &\sim \quad \frac{(n+1)(n+2)}{(y+1)^2R_2^2}\sqrt{1-\frac{2(y+1-e^{yR_2})^2}{(n+2)(y+1-e^{yR_2})^2+y^2(y+1)R_2^2e^{yR_2}}} \\
&\quad -\frac{(n+1)}{(y+1)^2R_2}\sqrt{1-\frac{(y+1-e^{yR_2})^2}{(n+1)(y+1-e^{yR_2})^2+y^2(y+1)R_2^2e^{yR_2}}} \\
&\quad -\frac{(n+1)^2}{(y+1)^2R_2^2}\left(1-\frac{(y+1-e^{yR_2})^2}{(n+1)(y+1-e^{yR_2})^2+y^2(y+1)R_2^2e^{yR_2}}\right)+\frac{z-y}{(y+1)^2} \\
&\sim \quad \frac{(1-R_2)(n+1)}{(y+1)^2R_2^2}+\frac{z-y}{(y+1)^2}.
\end{aligned}$$

Thus $\sigma_n^2 \to \infty$, as $n \to \infty$. This completes the proof of Theorem 1.2. □

3. Applications

In this section, we give the asymptotic formulas and the asymptotic normality of polynomials related to Dowling lattices from Theorems 1.1 and 1.2.

Corollary 3.1. *The following asymptotic formulas hold.*

(1) *The Bell numbers*

$$B_n = \sum_{k=0}^{n} S(n,k) \sim \frac{n!\exp\left(e^{R_1}-1\right)}{R_1^n\sqrt{2\pi(n+R_1e^{R_1})}},$$

where R_1 is the unique positive solution of $Re^R = n$;

(2) *The Dowling numbers*

$$W_{m,n} = \sum_{k=0}^{n} W_m(n,k) \sim \frac{n!\exp\left(R_1+\frac{e^{mR_1}-1}{m}\right)}{R_1^n\sqrt{2\pi(n+mR_1e^{mR_1})}},$$

where R_1 is the unique positive solution of $R(1+e^{mR}) = n$;

(3) *The r-Bell numbers*

$$B_{n,r} = \sum_{k=0}^{n} S_r(n,k) \sim \frac{n!\exp\left(rR_1+e^{R_1}-1\right)}{R_1^n\sqrt{2\pi(n+R_1e^{R_1})}},$$

where R_1 is the unique positive solution of $R(r+e^R) = n$;

(4) *The r-Dowling numbers*

$$W_{m,r,n} = \sum_{k=0}^{n} W_{m,r}(n,k) \sim \frac{n!\exp\left(rR_1+\frac{e^{mR_1}-1}{m}\right)}{R_1^n\sqrt{2\pi(n+mR_1e^{mR_1})}},$$

where R_1 is the unique positive solution of $R(r+e^{mR}) = n$;

(5) *The ordered Bell numbers*

$$F_n = \sum_{k=0}^{n} k!S(n,k) \sim \frac{n!}{R_2^n\sqrt{2\pi(n(2-e^{R_2})^2+2R_2^2e^{R_2})}},$$

where R_2 is the solution of $\frac{Re^R}{2-e^R} = n$ satisfying $0 < R_2 < 1$;

(6) *The numbers*

$$F_{n,m,1} = \sum_{k=0}^{n} m^k k! W_m(n,k) \sim \frac{mn! e^{R_2}}{R_2^n \sqrt{2\pi(nm(2 - e^{mR_2})^2 + 2m^4 R_2^2 e^{mR_2})}},$$

where R_2 *is the solution of* $R + \frac{mRe^{mR}}{2-e^{mR}} = n$ *satisfying* $0 < R_2 < 1$*;*

(7) *The numbers*

$$F_{n,m,2} = \sum_{k=0}^{n} k! W_m(n,k) \sim \frac{mn! e^{R_2}}{R_2^n \sqrt{2\pi(n(m + 1 - e^{mR_2})^2 + m^2(m+1) R_2^2 e^{mR_2})}},$$

where R_2 *is the solution of* $R + \frac{mRe^{mR}}{m+1-e^{mR}} = n$ *satisfying* $0 < R_2 < 1$.

Corollary 3.2. *The sequences* $(S(n,k))$, $(W_m(n,k))$, $(S_r(n,k))$, $(W_{m,r}(n,k))$ *and* $(k!S(n,k))$, $(k!W_m(n,k))$, $(m^k k! W_m(n,k))$ *are asymptotically normal respectively.*

4. Remarks

Let $a_0, a_1, \ldots, a_n$ be a sequence of positive numbers. The sequence is *unimodal* if there is an index $0 \leq m \leq n$ such that $a_0 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n$ (such an index m is called a mode of the sequence). The sequence is *log-concave* if $a_{i-1}a_{i+1} \leq a_i^2$ for $i = 1, \ldots, n-1$. Clearly, the log-concavity implies the unimodality. Unimodal and log-concave sequences occur naturally in combinatorics, analysis, algebra, geometry, probability and statistics. We refer the reader to Stanley [24], Brenti [7] and Brändén [6] for surveys and [8, 9, 22, 25] for some recent progress on this subject.

One classical approach to unimodality and log-concavity of a finite sequence is to use Newton's inequality: if the polynomial $\sum_{i=0}^{n} a_i x^i$ with positive coefficients has only real zeros, then

$$a_i^2 \geq a_{i-1}a_{i+1}\left(1 + \frac{1}{i}\right)\left(1 + \frac{1}{n-i}\right)$$

for $1 \leq i \leq n-1$, and the sequence $a_0, a_1, \ldots, a_n$ is therefore unimodal and log-concave (see Hardy, Littlewood and Pólya [17, p. 104] for instance). So by Theorem 2.4, we have the sequences $(S_{y,z}(n,k))_{k=0}^{n}$ and $(F_{y,z}(n,k))_{k=0}^{n}$ are unimodal and log-concave respectively. Recently, Gyimesi and Nyul [19] presented a combinatorial interpretation of r-Whitney numbers with colored set partitions. It is possible to find combinatorial interpretations of $(S_{y,z}(n,k))_{k=0}^{n}$ and $(F_{y,z}(n,k))_{k=0}^{n}$. Furthermore, we probably can present a combinatorial proof of the unimodality and the log-concavity of these two sequences.

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