Fractional-Order Iterative Learning Control
for Robotic Arm-$\text{PD}^2\text{D}^\alpha$ Type

Boško P. Cvetković[a], Mihailo P. Lazarević[a]

[a]University of Belgrade, Faculty of Mechanical Engineering, Belgrade 11120, Serbia

Abstract. In this paper, a new open-loop $\text{PD}^2\text{D}^\alpha$ type a fractional order iterative learning control (ILC) is studied for joint space trajectory tracking control of a linearized uncertain robotic arm. The robust convergent analysis of the tracking errors has been done in time domain where it is theoretically proven that the boundednesses of the tracking error are guaranteed in the presence of model uncertainty. The convergence of the proposed open-loop ILC law is proven mathematically using Gronwall integral inequality for a linearized robotic system and sufficient conditions for convergence and robustness are obtained.

1. INTRODUCTION

The science of robotics has grown tremendously over the past twenty years, fueled by rapid advances in computer and sensor technology, as well as theoretical advances in control theory, [1]. At present time, the vast majority of robot applications deal with industrial robot arms operating in factory environments. Robots are being deployed to accomplish tasks having strict requirements of accuracy, precision, repeatability, mass production and quality in addition to ease of human effort and cost-effectiveness, [2]. The investigation into the modeling of the dynamics and control of the robotic systems and mechanisms has been an active topic of research which is stimulated by the different applications and by the increasing demand for better performance of robotic systems, [3].

Taking advantage of the fact that robot manipulators are generally used in repetitive tasks, iterative learning control (ILC) schemes [4-7] have been proposed for robot manipulators in the past three decades to enhance the tracking accuracy from operation to operation for given systems. Also, various version ILC strategies are proposed for different type dynamical systems [8-12] that utilize given objective systems data and past information in the form of repetitions to track a reference trajectory over finite time intervals without detail modeling. The basic idea of ILC is to improve the current tracking performance by fully taking advantage of the past control experience. Since the structure of ILC can be viewed as a feedforward control methodology in the time domain and it is simple and easy to implement, the ILC method is suitable for designing a nonlinear tracking control system [13,14]. Also, many classical ILC algorithms have a restriction that involves the assumption that the coupling matrix [CB] has a full-column rank where B denotes the control matrix and C denotes the output matrix of state space for the linearized system. To

overcome this lack, we introduced [14], open-closed ILC PIDD2/PID type which successfully resolves this drawback.

Recently, increasing attention is dedicated to fractional differential equations and their applications in various science and engineering fields [15-17]. Moreover, increasing attention has been dedicated to fractional calculus (FC) and its application in the control and modeling of fractional-order systems [18-20].

Also, the application of ILC to the fractional-order system has become a new topic, [21-27] it does not only retains the advantages of the classical ILC, but also offers potential for better performances in a variety of complex physical processes. For example, a fractional-order D-type ILC algorithm was proposed in the frequency domain [21], and $PD\alpha$ type ILC in the time domain [22]. Most of the existing fractional-order ILC (FOILC) methods for fractional-order systems only focus on the regular and a few singular systems of fractional order.

On the other hand, in many practical robotic control applications, the robotic systems can be presented as nonlinear or linearized integer order dynamic systems. So, there is a difficulty to apply fractional order ILC scheme to integer order system. Recently, authors [28] proposed $PD\alpha$ type FOILC schemes for already obtained integer-order model (given as linear time-varying mechanical system) which is more practical for control practice. Motivated by the search for new FOILC law and their application to mechanical systems, the new open-loop PD^2D^α-type FOILC law for a class of linearized robotic systems is suggested. Particularly, term D2 in the proposed FOILC serves to overcome well-known restriction the coupling matrix $[CB] = 0$ for a linear or linearized system [29], [30]. Sufficient conditions are derived in the time domain which are our main contributions. Consequently, one may conclude proposed FOILC type is good basis for further real-time robot control application.

"*Preliminaries*" section presents some of the norms and definitions of fractional derivative that are considered in this contribution. In "*Main results*" section, first problem formulation is stated and then convergence results for the proposed open-loop PD^2D^α-type learning law are given. Finally, the "Conclusion" section concludes this contribution.

2. PRELIMINARIES

2.1. Notations, the λ-norm

In this paper, the following norms are adopted [6] for n-dimensional Euclidean space R^n : sup-norm $\|x\|_\infty = \sup\limits_{1 \le i \le n} |x_i|$, $x = [x_1, x_2, \ldots, x_n]^T$ matrix norm as $\|A\|_\infty = \max\limits_{1 \le i \le m}(\sum\limits_{j=1}^{n} |a_{ij}|)$, $A = [a_{ij}]_{m \times n}$ where $\|A\|_\infty$ denotes the infinite norm of a matrix. Particularly, the standard λ-norm for a real function $g(t), (t \in [0, T]), g : [0, T] \to \Re^n$ is defined as:

$$\|g(t)\|_\lambda = \sup\limits_{t \in [0,T]} e^{-\lambda t}\|g(t)\|, \ \lambda > 0 \tag{1}$$

Especially when is norm $\|(.)\|_\infty$, it follows $\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty$. Also, for the previous norms, one can easily prove that the λ-norm is equivalent to the ∞-norm.

(i) *Property 1*: λ norm has the next property

$$\sup\limits_{t \in [0,T]} e^{-\lambda t} \int_0^t \|g(.)\| e^{a(t-\tau)} d\tau = \sup\limits_{t \in [0,T]} \int_0^t e^{-\lambda t} e^{\lambda \tau} e^{a(t-\tau)} e^{-\lambda \tau} \|g(.)\| \tau d\tau$$
$$\le \|g(.)\|_\lambda \sup\limits_{t \in [0,T]} \int_0^t e^{(a-\lambda)(t-\tau)} d\tau \le \|g(.)\|_\lambda \frac{1 - e^{-(a-\lambda)T}}{\lambda - a} \le \frac{1}{\lambda - a} \|g(.)\|_\lambda \tag{2}$$

Before presenting the main results, we first introduce the following Lemma 2.1 [6].

Lemma 2.1. *[6] Suppose a real positive series $\{q_k\}_1^\infty$ satisfies*

$$q_k \le \rho q_{k-1} + \tilde{\varepsilon} \tag{3}$$

where $\rho \geq 0, \tilde{\varepsilon} > 0$ and $p < 1$. Then the following holds:

$$\lim_{k \to \infty} q_k \leq \tilde{\varepsilon}/(1 - \rho) \tag{4}$$

One can notice that in case of $\tilde{\varepsilon} = 0$, $\lim_{k \to \infty} q_k \to 0$.

Lemma 2.2. *(Bellman-Gronwall inequality) [31] Let $f_1(t), g_1(t)$, and $h_1(t)$ be nonnegative continuous functions in the interval $[0, T]$. Moreover, if there is a nonnegative constant a such that left inequality holds, then the right inequality also holds.*

$$f(t) \leq h(t) + \int_0^t a f(\tau) d\tau + \int_0^t g(\tau) d\tau, \Rightarrow f(t) \leq h(t) + \int_0^t \exp(a \cdot (t - \tau))[ah(\tau) + g(\tau)] d\tau \tag{5}$$

2.2. Preliminary notes on fractional calculus

The theory of fractional order calculus can be traced back to 300 years ago, and now, it plays role in modern science particularly in the field of control engineering [16,32]. In the present section, we shortly review some basic definitions and properties of Riemann-Liouville (RL) and Caputo fractional operators [16,17]. Let $f(.) \in AC[a, b]$ be a continuous function over the finite interval $[a, b]$ having the first derivative almost everywhere in $[a, b]$, being integrable, that is, it is in $L^1[a, b]$. Also, the space $AC^m([a, b])$is the space of functions with continuous $m - 1$ derivatives on $[a, b]$, and m-th derivative in $L^1[a, b]$. The definition of RL fractional integral of order μ is described by:

$$_aI_t^\mu f(t) =_a D_t^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t - \tau)^{\mu-1} f(\tau) d\tau, \ \mu \in R^+ \tag{6}$$

where $\Gamma(.)$ is the well-known Euler's gamma function, which is defined by $\Gamma(z) = \int_0^\infty e^{-z} t^{z-1} dt$. The RL operator $_aI_t^\mu(.)$ possesses the semigroup property, i.e. $_aI_t^\mu \cdot_a I_t^\xi =_a I_t^{\mu+\xi}$ The RL derivative of the fractional order μ of a function $f(t)$ is given as

$$_a^{RL}D_t^\mu f(t) = D^n(_aI_t^{-(n-\mu)} f(t)) = \frac{d^n}{dt^n}(_aI_t^{-(n-\mu)} f(t)) = \frac{1}{\Gamma(n - \mu)} \frac{d^n}{dt^n} [\int_a^t (t - \tau)^{n-\mu-1} f(\tau) d\tau], \tag{7}$$

where $n - 1 \leq \mu < n \in Z^+$ and $D^n(.) = d^n(.)/dt^n, n \in \mathbb{N}$ is the classical n-order derivative. If we introduce $n = [\mu] \in \mathbb{N}$ which stands for integer part of μ , and $\alpha = (\mu - [\mu]) \in (0, 1)$ it follows next form for RL fractional order:

$$_a^{RL}D_t^\mu f(t) =_a^{RL} D_t^{[\mu]+\alpha} f(t) =_a^{RL} D_t^{n+\alpha} f(t) = \frac{d^{n+1}}{dt^{n+1}}(_aI_t^{-(1-\alpha)} f(t)) \tag{8}$$

Also, the definition of Caputo derivative of fractional order μ is introduced [16,17] as follows:

$$_a^C D_t^\mu f(t) =_a D_t^{\mu-n} D^n f(t) =_a I_t^{n-\mu}(f^{(n)}(t)) = \frac{1}{\Gamma(n - \mu)} \int_a^t (t - \tau)^{n-\mu-1} f^{(n)}(\tau) d\tau, \tag{9}$$

where $n - 1 < \mu < n \in Z^+$. Also, we have:

$$_a^C D_t^\mu f(t) =_a D_t^{\mu-[\mu]-1} D^{[\mu]+1} f(t) =_a I_t^{n+1-\mu}(f^{(n+1)}(t)) =_a I_t^{1-\alpha}(f^{(n+1)}(t)) \tag{10}$$

In this case $n = 1$, for $0 \leq \alpha < 1$ we obtain:

$$_a^{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} [\int_a^t (t - \tau)^{-\alpha} f(\tau) d\tau,]$$

$$_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} [\int_a^t (t - \tau)^{-\alpha} \frac{df(\tau)}{dt} d\tau,] \tag{11}$$

In the following sections, D^α will denote $_CD_t^\alpha$, $^{RL}D_t^\alpha {}_{to}^CD_t^\alpha$ for the brevity of notation. If $f(0) = 0$, we can easily prove that in case $a = t_0$ it yields:

$$t_0D_t^{-\alpha}(_{t_0}D_t^\alpha f(t)) = f(t) \tag{12}$$

3. MAIN RESULTS

3.1. Problem formulation

It is well known that many different schemes of robot control exist in literature. Here we consider robot arm control, where dynamics of our robot can be obtained using the procedure for symbolic form computation of the complete dynamics of a robotic system with kinematic chain structures based on the Rodriquez approach, [33]. Differential equations of a robotic system can be presented in the covariant form of Lagrange equations with corresponding external generalized forces. Moreover, the robot arm dynamics can be obtained in compact form as:

$$a(q)\ddot{q} + (N(q,\dot{q}) - Q^g) = a(q)\ddot{q} + n(q,\dot{q}) = Q^m \tag{13}$$

where $a(q) = [a_{\alpha\beta}] \in R^{nxn}$ denotes inertia matrix, $N(q,\dot{q})$ is the matrix that includes centrifugal and Coriolis effects, and Q^g and $Q^m = U$ are gravity term and motor torque vectors, respectively [2]. Here, a model based fractional order control scheme is introduced and implemented, which includes the application of the feedback linearization technique [34]. As result one can linearize the dynamics of robot as

$$\ddot{q}(t) = u(t) \tag{14}$$

where $u(t)$ is the new control signal, or in the case of the existence of the model uncertainty $\eta = \eta(t)$, we have:

$$\ddot{q}(t) = u(t) + \eta(t) \tag{15}$$

or in state-space

$$\dot{x}(t) = Ax(t) + Bu(t) + D\eta(t) , A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}_{2nx2n} , B = D = \begin{bmatrix} 0 \\ I \end{bmatrix}_{2nx2n} \tag{16}$$

$$y(t) = Cx(t) = \begin{bmatrix} I & 0 \end{bmatrix}_{nx2n} \cdot x(t) \tag{17}$$

We introduce the following assumptions:

A1. The desired trajectories $y_d(t)$, $x_d(t)$ are continuously differentiable on $[0, T]$ A1 is a reasonable assumption that makes possible calculating $\dot{e}_i(t) = \dot{y}_d(t) - \dot{y}_i(t), \ddot{e}_i(t) = C\delta\ddot{x}_i(t) = C(\ddot{x}_d(t) - \ddot{x}_i(t)), e_i^\alpha(t) = C\delta e_i^\alpha(t)$.

A2. The system (16), (17) is causal. Specifically, the existence of unique bounded control input $u_d(t)$ and $x_d(t)$ makes the output of the system (17) to be $u_d(t)$, i.e.

$$\dot{x}_d(t) = Ax_d(t) + Bu_d(t), \tag{18}$$

$$y_d(t) = Cx_d(t), \tag{19}$$

A3. The initial resetting conditions hold for all iterations, i.e.

$$x_i(0) = x_d(0), \ i = 0, 1, 2 \ldots, \tag{20}$$

Assumption A3. restricts the initial states or the initial outputs in each repetition operation should be equal with the desired initial ones so one can achieve a perfect tracking, from the beginning demands the perfect initial condition, that is, $\forall i, \delta x_i(0) = x_d(0) - x_i(0) = 0, e_i(0) = y_d(0) - y_i(0) = 0$. Many practical control problems require such a perfect tracking over the entire transient period including the initial one.

A4. The uncertainty $\eta_i(t) \in R^n$, is uniformly bounded. In the sequel, we use positive constant d_η, to denote the upper bounds for $\eta_i(t)$, i.e, $\forall t \in [0, T]$ and $\forall i \rightarrow \|\eta_i(t)\| \le d_\eta$ Assumption 4. puts the boundedness restrictions of the $\eta_i(t)$ on given time interval $\forall t \in [0, T]$. Based on contraction mapping, all existing ILC methods require Assumption 4. The reason is as follows. ILC methodology tries to use as little system prior knowledge as possible in its design, and the lack of such system knowledge, however, gives rise to a difficulty in designing a suitable (stable) closed-loop controller.

3.2. Convergence results

3.2.1. Open-loop PD^2D^α learning law

As we stated previously, many ILC laws have a restriction which involves the assumption that the coupling matrix [CB] has a full-column rank where B denotes the control matrix and C denotes the output matrix of state space for the linearized system (18),(19). To resolve this drawback, as well as improving the performance of robot control taking into account the properties of fractional operators we consider the following open-loop PD^2D^α learning law, see Figure 1:

$$u_{i+1}(t) = u_i(t) + \Gamma_1 \ddot{e}_i(t) + R_1 e_i(t) + R_2 e_i^{(\alpha)}(t) \tag{21}$$

where $e_i(t) = y_d(t) - y_i(t)$ is the trajectory tracking error in $i-$ th iteration, $e_{i+1}(t) = y_d(t) - y_{i+1}(t)$ is the trajectory tracking error in $i+1-$ th iteration. r_1, $R_1, R_2 \in R^{nxn}$ denote open-closed-loop positive-definite diagonal learning matrices ,i.e. $R_{(.)} = diag(r_{(.)1}, r_{(.)2}, \ldots, r_{(.)n}), \Gamma_{(.)} = diag(g_{(.)1}, g_{(.)2}, \ldots, g_{(.)n})$.

Lemma 3.1. *For the system (16), (17) and the reference system (18), (19) then there exists a sufficient large λ satisfying*

$$\|\delta x_i^{(\alpha)}(t)\|_\lambda \le \rho_0'\|\delta u_i(t)\|_\lambda + \varepsilon_\eta \tag{22}$$

where $o(\lambda^{-1}) = \dfrac{(1 - e^{-\lambda T})}{\lambda} \le \dfrac{1}{\lambda}$ and $\rho_0' = \rho_0 + h_0 o(\lambda^{-1}), \varepsilon_\eta = (h_1 + h_2 T)d_\eta$, ρ_0 , h_0 , h_1 , h_2 , d_η some positive constants.
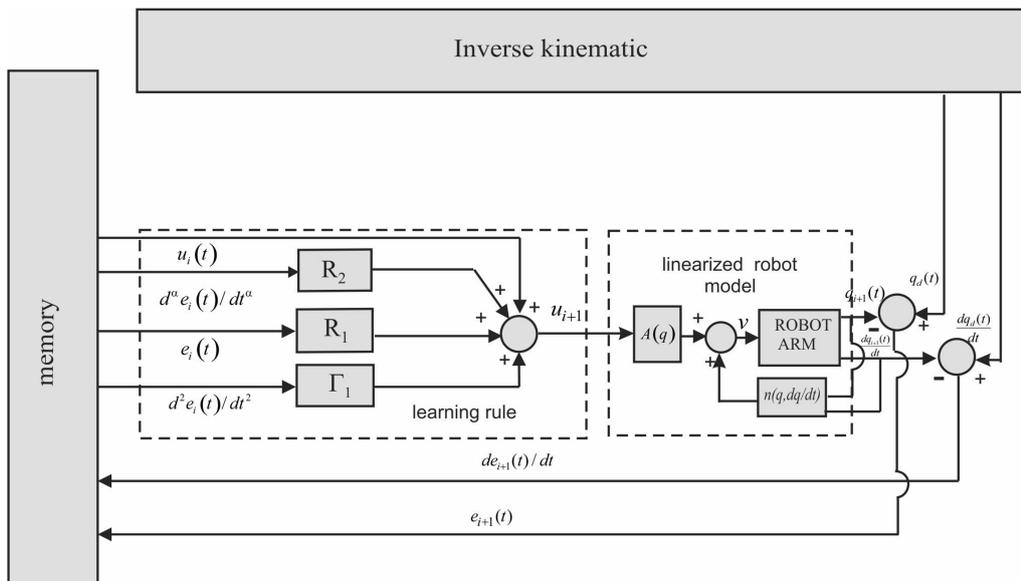

Figure 1: Block diagram of the open-loop PD^2D^α law for the robot control

Proof.
We can obtain the solution of equation (17), that is, the state response $x_i(t)$ is expressed by

$$
\begin{aligned}
x_i(t) &= \Phi(t)x_i(0) + \int_0^t \Phi(t-\tau)Bu_i(\tau)d\tau + \int_0^t \Phi(t-\tau)D\eta_i(\tau)d\tau \\
&= g_i(t) + \int_0^t \Phi(t-\tau)Bu_i(\tau)d\tau + \int_0^t \Phi(t-\tau)D\eta_i(\tau)d\tau
\end{aligned}
\tag{23}
$$

as well as output response $y_i(t)$

$$
y_i(t) = Cx_i(t)
\tag{24}
$$

Where $g_i(t) \in R^r$ and $\Phi(t-\tau) \in R^{r\times r}$ are continuously differentiable in t and τ, and $x_i(0)$ is an initial condition of state variable $x_i(t)$ for the i-th iteration. In similar manner, we have solution for the equation (18),

$$
x_d(t) = g_d(t) + \int_0^t \Phi(t-\tau)Bu_d(\tau)d\tau
\tag{25}
$$

Applying α-th order fractional derivative to the (25), (27) we obtain:

$$
x_d^{(\alpha)}(t) = g_d^{(\alpha)}(t) + \lim_{\tau\to t-0}[_\tau D_t^{\alpha-1}\Phi(t-\tau)Bu_d(\tau)] - \int_0^t {}_\tau D_t^\alpha \Phi(t-\tau)Bu_d(\tau)d\tau
\tag{26}
$$

Let

$$
\begin{aligned}
\delta x_i &= x_d(t) - x_i(t), \quad \delta\dot{x}_i = \dot{x}_d(t) - \dot{x}_i(t) \\
\delta u_i &= u_d(t) - u_i(t), \quad \delta\dot{u}_i = \dot{u}_d(t) - \dot{u}_i(t)
\end{aligned}
\tag{27}
$$

$$
D^\alpha(\delta x_i(t)) = \delta x_i^{(\alpha)} = x_d^{(\alpha)}(t) - x_i^{(\alpha)}(t)
\tag{28}
$$

Also, it follows from (26)-(28) it yields, [35]:

$$
\delta x_i^{(\alpha)}(t) = x_d^{(\alpha)}(t) - x_i^{(\alpha)}(t) = g_d^{(\alpha)}(t) - g_i^{(\alpha)}(t) + \lim_{\tau\to t-0}[_\tau D_t^{\alpha-1}\Phi(t-\tau)Bu_d(\tau)] -
$$

$$
\lim_{\tau\to t-0}[_\tau D_t^{\alpha-1}\Phi(t-\tau)Bu_i(\tau)] + \int_\tau^t {}_\tau D_t^\alpha \Phi(t-\tau)Bu_d(\tau)d\tau - \int_\tau^t {}_\tau D_t^\alpha \Phi(t-\tau)Bu_i(\tau)d\tau
\tag{29}
$$

$$
- \lim_{\tau\to t-0}[_\tau D_t^{\alpha-1}\Phi(t-\tau)D\eta_i(\tau)] - \int_0^t {}_\tau D_t^\alpha \Phi(t-\tau)D\eta_i(\tau)d\tau
$$

Taking into account assumption A3, it follows:

$$
g_d^{(\alpha)}(t) - g_i^{(\alpha)}(t) = 0.
\tag{30}
$$

Hence, expression (29) takes the form:

$$
\delta x_i^{(\alpha)}(t) = {}_\tau{}^{D_t^{\alpha-1}}(\Phi(t-\tau))B|_{\tau=t}\delta u_i(t) + \int_0^t {}_\tau D^\alpha \Phi(t-\tau)B\delta u_i(\tau)d\tau
$$

$$
- [_\tau D_t^{\alpha-1}\Phi(t-\tau)D\eta_i(\tau)|_{\tau=t}] - \int_0^t {}_\tau D^\alpha \Phi(t-\tau)D\eta_i(\tau)d\tau
\tag{31}
$$

Taking $\lambda-$ norm of (31) it yields:

$$\|\delta x_i^{(\alpha)}(t)\|_\lambda \leq \sup_{0 \leq t \leq T} \|_\tau D_t^{\alpha-1}\Phi(t-\tau)B|_{\tau=t}\|_\infty \|\delta u_i(t)\|_\lambda + \sup_{0 \leq t, \tau \leq T} \|_\tau D_t^\alpha\Phi(t-\tau)B\|_\infty \int_0^t e^{-\lambda t} \cdot \|\delta u_i(\tau)\|_\infty d\tau$$

$$+ \sup_{0 \leq t \leq T} \|_\tau D_t^{\alpha-1}\Phi(t-\tau)D\|_\infty \|\eta_i(t)\|_\infty + \sup_{0 \leq t, \tau \leq T} \|_\tau D_t^\alpha\Phi(t-\tau)D\|_\infty \int_0^t e^{-\lambda t}\|\eta_i(\tau)\|_\infty d\tau$$

$$\|\delta x_i^{(\alpha)}(t)\|_\lambda \leq \sup_{0 \leq t \leq T} \|_\tau D_t^{\alpha-1}\Phi(t-\tau)B|_{\tau=t}\|_\infty \|\delta u_i(t)\|_\lambda \tag{32}$$

$$+ \sup_{0 \leq t, \tau \leq T} \|_\tau D_t^\alpha\Phi(t-\tau)B\|_\infty \int_0^t e^{-\lambda(t-\tau)} \cdot \sup_{0 \leq \tau \leq T} e^{-\lambda\tau}\|\delta u_i(\tau)\|_\infty d\tau$$

$$+ \sup_{0 \leq t \leq T} \|_\tau D_t^{\alpha-1}\Phi(t-\tau)D\|_\infty \cdot d_\eta + \sup_{0 \leq t, \tau \leq T} \|_\tau D_t^\alpha\Phi(t-\tau)D\|_\infty \int_0^t e^{-\lambda t}d_\eta d\tau$$

or

$$\|\delta x_i^\alpha(t)\|_\lambda \leq \rho_0\|\delta u_i(t)\|_\lambda + h_0 \cdot \|\delta u_i(\tau)\|_\lambda \sup_{0 \leq t \leq T} \int_0^t e^{-\lambda(t-\tau)} \cdot d\tau + h_1 d_\eta + h_2 d_\eta \cdot \sup_{0 \leq t \leq T} \int_0^t e^{-\lambda t}d\tau \tag{33}$$

$$\|\delta x_i^\alpha(t)\|_\lambda \leq \rho_0\|\delta u_i(t)\|_\lambda + h_0\|\delta u_i(\tau)\|_\lambda O(\lambda^{-1}) + h_1 d_\eta + h_2 d_\eta T$$

where

$$\rho_0 = \sup_{0 \leq t \leq T} \|_\tau D_t^{\alpha-1}\Phi(t-\tau)B|_{\tau=t}\|_\infty, h_0 = \sup_{0 \leq t, \tau \leq T} \|_\tau D_t^\alpha\Phi(t-\tau)B\|_\infty,$$

$$h_1 = \sup_{0 \leq t \leq T} \|_\tau D_t^{\alpha-1}\Phi(t-\tau)D\|_\infty , h_2 = \sup_{0 \leq t, \tau \leq T} \|_\tau D_t^\alpha\Phi(t-\tau)D\|_\infty \tag{34}$$

$$o(\lambda^{-1}) = \frac{(1-e^{-\lambda T})}{\lambda} \leq \frac{1}{\lambda}, \|\eta_i(t)\|_\infty \leq d_\eta$$

Thus, we have:

$$\|\delta x_i^\alpha(t)\|_\lambda \leq \rho_0'\|\delta u_i(t)\|_\lambda + \varepsilon_\eta \tag{35}$$

where

$$\rho_0 = \rho_0 + h_0 O(\lambda^{-1}) , \varepsilon_\eta = (h_1 + h_2 T)d_\eta \tag{36}$$

This completes the proof.

A sufficient condition for convergence of a proposed open-loop ILC is given by the Theorem 3.2 and proved as follows.

Theorem 3.2. *Considering system (16), (17) under assumptions A1-A4 . If*

$$\rho + \tilde\rho_0 < 1 \tag{37}$$

where $\rho = \|(I - \Gamma_1 CAB)\|_\infty$, $\tilde\rho_0 = \beta_3 \cdot \rho_0 = \|R_2 C\| \cdot \sup_{0 \leq t \leq T} \|_\tau D_t^{\alpha-1}(\Phi(t-\tau)B|_{\tau=t}\|_\infty$, *then, the open-loop* $PD^2 D^\alpha$
learning law (21) guarantees that when $i \to \infty$ *bounds of the tracking errors* $\|x_d(t) - x_i(t)\|$, $\|y_d(t) - y_i(t)\|$, *and*
$\|u_d(t) - u_i(t)\|$ *converge asymptotically into the specified bounds. The bounds* $\tilde\varepsilon_x, \tilde\varepsilon_y, \tilde\varepsilon_u$ *are given by* $\tilde\varepsilon_u = \frac{1}{1-\tilde\rho}\tilde\varepsilon, \tilde\varepsilon_x =$
$d_1 d_\eta T + bO(\lambda_1^{-1})\frac{1}{1-\tilde\rho}\tilde\varepsilon$, $\tilde\varepsilon_y = c \cdot \tilde\varepsilon_x$ *where* $\tilde\rho = \rho + \beta_3(\rho_0 + h_0 O(\lambda^{-1})) + \beta_1 bO(\lambda_1^{-1})$, $\tilde\varepsilon = \beta_1 d_1 d_\eta T + \beta_2 d_\eta + \beta_3 \varepsilon_\eta$

Proof: Note that the tracking error and its derivatives integer and fractional can be obtained:

$$e_i = C\delta x_i \, , \; e_i^{(\alpha)} = C\delta x_i^{(\alpha)} \, , \; \dot{e}_i = C\delta \dot{x}_i = C(A\delta x_i + B\delta u_i - D\eta_j)$$
$$\ddot{e}_i = C\delta_i \ddot{x}_i = CA^2\delta x_i + CAB\delta u_i - CD\dot{\eta}_i - CAD\eta_i \tag{38}$$

According to the proposed ILC law (21), we obtain

$$\delta u_{i+1}(t) = u_d(t) - u_{i+1}(t) = \delta u_i(t) - \Gamma_1\ddot{e}_i(t) - R_1 e_i(t) - R_2 e_i^{(\alpha)}(t) \tag{39}$$

Substituting (38) into (39) we can get

$$\delta u_{i+1}(t) = [I - \Gamma_1 CAB]\delta u_j(t) - (\Gamma_1 CA^2 + R_1 C)\delta x_i(t) - R_2 C\delta x_i^{(\alpha)}(t) + \Gamma_1 CAD\eta_i(t) + \Gamma_1 CD\dot{\eta}_i(t) \tag{40}$$

One can observe, that in our case that the matrices CAB, CAD have a full column rank and the matrices CB, CD have not a full column rank, $([C][B] = 0)$, $([C][D] = 0)$. From previous eq. (28), we get:

$$\delta u_{i+1}(t) = [I - \Gamma_1 CAB]\delta u_i(t) - (\Gamma_1 CA^2 + R_1 C)\delta x_i(t) - R_2 C\delta x_i^{(\alpha)}(t) + \Gamma_1 CAD\eta_i(t) \tag{41}$$

Taking the standard norm for equation (41) and using the condition of Theorem 3.2 as well as assumption A4 we have

$$\|\delta u_{i+1}(t)\| \le \rho\|\delta u_i(t)\| + \|[\Gamma_1 CA^2 + R_1 C]\|\|\delta x_i(t)\| + \|\Gamma_1 CAD\|\|\eta_i(t)\| + \|R_2 C\|\|\delta x_i^{(\alpha)}(t)\| \tag{42}$$

Then we can arrive

$$\|\delta u_{i+1}(t)\| \le \rho\|\delta u_i(t)\| + \beta_1\|\delta x_i(t)\| + \beta_2 d_\eta + \beta_3\|\delta x_i^{(\alpha)}(t)\| \tag{43}$$

where

$$\beta_1 = \|[\Gamma_1 CA^2 + R_1 C]\| \, , \; \beta_2 = \|\Gamma_1 CAD\| \, , \; \beta_3 = \|R_2 C\| \tag{44}$$

Also, we have from (17) and (38)

$$\delta x_i = \int_0^t (A\delta x_i(\tau) + B\delta u_i(\tau) - D\eta_i(\tau))d\tau \tag{45}$$

Taking the norm on the both sides of(33) it follows:

$$\|\delta x_i\| = \int_0^t (a\|\delta x(\tau)\| + b\|\delta u_i(\tau)\| + d_1\|\eta_i(\tau)\|\mu\tau \tag{46}$$

where $a = \|A\|$, $b = \|B\|$, $d_1 = \|D\|$. From assumption A4, we have:

$$d_1 \int_0^t (\|\eta_i(\tau)\| \, d\tau \le d_1 \cdot d_\eta T \tag{47}$$

which implies that

$$\|\delta x_i\| \le d_1 d_\eta T + \int_0^t (a\|\delta x(\tau)\| + b\|\delta u_i(\tau)\| \, d\tau \tag{48}$$

Applying the Bellman-Grownwall lemma [28], we obtain

$$\|\delta x_i\| \le d_1 d_\eta T e^{at} + \int_0^t e^{a(t-\tau)} b\|\delta u_i(\tau)\| d\tau \tag{49}$$

and multiplying the previous inequality by $e^{-\lambda t}, \lambda > a$ and taking the λ-norm, we get

$$\|\delta x_i\|_\lambda \le d_1 d_\eta T + \frac{1 - e^{(a-\lambda)T}}{\lambda - a} b \|\delta u_i\|_\lambda \le d_1 d_\eta T + bo(\lambda_1^{-1})\|\delta u_j\|_\lambda \, , O(\lambda_1^{-1}) = \frac{1}{\lambda - a} \tag{50}$$

Taking into account (43), and applying the λ-norm, we have

$$\|\delta u_{i+1}(t)\|_\lambda \le \rho\|\delta u_i(t)\|_\lambda + \beta_1\|\delta x_i(t)\|_\lambda + \beta_2 d_\eta + \beta_3\|\delta x_i^{(\alpha)}(t)\|_\lambda \tag{51}$$

Now, linking (22), (50) with (51), we can get

$$\|\delta u_{i+1}(t)\|_\lambda \le \rho\|\delta u_i(t)\|_\lambda + \beta_1(d_1 d_\eta T + bO(\lambda_1^{-1})\|\delta u_i\|_\lambda) + \beta_2 d_\eta + \beta_3(\rho'_0 \|\delta u_i(t)\|_\lambda + \varepsilon_\eta) \tag{52}$$

$$\|\delta u_{i+1}(t)\|_\lambda \le \tilde{\rho}\|\delta u_i(t)\|_\lambda + \tilde{\varepsilon} \tag{53}$$

where are

$$\tilde{\rho} = \rho + \beta_3(\rho_0 + h_0 O(\lambda^{-1})) + \beta_1 bO(\lambda_1^{-1}), \tilde{\varepsilon} = \beta_1 d_1 d_\eta T + \beta_2 d_\eta + \beta_3 \varepsilon_\eta \tag{54}$$

Since $\rho + \beta_3\rho_0 < 1$ by assumption, it is possible to choose λ sufficiently large so that $\tilde{\rho} < 1$. Therefore, according to Lemma 2.1, it implies that:

$$\lim_{i\to\infty} \|\delta u_i\|_\lambda \le \frac{1}{1 - \tilde{\rho}} \tilde{\varepsilon} = \tilde{\varepsilon}_u \tag{55}$$

Submitting eq. (55) into eq. (59), as well as into eq. (38) we have

$$\begin{aligned}
\lim_{i\to\infty} \|\delta x_i\|_\lambda &\le d_1 d_\eta T + bO(\lambda_1^{-1})(\frac{1}{1 - \tilde{\rho}} \tilde{\varepsilon}) = \tilde{\varepsilon}_x, \\
\lim_{i\to\infty} \|e_i\|_\lambda &\le c(d_1 d_\eta T + bO(\lambda_1^{-1})(\frac{1}{1 - \tilde{\rho}} \tilde{\varepsilon})) = \tilde{\varepsilon}_y
\end{aligned} \tag{56}$$

This concludes the proof.

Remark 3.3. *In the case of without disturbance and uncertainty of the system, $\xi_i(t) = 0, \eta_i(t) = 0$. i.e d_ξ , d_η , $d_{\eta\alpha}$ tend to zero, these bounds also tend to zero. Namely, one can obtain when $i \to \infty$ the bounds of the tracking errors $\|x_d(t) - x_i(t)\|$, $\|y_d(t) - y_i(t)\|$, $\|u_d(t) - u_i(t)\|$, converge asymptotically to zero. Proof is similar with the proof of Theorem 3.2, taking into account $\xi_i(t) = 0, \eta_i(t) = 0$, i.e. $\varepsilon = \delta = 0$.*

$$\lim_{i\to\infty} \|\delta u_i\|_\lambda = 0 \tag{57}$$

According to the existence and uniqueness theorem of integer-order differential equation, it is obtained that

$$\lim_{i\to\infty} y_i(t) = y_d(t) \tag{58}$$

4. Conclusion

In this article, we studied the tracking problem of the robotic arm via ILC. For the first time the open-loop PD^2D^α type fractional-order ILC law is proposed for a given linearized robotic arm. The proposed FOILC scheme for the already obtained integer-order model of robotic arm is more practical for control practice. In particular, the sufficient conditions for the robust convergence in time domain of the proposed FOILC were defined, by the corresponding theorem, and proved. For analysis, it is found that the sufficient conditions of convergence not only depend on all of system dynamics, but also rely on learning matrices.

References

[1] Kurdila J.A, Ben-Tzvi P. Dynamics and Control of Robotic Systems,John Wiley & Sons,NJ, USA, 2020.

[2] Siciliano B., Sciavicco L., Villani L.,OrioloG. Robotics, Springer-Verlag, London, 2009.

[3] Lynch M. K., Park C.F.,Modern Robotics Mechanics, Planning, and Control, Cambridge University Press, Cambridge 2017.

[4] Uchiyama M. Formulation of high-speed motion pattern of a mechanical arm by trial. Trans Soc Inst Cont Eng. 1978; 14:706-712.

[5] Arimoto S, Kawamura S. Bettering operation of robots by learning. J Robotic Syst. 1984;1:123-140.

[6] Ahn HS, Moore K and Chen Y. Iterative learning control robustness and monotonic convergence for interval systems. 1st ed.London:Springer-Verlag London Limited, 2007.

[7] Xu JX, Panda SK, Lee TH.Real-time Iterative Learning Control, Design and Applications. 1st ed. London: Springer-Verlag, 2009.

[8] Tan KK, Dou H, Chen Y, et al. High precision linear motor control via relay-tuning and iterative learning based on zero-phase Eltering. IEEE Transactions on Control Systems Technology 9(2): 244–253,2001.

[9] Hou,Z.,Xu,J.,Yan,J.An iterative learning approach for density control of freeway traffic low via ramp metering. Transportation Research-PartC.16(1),71–97(2008)

[10] Meng,D.,Jia.Y.,Du.J,Zhang,J.On iterative learning algorithms for the formation control of nonlinear multi-agent systems.Automatica.50(1),291-295(2014)

[11] FravoliniMand Fabietti P. An iterative learning strategy for the auto-tuning of the feedforward and feedback controller in type-l diabetes. Computer Methods in Biomechanics and Biomedical Engineering 17(13): 1464-1482.2014.

[12] Y. Zhang, J.Wei, H. Wang, J. Liu,Adaptive Fuzzy Iterative Learning Control of Robotic Systems with Time-delay Outputs and Input Dead Zone via Observer, Filomat 32:5 (2018), 1571-1580

[13] Isao T and Hunag PH. Iterative Learning Control for Trajectory Tracking of Robot Manipulators. International Journal of Automation and Smart Technology, 7(3) 133-139, 2017.

[14] Lazarević P.M,Mandić D. P., Ostojić S. Further results on advanced robust iterative learning control and modeling of robotic systems, Proc IMechE Part C: J Mechanical Engineering Science 0(0) 1-16, 2020. DOI: 10.1177/0954406220965996.

[15] Podlubny I., Fractional Differential Equations. Academic Press, San Diego,1999.

[16] Kilbas, AA, Srivastava, H, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, Elsevier, Amsterdam, NEDERLAND, 2006,Vol.204.

[17] Nakhushev A. M., Fractional Calculus and Its Applications, FIZMATLIT, Moscow, 2003 (in Russian).

[18] Monje C.A, Chen Y. Q, Vinagre B. M, Xue D. Y, Feliu-Batlle V. Fractional-order Systems and Controls: Fundamentals and Applications. London:Springer-Verlag, 2010.

[19] Bahaa G.,Fractional Optimal Control Problem for Differential System with Control Constraints, Filomat 30:8 (2016), 2177–2189, DOI 10.2298/FIL1608177B

[20] Petras I.(Ed.) Handbook of Fractional Calculus with Applications,Vol. 6: Applications in Control, Boston: Walter de Gruyter GmbH,2019.

[21] Chen Y.Q, Moore K.L., On type iterative learning control, In Proc. 40th IEEE Conference on Decision and Control, Orlando, FL USA,pp.4451-4456,2001.

[22] Lazarević,P.M., PD^α -type iterative learning control for fractional LTI system-16th Int.Cong. CHISA 2004, G2.9, Praha,Czech Republic, 22-26, August, 2004.

[23] Li Y., Chen Y.Q., Ahn H.S., Fractional-order iterative learning control for fractional-order linear systems,Asian J. Contro1,13 (1), $54 - 63, 2011.$

[24] Lazarević P.M, P.Mandić, Feedback-feedforward iterative learning control for fractional-order uncertain time delay system-PD alpha type, ICFDA'14 Catania, 23-25 June 2014, IE86, 2014.

[25] Liu X., Yanfang Li Y., Yanmin Liu Y.,The convergence of iterative leaming control for some fractional system,Advances in Difference Equations (2017), 2017:132.

[26] Lazarević M and Panagiotis T. Robust second-order PDalpha type iterative learning control for a class of uncertain fractional order singular systems,J Vibrat. Control 2016; 22: 2004-2018. doi:10.1177/1077546314562241

[27] Wang X,Wang J., Shen D., Zhou Y.Convergence analysis for iterative learning control of conformable fractional differential equations. Math Meth Appl Sci. 2018;(41):8315-8328. https://doi.org/ 10.1002/mma.5291.

[28] Zhao Y., Zhou F., Wang D., Li Y., Design and Analysis of Fractional Order PDalpha-Type Iterative Learning Control , IFAC PapersOnLine 50-1 (2017) 8077–8083.

[29] Li Y., Chen Y. Q., Ahn S., Tian G., A survey on fractional-order iterative learning control, J. Optim. Theory Appl. 1, 156 (2013), 127-140.

[30] Lazarević M., Iterative Learning Control of Integer and Noninteger Order: an Overview, Scientific Technical Review, 2014,Vol.64,No.1,pp.35-47

[31] Xiaoe, Zhao J., Convergence monotonicity and speed comparison of iterative learning control algorithms for nonlinear systems,IMA Journal of Mathematical Control and Information (2013) 30, 473–486 doi:10.1093/imamci/dns034

[32] Kochubei A and Luchko Y.(Eds.) Handbook of Fractional Calculus with Applications,vol.1: Basic Theory, Boston:Walter de Gruyter GmbH, 2019.

[33] Čović V., Lazarević M., Robot Mechanics, Belgrade, Faculty of Mechanical Engineering (in Serbian), 2009.

[34] Slotine J.J., Li, Applied nonlinear control. Vol. 199: Prentice-Hall Englewood Cliffs, NJ.,1991.

[35] Zhao Y, Zhou F., Wang Y. ·Li Y., Fractional-order iterative learning control with initial state learning design , Nonlinear Dyn (2017) 90:1257–1268 DOI 10.1007/s11071-017-3724-6.