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Weak KKM Set-Valued Mappings in Hyperconvex Metric Spaces

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Abstract. In this paper, the concept of weak KKM set-valued mapping is extended from topological vector spaces to hyperconvex metric spaces. For these mappings we obtain several intersection theorems that prove to be useful in establishing existence criteria for weak and strong solutions of the general variational inequality problem and minimax inequalities.

1. Introduction

Since Ky Fan [1] extended the Knaster-Kuratowski-Mazurkiewicz theorem to topological vector spaces, establishing the so-called Fan-KKM principle, this became a fundamental result in nonlinear analysis. This motivated Sehie Park [2] to introduce the concept of KKM theory, in which, the notion of KKM mappings plays a central role. Recall that a mapping is called a KKM mapping if it satisfies the assumption of the Fan-KKM principle. Subsequently, having introduced concepts more general than that of a KKM mapping, significant generalizations of several results from nonlinear analysis and optimization were obtained. We need to recall two of these concepts.

Let X be a nonempty set in a vector space, Y be a nonempty set and $T, S : X \rightrightarrows Y$ be two set-valued mappings. We say that:

- S is a KKM mapping with respect to T (w.r.t. T, briefly) [3] if for each nonempty finite subset A of X, $T(\text{conv } A) \subseteq S(A)$ (here and throughout the paper, conv A designates the convex hull of A).

- S is a weak KKM mapping w.r.t. T [4] if for each nonempty finite subset A of X and any $x \in \text{conv } A$, $T(x) \cap S(A) \neq \emptyset$.

The KKM theory was developed in topological spaces endowed with various abstract convex structures (for instance, the c-spaces of Horvath [5], the generalized convex spaces of Park and Kim [6], or the FC spaces of Ding [7]). Khamsi [8] established a hyperconvex version of the KKM-Fan principle and the KKM theory in hyperconvex metric spaces was developed. Within this framework, interesting KKM type

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theorems with applications in fixed point theory, minimax theory and best approximation theory have been obtained in a large number of papers (see, for example [9]-[14]).

The present paper fits into this interesting group of works. Our aim is to obtain, in hyperconvex metric spaces, KKM type theorems for a mapping which is weak KKM w.r.t. the other one. The layout of the paper is as follows. In the next section we recall some needed definitions and results, mainly concerning metric spaces. In Section 3, the aforementioned concept of weak KKM mapping will be adapted to hyperconvex metric spaces. Using this concept, in Section 3 we establish several intersection theorems in hyperconvex metric spaces. The last two sections are devoted to applications of our intersection theorems. In Section 4 we derive existence criteria for weak and strong solutions of the general variational inequality. In Section 5, several minimax inequalities are obtained.

2. Basic concepts

We begin by explaining Aronszajn and Panitchpakdi's notion of hyperconvex metric space [15] and some related concepts.

Definition 2.1. *A metric space (X, d) is said to be hyperconvex if $\bigcap_{\alpha\in\Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$, for any family of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ in X for which $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$, for any $\alpha, \beta \in \Gamma$ (here, $B(x, r)$ designates the closed ball with center x and radius r).*

Examples of hyperconvex metric spaces are the real line with the usual metric, $l^\infty(I)$ for any set I and $L^\infty(\mu)$ for a finite measure μ. It is worth noting that for any $n \ge 2$, $\mathbb{R}^n$ with the Euclidean metric is not hyperconvex, but it is hyperconvex with respect to the Chebyshev distance. In opposition to the lack of linearity, hyperconvexity provides a rich metric structure that leads to a collection of interesting properties. The concepts introduced below are metric correspondents of the notions of convex hull and convex set from vector spaces.

Definition 2.2. *Let (X, d) be a metric space. Let A be a bounded nonempty subset of X. Set*

$$co(A) = \cap \{B : B \text{ is a closed ball such that } A \subseteq B\}.$$

If $A = co(A)$, we will say A is an admissible subset of A. A is called sub-admissible if for each finite subset D of A, $co(D) \subseteq A$.

From the previous definition, we infer that every admissible set is closed, as it is an intersection of closed sets. It follows also easily that an intersection of admissible (resp., sub-admissible) sets is admissible (resp., sub-admissible). Obviously, if A is an admissible subset of a metric space X, then A is sub-admissible; when A is compact, the converse is also true in hyperconvex metric spaces [16]. It is also worth noting that if M is hyperconvex, then each admissible set in M is also hyperconvex.

Regarding metric spaces it is shown in [8] that

Proposition 2.3. *Let (X, d) be a metric space.*

(i) There exists an isometric embedding $i : X \to l^\infty(X)$.

(ii) X is hyperconvex iff for any metric space M which contains isometrically X, there exists a nonexpansive retraction $r : M \to X$, i.e. r is nonexpansive and $r(x) = x$ for any $x \in X$.

Remark 2.4. *Let X be hyperconvex metric space. Based on the previous proposition we can identify X with $i(X)$ (its copy in the Banach space $l^\infty(X)$). For a nonempty finite subset A of X denote by conv A the convex hull of A, regarded as a subset of $l^\infty(X)$. From (ii) in Proposition 2.3, there exists a nonexpansive retraction $r : l^\infty(X) \to X$. Then, $r(conv\ A) \subseteq co\ A$ (see [11, p. 616]).*

The concepts of quasiconvex respectively, quasiconcave function were extended to the case of metric spaces in [16] as follows:

Definition 2.5. *Let X be a metric space. A function $f : X \to \mathbb{R}$ is said to be metric quasiconvex (resp., metric quasiconcave) if for each $\lambda \in \mathbb{R}$, the sublevel set $\{x \in X : f(x) \leq \lambda\}$ (resp., the set $\{x \in X : f(x) \geq \lambda\}$) is sub-admissible.*

Notice that for a metric quasiconvex function defined on a hyperconvex metric space, Kirk et al [11] used the term hyper quasi-convex function.

Remark 2.6. *(a) It is easy to see that a function $f : X \to \mathbb{R}$ is metric quasiconvex if and only if for each nonempty finite subset A of X and any $x \in$ co A, $f(x) \leq \max_{u \in A} f(u)$. A similar characterization can be given for metric quasiconcave fuctions replacing "$\leq$ max" with $\geq$ min. (b) It is worth pointing out that if $f : X \to \mathbb{R}$ is a metric quasiconvex function then the sets of the form $\{x \in X : f(x) < \lambda\}$ are sub-admissible (the proof is elementary).*

We close this section with some notations and notions regarding set-valued mappings. Let $T : X \rightrightarrows Y$ be a set-valued mapping. If $A \subseteq X$ we denote by $T(A)$ the image of A under T, that is, $T(A) = \bigcup_{x \in X} T(x)$. We associate to T other two mappings $T^c : X \rightrightarrows Y$, the complementary of T and $T^* : Y \rightrightarrows X$, the dual of T, defined by $T^c(x) = Y \setminus T(x)$ and $T^*(y) = \{x \in X : y \notin T(x)\}$. The values of T^* are called the cofibers of T.

For topological spaces X and Y, a set-valued mapping $T : X \to 2^Y$ is said to be

(i) upper semicontinuous if for every open subset G of Y, the set $\{x \in X : T(x) \subseteq G\}$ is open;
(ii) lower semicontinuous if for every open subset G of Y, the set $\{x \in X : T(x) \cap G \neq \emptyset\}$ is open;
(iii) continuous if it is both upper and lower semicontinuous;
(iv) closed if its graph (that is, the set Gr $T = \{(x, y) \in X \times Y : y \in T(x)\}$) is a closed subset of $X \times Y$.

Note that if the space Y is Hausdorff and compact, a set-valued mapping $T : X \rightrightarrows Y$ is closed if and only if it is upper semicontinuous and closed-valued.

3. Intersection theorems

The concepts of weak KKM mapping can be extended to hyperconvex metric spaces as follows.

Definition 3.1. *Let X be a sub-admissible subset of a metric space, Y be a nonempty set and $S, T : X \rightrightarrows Y$ be two set-valued mappings. We say that S is a weak KKM mapping w.r.t. T if for each nonempty finite subset A of X and any $x \in co\, A$, $T(x) \cap S(A) \neq \emptyset$.*

Note that if S is weak KKM mapping w.r.t. T, then for every $x \in X$, $T(x) \cap S(x) \neq \emptyset$; particularly both mappings T ans S are nonempty-valued (to see this is sufficient to take in Definition 3.1, $A = \{x\}$).

The first result from this section was established in [4, Theorem 2] in the framework of generalized convex spaces, but the proof given here is different.

Theorem 3.2. *Let X be a compact hyperconvex metric space, Y be a nonempty set and $S, T : X \rightrightarrows Y$ two set-valued mappings satisfying the following conditions:*

(i) S is weak KKM mapping w.r.t. T;

(ii) for each $u \in X$, the set $\{x \in X : T(x) \cap S(u) \neq \emptyset\}$ is closed.

Then, there exists an $x_0 \in X$ such that $T(x_0) \cap S(u) \neq \emptyset$ for all $u \in X$.

Proof. For each $u \in X$, set
$$G(u) = \{x \in X : T(x) \cap S(u) = \emptyset\}.$$
By way of contradiction, assume that the conclusion of the theorem is not true. Then $X = \bigcup_{u \in X} G(u)$. Since X is compact and the sets $G(u)$ are open (by (ii)), there exists a finite set $\{u_1, \cdots, u_n\} \subseteq X$ such that $X = \bigcup_{i=1}^{n} G(u_i)$. We will identify X with its copy in the Banach space $l^\infty(X)$. Let
$$C = \text{ conv } \{u_1, \cdots, u_n\} \subseteq l^\infty(X) \text{ and } L = \text{ co } \{u_1, \cdots, u_n\} \subseteq X.$$
By Proposition 2.3 (ii), there exists nonexpansive retraction $r : l^\infty(X) \to X$ and, in view of Remark 2.4, $r(C) \subset L$.

For every $x \in C$, $r(x) \in r(C) \subseteq L \subseteq \bigcup_{i=1}^{n} G(u_i)$. Hence, there exists an index $i_0 \in \{1, \ldots, n\}$ such that $r(x) \in G(u_{i_0})$ which implies that $d\Big(r(x), G^c(u_{i_0})\Big) > 0$. Therefore, we have

$$\alpha(x) = \sum_{i=1}^{n} d\Big(r(x), G^c(u_i)\Big) > 0,$$

for any $x \in C$. Consider the function $f : C \to C$ defined by

$$f(x) = \frac{1}{\alpha(x)} \sum_{i=1}^{n} d\Big(r(x), G^c(u_i)\Big)\, u_i\ .$$

Clearly, f is continuous. From Schauder's fixed point theorem, there exists $x_0 \in C$ such that $f(x_0) = x_0$. Set $I = \Big\{i \in \{1, \ldots, n\} : d\Big(r(x_0), G^c(u_i)\Big) > 0\Big\}$. Then $x_0 \in$ conv $\{u_i : i \in I\}$, whence

$$r(x_0) \in r\Big(\text{conv}\ \{u_i;\ i \in I\}\Big) \subseteq\ \text{co}\ \{u_i;\ i \in I\}.$$

Since S is weak KKM mapping w.r.t. T, there exists an index $j \in I$ such that $T(r(x_0)) \cap S(u_j) \neq \emptyset$, which implies that $r(x_0) \in G^c(u_j)$. This contradicts the fact that $d(r(x_0), G^c(u_j)) > 0$. □

Remark 3.3. *Note condition (ii) in Theorem 3.2 holds whenever Y is topological space, T is upper semicontinuous and S is closed-valued.*

We give below a noncompact version of Theorem 3.2. As usual, in this case a coercivity condition is needed.

Theorem 3.4. *Let X be a hyperconvex metric space, Y be a nonempty set and $S, T : X \rightrightarrows Y$ two set-valued mappings satisfying the following conditions:*

(i) S is weak KKM mapping w.r.t. T;
(ii) for each $u \in X$, the set $\{x \in X : T(x) \cap S(u) \neq \emptyset\}$ is closed;
(iii) there exists a nonempty compact subset K of X such that for every nonempty finite subset A of X there exists a compact admissible subset $M_A \subseteq X$, containing A, such that

$$x \in M_A \setminus K \Rightarrow T(x) \cap S(u) = \emptyset \text{ for some } u \in M_A.$$

Then, there exists an $x_0 \in X$ such that $T(x_0) \cap S(u) \neq \emptyset$ for all $u \in X$.

Proof. Let $G : X \rightrightarrows X$ be the set-valued mapping considered in the proof of Theorem 3.2. Assume that the conclusion does not hold. Then $X = \bigcup_{u \in X} G(u)$. Since K is compact there exists a finite set $A \subseteq X$ such that

$$K \subseteq \bigcup_{u \in A} G(u). \tag{1}$$

By (iii), there exists a compact admissible set M such that $A \subseteq M \subseteq X$ and

$$M \setminus K \subseteq \bigcup_{u \in M} G(u). \tag{2}$$

Since $A \subseteq M$, by (1) and (2), we have $M \subseteq \bigcup_{u \in M} G(u)$. As M is admissible and compact, it is a hyperconvex metric subspace of X. Clearly $T_{|M}$ and $S_{|M}$ (the restrictions of T and S to M) satisfy the assumptions of Theorem 3.2. From this theorem, it follows that there exists $x_0 \in M$ such that $T(x_0) \cap S(u) \neq \emptyset$ for all $u \in M$. This means that $x_0 \notin \bigcup_{u \in M} G(u)$; a contradiction. □

To prove the next intersection theorem we need the following lemma:

Lemma 3.5. *[17, Proposition 3.2] Let X and Y be two hyperconvex metric spaces such that Y is compact. If $P : X \rightrightarrows Y$ is a closed set-valued mapping with nonempty admissible values and sub-admissible cofibers, then $\bigcap_{u\in X} P(u) \neq \emptyset$.*

Theorem 3.6. *Let X and Y be two hypermetric convex spaces such that X is compact and $S, T : X \rightrightarrows Y$ be two set-valued mappings that satisfy the following conditions:*

(i) S is closed with admissible values and sub-admissible cofibers;

(ii) T has compact admissible values;

(iii) S is weak KKM mapping w.r.t. T;

(iv) for each $u \in X$, the set $\{x \in X : T(x) \cap S(u) \neq \emptyset\}$ is closed.

Then, there exists an $x_0 \in X$ such that $T(x_0) \cap \bigcap_{u\in X} S(u) \neq \emptyset$.

Proof. From Theorem 3.2, via assumptions (iii) and (iv), there exists $x_0 \in X$ such that $T(x_0) \cap S(u) \neq \emptyset$ for all $u \in X$. This means that the set-valued mapping $P : X \rightrightarrows T(x_0)$ defined by

$$P(u) = T(x_0) \cap S(u)$$

has nonempty values. One can easily see that P is a closed mapping with admissible values. For each $y \in T(x_0)$,

$$P^*(y) = \{u \in X : y \notin T(x_0) \cap S(u)\} = \{u \in X : y \notin S(u)\} = S^*(y).$$

Thus, in view of (i), P has sub-admissible cofibers. From Lemma 3.5,

$$\emptyset \neq \bigcap_{u\in X} P(u) = T(x_0) \cap \bigcap_{u\in X} S(u).$$

□

Remark 3.7. *(a) In view of Remark 3.3, assumption (iv) in Theorem 3.6 holds if the set-valued mapping T is upper semicontinuos. This fact will be used in applications, in the next sections. (b) Since any admissible set is closed and any closed subset of a compact space is compact, when Y is a compact space, assumption (ii) in Theorem 3.6 can be more simply formulated as follows: T has admissible values.*

4. First applications

As first applications of the intersection results obtained in the previous section we establish existence criteria of weak and respectively strong solutions for a generalized variational inequality of Stampacchia type.

Theorem 4.1. *Let X be a compact hyperconvex metric space and Y be a topological space. Let $P : X \rightrightarrows Y$ be an upper semicontinuous set-valued mapping with nonempty values and f be a real function defined on $X \times Y \times X$ that satisfies the following conditions:*

(i) for each $u \in X$, the set $\{(x, y) \in X \times Y : f(x, y, u) \geq 0\}$ is closed;

(ii) for each $(x, y) \in X \times Y$, the set $\{u \in X : f(x, y, u) < 0\}$ is sub-admissible;

(iii) for every $x \in X$, there exists $y \in P(x)$ such that $f(x, y, x) \geq 0$;

(iv) there exists a nonempty compact subset K of X, such that for every nonempty finite subset A of X there exists a compact admissible subset $M_A \subseteq X$, containing A, such that

$$x \in M_A \setminus K \Longrightarrow f(x, y, u) < 0 \text{ for some } y \in P(x) \text{ and } u \in M_A.$$

Then, there exists $x_0 \in X$ such that for each $u \in X$ $f(x_0, y, u) \geq 0$ for some $y \in P(x_0)$ (depending of x).

Proof. Consider the set-valued mappings $T, S : X \rightrightarrows X \times Y$ defined by

$$T(x) = \{x\} \times P(x), \quad S(u) = \{(x, y) \in X \times Y : f(x, y, u) \geq 0\}.$$

We prove that S is weak KKM w.r.t. T. Suppose it is not. Then there exists a nonempty finite subset A of X and $x \in$ co A such that $T(x) \cap S(A) = \emptyset$. Consequently, for every $y \in P(x)$ and $u \in A$, $f(x, y, u) < 0$. By (ii), it follows that $f(x, y, x) < 0$, for all $y \in P(x)$. On the other hand, by (iii), $f(x, y, x) \geq 0$ for at least one $y \in P(x)$; a contradiction.

Since P is upper semicontinuous so will be T. From (i), the set-valued mapping S is closed-valued. Taking into account Remark 3.3, condition (ii) from Theorem 3.4 is fulfilled. We note also that assumption (iv) of the theorem is just a reformulation of condition (iii) of Theorem 3.4.

From Theorem 3.4, there exists $x_0 \in X$ such that for each $u \in X$, $T(x_0) \cap S(u) \neq \emptyset$. Thus, for each $u \in X$ there exists $y_u \in T(x_0)$ such that $f(x_0, y_u, u) \geq 0$. $\square$

Recall (see [18, Theorem 3.1]) that if (X, d_X) and (Y, d_Y) are two hyperconvex metric spaces, then $X \times Y$ equipped with the metric d defined by

$$d\Big((x_1, y_1), (x_2, y_2)\Big) = max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

is also a hyperconvex metric space. Moreover, if $B\Big((x, y), r\Big)$ is the closed ball from $X \times Y$ of center (x, y) and radius r, then $B\Big((x, y), r\Big) = B_X(x, r) \times B_Y(y, r)$, where $B_X(x, r)$ ($B_Y(y, r)$, resp.) is the closed ball centered at x with radius r in X (centered at y with radius r in Y, respectively). It follows immediately that the product of two admissible sets is admissible in the space $X \times Y$.

Theorem 4.2. *Assume that X and Y are two hyperconvex metric spaces such that X is compact. Let $P : X \rightrightarrows Y$ be an upper semicontinuous set-valued mapping with compact admissible values and f be a real function defined on $X \times Y \times X$ that satisfies the following conditions:*

(i) the set $\{(x, y, u) \in X \times Y \times X : f(x, y, u) \geq 0\}$ is closed;

(ii) for each $u \in X$, the set $\{(x, y) \in X \times Y : f(x, y, u) \geq 0\}$ is admissible;

(iii) for each $(x, y) \in X \times Y$, the set $\{u \in X : f(x, y, u) < 0\}$ is sub-admissible;

(iv) for every $x \in X$, there exists $y \in P(x)$ such that $f(x, y, x) \geq 0$.

Then, there exists $(x_0, y_0) \in X \times Y$ such that $y_0 \in P(x_0)$ and $f(x_0, y_0, u) \geq 0$ for all $u \in X$.

Proof. The proof is similar to that of the previous theorem using in the argument Theorem 3.6 instead of Theorem 3.4. If T and S are the set-valued mappings defined above one can see that now T is upper semicontinuous with compact admissible values. Moreover, S is closed (by (i)), its values are nonempty and admissible (by (ii)) and its cofibers are sub-admissible (in view of (iii)).

From Theorem 3.6, there exist $x_0 \in X$ and $y_0 \in P(x_0)$ such that $(x_0, y_0) \in \bigcap_{u \in X} S(u)$. This means that (x_0, y_0) satisfies the desired conclusion. $\square$

The conclusion of Theorem 4.2 is the same as that of Theorem 4.4 in [17] and of Theorem 2.7 in [19], but its assumptions are different. For instance, in the first aforementioned theorem, assumption (iv) is replaced with the following one: for each $(x, u) \in X \times X$, there exist $y \in P(x)$ and $z \in X$ such that $f(z, y, u) \geq 0$; then, in Theorem 2.7 in [19], the function $f(\cdot, \cdot, u)$ is assumed to be lower semicontinuous on $X \times Y$, for all $u \in X$.

Remark 4.3. *Some of the assumptions of Theorem 4.2 can be replaced by others, stronger but easier to check in concrete situations. Thus, it is clear that condition (i) is satisfied when f is upper semicontinuou on $X \times Y \times X$. Then, (iii) holds if for every $(x, y) \in X \times Y$ the function $f(x, y, \cdot)$ is metric quasiconvex on X.*

For a subset A of a metric space X and $r > 0$ we denote by $B(A, r)$ ($B^{\circ}(A, r)$, respectively) the union of all closed (open, respectively) balls of radius r centred in a point from A.

Lemma 4.4. *If A is an admissible subset of a metric space and $r > 0$, then*

(i) $B(A, r)$ is an admissible set;

(ii) $B^{\circ}(A, r)$ is a sub-admissible set.

Proof. We prove only statement (ii), because (i) is well-known (e.g., see [20, p.864]) Let $\{x_1, \dots x_n\}$ be a finite subset of $B^{\circ}(A, r)$. Hence for each index i there exists a point $y_i \in A$ such that $d(x_i, y_i) < r$. Set $r_0 = \max_{1\le i\le n} d(x_i, y_i)$. Then,

$$\mathrm{co}\,\{x_1, \dots x_n\} \subseteq B(\{y_1, \dots, y_n\}, r_0) \subseteq B(A, r_0) \subseteq B^{\circ}(A, r).$$

□

Let (X, d) be a metric space. For two closed subsets A and B of X, the Hausdorff distance between them will be denoted by $h(A, B)$. It is known that

$$\Big|d(x, A) - d(y, A)\Big| \le d(x, y) \text{ and}$$

$$\Big|d(x, A) - d(x, B)\Big| \le h(A, B),$$

for every closed sets A and B in X and each points $x, y \in X$.

A continuous (actually, only upper-semicontinuous it suffices) set-valued mapping $F : X \rightrightarrows X$ with nonempty, compact and admissible values, defined on a compact hypermetric convex space X has a fixed point (see [16, Theorem 2.2]). The theorem below offers some information about the position of the fixed point.

Theorem 4.5. *Let (X, d) be a compact hypermetric convex space and $T, F : X \rightrightarrows X$ be two set-valued mappings with nonempty values satisfying the following conditions:*

(i) T is upper semicontinuous with compact admissible values and F is continuous with admissible compact values;

(ii) for each $x \in X$, the set $\{y \in X : d\big(y, F(y)\big) \le d(x, F(y))\}$ is sub-admisible;

(iii) for each $x \in X$ there exists $y \in T(x)$ such that $d\big(y, F(y)\big) \le d\big(x, F(y)\big)$.

Then, there exists $x_0 \in X$ such that the restriction of the mapping F to $T(x_0)$ has a fixed point.

Proof. For $u \in X$, set

$$S(u) = \{y \in X : d\big(y, F(y)\big) \le d(u, F(y))\}.$$

We claim that S is a closed mapping, that is, its graph $\mathrm{Gr}\, S = \{(u, y) \in X \times X : d\big(y, F(y)\big) \le d(u, F(y))\}$ is a closed set This statement follows as soon as we prove that the functions $p : X \to \mathbb{R}$ and $q : X \times X \to \mathbb{R}$ defined by

$$p(y) = d\big(y, F(y)\big), \quad q(u, y) = d(u, F(y))$$

are continuous. As F is continuous and compact-valued, the continuity of p follows from Corollary 1.4.17 in [21].

Let $(u, y) \in X \times X$ and $\{(u_n, y_n)\}$ be a sequence converging to (u, y). Then

$$|q(u, y) - q(u_n, y_n)| = \Big|d(u, F(y)) - d(u_n, F(y_n)\Big| \le \Big|d(u, F(y)) - d(u_n, F(y))\Big|$$

$$+\Big|d(u_n, F(y)) - d(u_n, F(y_n)\Big| \le d(u, u_n) + H\big(F(y), F(y_n)\big).$$

Passing to the limit, we get $\lim_{n\to\infty} q(u_n, y_n) = q(u, y)$, hence the function q is continuous.

Summing up all of the above, we infer that S is a closed mapping. Moreover, its values are nonempty (since $u \in S(u)$) and admissible (by (ii)). For each $y \in X$,

$$S^*(y) = \{u \in X : d(u, F(y)) < d\big(y, F(y)\big)\} = B^\circ\Big(F(y), d\big(y, F(y)\big)\Big).$$

From Lemma 4.4 (ii), it follows that $S^*(y)$ is a sub-admissible set, hence S has sub-admissible cofibers.

We next show that the mapping S is weak *KKM* w.r.t. T. Let us assume that there exist a finite set $A \subseteq X$ and a point $x \in \operatorname{co} A$ such that $T(x) \cap S(A) = \emptyset$. Then for every $u \in A$ and each $y \in T(x)$, $y \notin S(u)$, that is, $d\big(u, F(y)\big) < d\big(y, F(y)\big)$. Consequently,

$$A \subseteq B^\circ(F(y), d(y, F(y)),$$

for every $y \in T(x)$. In view of Lemma 4.4, for any $y \in T(x)$,

$$x \in \operatorname{co} A \subseteq B^\circ(F(y); d(y, f(y)),$$

whence $d(x, F(y)) < d\big(y, F(y)\big)$, which contradicts assumption (iii).

From Theorem 3.6, there exist two points $x_0, y_0 \in X$ such that $y_0 \in T(x_0) \cap \bigcap_{u\in X} S(u)$. Choose a point $u_0 \in F(y_0)$. Then, from $y_0 \in S(u_0)$, we get $d\big(y_0, F(y_0)\big) \le d\big(u_0, F(y_0)\big) = 0$. Hence y_0 is fixed point of F situated in $T(x_0)$. □

Remark 4.6. *Theorem 4.5 is a hyperconvex version of Theorem 5.5 in [22].*

5. Minimax inequalities

Theorem 5.1. *Let X be a compact hypermetric convex space and Y be a topological space. Let $T : X \rightrightarrows Y$ be an upper semicontinuous set-valued mapping with nonempty values and $f, g : X \times Y \to \mathbb{R}$ be two functions that satisfy the following conditions:*

(i) $f(x, y) \le g(x, y)$ for all $(x, y) \in X \times T(X)$;

(ii) for each $x \in X$, the function $g(x, \cdot)$ is upper semicontinuous on Y;

(iii) for each $y \in Y$, (at least) one of the functions $f(\cdot, y)$, $g(\cdot, y)$ is metric quasiconvex.

Then, $\inf_{x\in X} \sup_{y\in T(x)} f(x, y) \le \sup_{x\in X} \inf_{u\in X} \sup_{y\in T(x)} g(u, y)$.

Proof. We may assume that $\inf_{x\in X} \sup_{y\in T(x)} f(x, y) > -\infty$. For $\lambda < \inf_{x\in X} \sup_{y\in T(x)} f(x, y)$ arbitrarily fixed, define the set-valued mapping $S : X \rightrightarrows Y$ by

$$S(u) = \{y \in Y : g(u, y) \ge \lambda\}.$$

By (ii), $S(u)$ is a closed set, for each $u \in X$. Since T is upper semicontinuous, according to Remark 2.4 condition (ii) in Theorem 3.2 is fulfilled. We show that S is weak *KKM* mapping w.r.t. T. Suppose to the contrary, that there exist a nonempty finite subset A of X and $x \in \operatorname{co} A$ such that $T(x) \cap S(A) = \emptyset$. Then, for each $y \in T(x)$ and any $u \in A$, $g(u, y) < \lambda$. By (i) and (iii), it follows that

$$f(x, y) \le \max\{g(u, y) : u \in A\} < \lambda,$$

for all $y \in T(x)$. Hence $\sup_{y\in T(x)} f(x, y) \le \lambda$; clearly this contradicts the assumption $\lambda < \inf_{x\in X} \sup_{y\in T(x)} f(x, y)$.

From Theorem 3.2, there exists a point $x_0 \in X$ such that $T(x_0) \cap S(u) \ne \emptyset$ for all $u \in X$. Thus, for each $u \in X$, $\sup_{y\in T(x_0)} g(u, y) \ge \lambda$, and thereby,

$$\lambda \le \inf_{u\in X} \sup_{y\in T(x_0)} g(u, y) \le \sup_{x\in X} \inf_{u\in X} \sup_{y\in T(x)} g(u, y).$$

As λ was an arbitrary real number less than $\inf_{x\in X} \sup_{y\in T(x)} f(x, y)$, we get the desired conclusion. □

Remark 5.2. *Theorem 5.1 generalizes Theorem 2.5 in [16] and Theorem 5.2 in [23]. To see this, it suffices to take in our theorem $X = Y$, $f = g$ and $T(x) \equiv Y$.*

Theorem 5.3. *Let X be a compact hypermetric convex space and Y be a compact topological space. Let $S : X \rightrightarrows Y$ be a set-valued mapping with nonempty closed values and $f, g : X \times Y \to \mathbb{R}$ be two functions that satisfy the following conditions:*

(i) $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in X \times Y$;

(ii) g *is upper semicontinuous on* $X \times Y$;

(iii) for each $x \in X$, *the function* $u \longrightarrow \sup_{y\in S(u)} f(x, y)$ *is metric quasiconvex on* X.

Then, there exists $x_0 \in X$ such that

$$\inf_{x\in X} \sup_{y\in S(x)} f(x, y) \leq \inf_{u\in X} \max_{y\in S(u)} g(x_0, y).$$

Proof. First, let us observe that since g is upper semicontinuous on $X \times Y$, then for each $x \in X$, $g(x, \cdot)$ is also an upper semicontinuous function of y on Y and therefore its maximum $\max_{y\in S(u)} g(x, y)$ on the compact set $S(u)$ exists. Let $s = \inf_{x\in X} \sup_{y\in S(x)} f(x, y)$. We intend to apply Theorem 3.2 when the set-valued mapping $T : X \rightrightarrows Y$ is defined as follows:

$$T(x) = \{y \in Y : g(x, y) \geq s\}.$$

By (ii), the graph of T is closed, and since Y is compact, T is upper semicontinuous. By Remark 2.4, it follows that condition (ii) in Theorem 3.2 is satisfied.

We claim that S is a weak KKM mapping w.r.t. T. By way of contradiction, suppose that there exist a finite set $A \subseteq X$ and $x \in \operatorname{co} A$ such that $T(x) \cap S(u) = \emptyset$ for each $u \in A$. Then, for each $u \in A$ and $y \in S(u)$, $g(x, y) < s$, whence $\max_{y\in S(u)} g(x, y) < s$. By (i), it follows that

$$\sup_{y\in S(u)} f(x, y) \leq \max_{y\in S(u)} g(x, y) < s$$

for all $u \in A$. Since the function $u \longrightarrow \sup_{y\in S(u)} f(x, y)$ is metric quasiconvex, we infer that $\sup_{y\in S(x)} f(x, y) < s$; a contradiction.

From Theorem 3.2, there exists a point $x_0 \in X$ such that $T(x_0) \cap S(u) \neq \emptyset$ for all $u \in X$. Then, for every $u \in X$ there exists $y_u \in S(u)$ such that $g(x_0, y_u) \geq s$, hence $\max_{y\in S(u)} g(x_0, y) \geq s$. It follows that $\inf_{u\in X} \max_{y\in S(u)} g(x_0, y) \geq s$. □

Let us now take a look over the hypotheses of Theorems 5.3. Observe that the first two assumptions are standard. The next proposition give conditions under which assumption (iii) holds.

Proposition 5.4. *Let x be an arbitrary point in X. The function $u \longrightarrow \sup_{y\in S(u)} f(x, y)$ is metric quasiconvex in any of the following two situations:*

(i) the cofibers of S are sub-admissible sets;

(ii) the following conditions are fulfilled:

(ii_1) for each nonempty finite subset A of X, $S(coA) \subset \bigcup\{ coB : B \in \langle S(A)\rangle\}$ (here $\langle S(A)\rangle$ denotes the family of all nonempty finite subsets of $S(A)$);

(ii_2) the function $f(x, \cdot)$ is metric quasiconvex.

Proof. Consider the function $h : X \to \mathbb{R}$ defined by

$$h(u) = \sup_{y\in S(u)} f(x, y).$$

Take arbitrarily a nonempty finite subset $A = \{u_1, \dots, u_n\}$ of X and an element u of $\operatorname{co}A$. We have to prove that $h(u) \le \max_{1\le i\le n} h(u_i)$.

Assume first that S has sub-admissible cofibers. We claim that in this case

$$S(\operatorname{co}A) = S(A).$$

As $A \subseteq \operatorname{co}A$, the inclusion $S(A) \subseteq S(\operatorname{co}A)$ is obvious. To prove the inverse inclusion, suppose on the contrary that there exists $y \in S(\operatorname{co}A) \setminus S(A)$. From $y \in S(\operatorname{co}A\}$ we get $y \in S(u')$ for some $u' \in \operatorname{co} A$. From $y \notin S(A)$, we infer that $A \subseteq S^*(y)$. Since $S^*(y)$ is a sub-admissible set, we infer that

$$u' \in \operatorname{co} A \subseteq S^*(y),$$

whence $y \notin S(u')$; a contradiction.

Based on the equality proved above we obtain

$$h(u) = \sup_{y\in S(u)} f(x,y) \le \sup_{y\in S(\operatorname{co}A)} f(x,y)$$

$$= \sup_{y\in\bigcup_{i=1}^{n} S(u_i)} f(x,y) = \max_{1\le i\le n} \sup_{y\in S(u_i)} f(x,y) = \max_{1\le i\le n} h(u_i).$$

Assume now that (ii) holds. If $y \in S(u)$, there exists a finite nonempty subset B of $S(A)$ such that $y \in \operatorname{co}B$. Since the function f is metric quasiconvex in the second variable,

$$f(x,y) \le \max_{y'\in B} f(x,y) \le \sup_{y'\in S(A)} f(x,y') = \max_{1\le i\le n} \sup_{y'\in S(u_i)} f(x,y') = \max_{1\le i\le n} h(u_i).$$

Consequently, $h(u) = \sup_{y\in S(u)} f(x,y) \le \max_{1\le i\le n} h(u_i)$. □

Theorem 5.5. *Let X and Y be two hyperconvex metric spaces such that X is compact. Let $T : X \rightrightarrows Y$ be an upper semicontinuous set-valued mapping with nonempty compact and admissible values and $f : X \times Y \to \mathbb{R}$ be a real function. Assume that:*

(i) f is upper semicontinuous on $X \times Y$;

(ii) for each $y \in Y$, the function $f(\cdot, y)$ is metric quasiconvex;

(iii) for each $x \in X$, the function $f(x, \cdot)$ is metric quasiconcave.

Then, $\inf_{x\in X} \max_{y\in T(x)} f(x,y) \le \max_{y\in T(X)} \inf_{x\in X} f(x,y)$.

Proof. From [24, Lemma 17.8], $T(X)$ is a compact subset of Y and by (i) and [24, Lemma 2.4] the function $y \longrightarrow \inf_{x\in X} f(x,y)$ is upper semicontinuous. Consequently, $\max_{y\in T(X)} \inf_{x\in X} f(x,y)$ exists.

As in the proof of Theorem 5.1, for $\lambda < \inf_{x\in X} \max_{y\in T(x)} f(x,y)$ arbitrarily fixed we consider the set-valued mapping $S : X \rightrightarrows Y$ defined by $S(u) = \{y \in Y : f(u,y) \ge \lambda\}$. We show that the assumptions of Theorem 3.6 are satisfied. As f is upper semicontinuous on $X\times Y$, it follows that S is a closed mapping. Since the function f is metric quasiconcave in the second variable, the values of S are sub-admissible sets. As any compact sub-admissible set is admissible, S has admissible values. Since the function f is metric quasiconvex in the first variable, for each $y \in Y$ the set $\{u \in X : f(u,y) < \lambda\}$ is sub-admissible. This means that S has sub-admissible cofibers. As we have already seen in the proof of Theorem 5.1, under assumption (ii), S is weak KKM mapping w.r.t. T.

From Theorem 3.6 there is $x_0 \in X$ such that $T(x_0) \cap \bigcap_{u\in X} S(u) \ne \emptyset$. If we pick a point y_0 from this intersection, then $y_0 \in T(x_0)$ and $\inf_{u\in X} f(u, y_0) \ge \lambda$. Thus,

$$\max_{y\in T(X)} \inf_{u\in X} f(u,y) \ge \inf_{u\in X} f(u,y_0) \ge \lambda,$$

whence we get the desired conclusion. □

Remark 5.6. *When $T(x) \equiv Y$,Theorem 5.5 reduces to Corollary 3.6 in [25].*

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