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Coefficients Estimate for a Subclass of Holomorphic Mappings on the Unit Polydisk in $\mathbb{C}^n$

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Abstract. The aim of this paper is to obtain the sharp solutions of Fekete-Szegö problems of high dimensional version for family of holomorphic mappings that are normalized on the unit polydisk $\mathbb{U}^n$ in $\mathbb{C}^n$. The main results unify some recent works, which are closely related to the starlike mappings. Moreover, some previous results are improved.

1. Introduction

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, z_2, \cdots, z_n)$ with the maximum norm $\|z\| = \max\{|z_1|, |z_2|, ..., |z_n|\}$. Also, let $\mathbb{U}^n$ be the unit polydisc in $\mathbb{C}^n$ and let $\mathbb{U}^1 = \mathbb{U}$ be the unit disc. Let $\partial_0\mathbb{U}^n = \prod_{k=1}^{n} \partial\mathbb{U}$ be the distinguished boundary of $\mathbb{U}^n$, and $\partial\mathbb{U}^n$ be the boundary of $\mathbb{U}^n$. We denote by $\mathcal{H}(\mathbb{U}^n)$ the family of holomorphic mappings from $\mathbb{U}^n$ into $\mathbb{C}^n$ with the standard topology of locally uniform convergence. Let $f \in \mathcal{H}(\mathbb{U}^n)$, we say that f is normalized if $f(0) = 0$ and $J_f(0) = I_n$, where $J_f(0)$ is the complex Jacobian matrix of f at the point 0 and I_n is the identity matrix.

Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded circular domain. The m $(m > 2)$-Fréchet derivative of a mapping $f \in \mathcal{H}(\Omega)$ at point $z \in \Omega$ is written as $D^m f(z)(a^{m-1}, \cdot)$. The matrix representation is (see, e.g. Liu-Xu [13])

$$D^m f(z)(a^{m-1}, \cdot) = \left(\sum_{l_1, l_2, ..., l_{m-1}=1}^{n} \frac{\partial^m f_p(z)}{\partial z_k \partial z_{l_1} ... \partial z_{l_{m-1}}} a_{l_1} ... a_{l_{m-1}} \right)_{1 \leqslant p,k \leqslant n},$$

where $f(z) = (f_1(z), f_2(z), ..., f_n(z))'$, $a = (a_1, a_2, ..., a_n)' \in \mathbb{C}^n$.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f(z) \prec g(z)$, provided there exists a analytic function $w(z)$ defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$.

Suppose that ψ is a convex Carathéodory function on the unit disk $\mathbb{U}$ such that $\psi(0) = 1$, $\psi'(0) > 0$, $\Re\big(\psi(\xi)\big) > 0$ and $\psi(\mathbb{U})$ is symmetric with respect to the real axis. Also, $\psi(\xi)$ has a series expansion of the form

$$\psi(\xi) = 1 + A_1\xi + A_2\xi^2 + A_3\xi^3 + ..., (A_1 > 0), \ \xi \in \mathbb{U}. \tag{1}$$

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Definition 1.1. *Let $f : \mathbb{U}^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping. If $f(0) = 0$ and $0 \leqslant \gamma < 1$, then*

$$f \in \mathcal{S}^*_{\psi,\gamma}(\mathbb{U}^n) \Longleftrightarrow \frac{1}{1-\gamma}\frac{z_j}{g_j(z)} - \frac{\gamma}{1-\gamma} \in \psi(\mathbb{U}),\ z \in \mathbb{U}^n\backslash\{0\}, \tag{2}$$

where $g(z) = (g_1(z), g_2(z), ..., g_n(z))' = (Df(z))^{-1}f(z)$, $|z_j| = \|z\| = \max\limits_{1\leqslant k\leqslant n}\{|z_k|\}$, $T_x \in T(x)$ and the function ψ is defined by (1).

Remark 1.2. *(I) If $\gamma = 0$, then the definition $\mathcal{S}^*_{\psi,0}(\mathbb{U}^n)$ due to Xu-Liu-Liu [31].*
(II) If $\gamma = 0$, $n = 1$ and $\psi(\xi) = \frac{1+\xi}{1-\xi}$, then the class $\mathcal{S}^*_{\frac{1+\xi}{1-\xi},0}(\mathbb{U})$ was the usual starlike function.
(III) Let $\alpha \in [0, 1)$, $c \in (0, 1)$, $n \in \mathbb{Z}^+$, $\xi \in \mathbb{U}$. Define the functions set by

$$\mathcal{M} = \left\{\frac{1+(1-2\alpha)\xi}{1-\xi}, \frac{1+\xi}{1+(2\alpha-1)\xi}, \Big(\frac{1+\xi}{1-\xi}\Big)^{\alpha}, \frac{1+c\xi}{1-c\xi}\right\},$$

then for different functions $\psi \in \mathcal{M}$ in Definition 1.1, we can get kinds of well-known subclasses of starlike mappings in $\mathbb{U}^n$ (see, e.g. [2, 12, 13, 29, 30]).

As well known, the coefficient functional $\rho_\mu(f) = a_3 - \mu a_2^2$ on the normalized analytic functions f plays an important role in one-dimensional function theory, where $f(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \cdots$, $\xi \in \mathbb{U}$. For details, we refer the reader to survey articles of Kanas [10] and Srivastava et al.[18] (also see, e.g. [4, 9, 15, 17, 19–23, 26, 27]). The problem of maximizing the absolute value of the functional $\rho_\mu(f)$ is called the Fekete-Szegö problem, which is related to the Bieberbach conjecture (see [1]). However, Cartan [3] stated that the Bieberbach conjecture does not hold in several complex variables. Until now, only a few complete results are known for the inequalities of homogeneous expansions for subclasses of biholomorphic mappings in $\mathbb{C}^n$ (see, e.g. Graham-Hamada-Kohr [5], Graham-Hamada-Honda-Shon [6], Hamada-Honda-Kohr [8], Kohr [11], Liu-Liu [14], Gong [7]). In 2014, Xu-Liu [30] extended the Fekete-Szegö inequality from the case of one dimension to higher dimensions for a subclass of starlike mappings defined on the unit ball in a complex Banach space or on the unit polydisk in $\mathbb{C}^n$. Furthermore, Luo-Xu [12] and Xu-Fang-Liu [29] consider the results related to strongly starlike mappings of order α and starlike mappings of order α ($0 \leqslant \alpha < 1$), respectively. Recently, Liu-Xu [13] established inequalities between the second and the third coefficients of homogeneous expansions for starlike mappings and starlike mappings of order α defined on bounded starlike circular domains in $\mathbb{C}^n$, respectively. Some more general works on coefficients inequalities in several complex variables can be found in Tu-Xiong [25] and Xu-Liu-Liu [31].

In this paper, we will obtain the sharp coefficients bounds on Fekete-Szegö problem for the class $\mathcal{S}^*_{\psi,\gamma}(\mathbb{U}^n)$. This is a continuation of the works in [25] and [31]. Our results extend some works that are related starlike mappings in $\mathbb{C}^n$, and give a positive answer to a conjecture proposed by Tu-Xiong [25]. Compare with the recent works on Fekete-Szegö problem(e.g., [12], [31]), the critical processes of proofs are different: our arguments in this paper are heavily based on the subordination techniques.

Throughout the paper, it is assumed that

$$\mathfrak{M}_1 = \frac{1}{2}\Big[\frac{1}{(1-\gamma)^2}\frac{A_2 - A_1}{A_1^2} + 1\Big],\ \mathfrak{M}_2 = \frac{1}{2}\Big[\frac{1}{(1-\gamma)^2}\frac{A_2 + A_1}{A_1^2} + 1\Big],$$

and $\psi(\xi) = 1 + A_1\xi + A_2\xi^2 + \cdots + A_n\xi^n + \cdots (A_1 > 0)$ is the function defined as (1).

2. Preliminaries

The following Lemma is needed in the proof of main theorems.

Lemma 2.1 ([16]). *Let $\mathcal{P}$ be the usual class of functions with positive real part in $\mathbb{U}$. Suppose that $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$, then $|c_n| \le 2$ for $n \ge 1$. If $|c_1| = 2$ then $p(z) \equiv p_1(z) = \frac{1+\gamma_1 z}{1-\gamma_1 z}$ with $\gamma_1 = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore we have*

$$|c_2 - \frac{c_1^2}{2}| \leqslant 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + z\frac{\gamma_2 z+\gamma_1}{1+\overline{\gamma_1}\gamma_2 z}}{1 - z\frac{\gamma_2 z+\gamma_1}{1+\overline{\gamma_1}\gamma_2 z}}$$

and $\gamma_1 = \frac{c_1}{2}, \gamma_2 = \frac{2c_2-c_1^2}{4-|c_1|^2}$. Conversely if $p(z) = p_2(z)$ for some $|\gamma_1| < 1$ and $|\gamma_2| = 1$, then $\gamma_1 = \frac{c_1}{2}, \gamma_2 = \frac{2c_2-c_1^2}{4-|c_1|^2}$ and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$.

3. Main results

In this section, we obtain the sharp unified solutions on Fekete-Szegö problem for $\mathcal{S}^*_{\psi,\gamma}(\mathbb{U}^n)$ with the the parameters $\mu \in \Re$ (also, parameters $\mu \in \mathbb{C}$).

Theorem 3.1. *Let $f \in \mathcal{S}^*_{\psi,\gamma}(\mathbb{U}^n)$. Suppose that $|z_k| = \|z\| = \max_{1\leqslant j\leqslant n}\{|z_j|\}$, $z_0 = \frac{z}{\|z\|}$, $z \in \mathbb{U}^n\backslash\{0\}$ and*

$$\frac{1}{2}D^2 f_k(0)\Big(z_0, \frac{D^2 f(0)(z_0^2)}{2!}\Big)\frac{z_k}{\|z\|} = \Big(\frac{D^2 f_k(0)(z_0^2)}{2!}\Big)^2, \tag{3}$$

then we have

$$\Big\|\frac{D^3 f(0)(z^3)}{3!} - \mu\frac{1}{2}D^2 f(0)\Big(z, \frac{D^2 f(0)(z^2)}{2!}\Big)\Big\| \leqslant \begin{cases} \frac{1}{2}A_1^2\|z\|^3\Big[\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1 - 2\mu + \frac{\gamma^2-2\gamma}{(1-\gamma)^2}\frac{1}{A_1}\Big], & \mu \leqslant \mathfrak{M}_1, \\ \frac{1}{2}A_1\|z\|^3, & \mathfrak{M}_1 \leqslant \mu \leqslant \mathfrak{M}_2, \\ \frac{1}{2}A_1^2\|z\|^3\Big[2\mu - \frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} - 1 + \frac{1}{A_1}\frac{\gamma^2-2\gamma}{(1-\gamma)^2}\Big], & \mu \geqslant \mathfrak{M}_2. \end{cases}$$

The above estimates are sharp for each real μ.

Proof. Fix $z \in \mathbb{U}^n \setminus \{0\}$, and set $z_0 = \frac{z}{\|z\|}$. We define a function $q_j : \mathbb{U} \to \mathbb{C}$ by

$$q_j(\xi) = \begin{cases} \frac{1}{1-\gamma}\frac{\xi z_j}{p_j(\xi z_0)\|z\|} - \frac{\gamma}{1-\gamma}, & \xi \neq 0, \\ 1, & \xi = 0, \end{cases} \tag{4}$$

where $p(z) = (Df(z))^{-1}f(z)$ and $|z_j| = \|z\| = \max_{1\leqslant k\leqslant n}\{|z_k|\}$. It is easy to see that $q_j(\xi) \in H(\mathbb{U})$. Since $f \in \mathcal{S}^*_{\psi,\gamma}(\mathbb{U}^n)$, using (4), then we have $q_j(\xi) \in \psi(\mathbb{U})$, $\xi \in \mathbb{U}$. Furthermore, the fact $q_j(0) = \psi(0) = 1$ implies that $q_j(\xi) \prec \psi(\xi)$, $\xi \in \mathbb{U}$. Taking a function

$$\mathcal{P}(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots \prec \frac{1+z}{1-z},\ z \in \mathbb{U}, \tag{5}$$

we note $\mathcal{P}(0) = 1$ and $\mathcal{P}$ is a function with positive real part. By (5), there is a function $w(z)$, such that

$$q_j(\xi) = \psi(w(\xi)) = 1 + \frac{1}{2}A_1 c_1 \xi + \Big(\frac{1}{2}A_1\Big(c_2 - \frac{c_1^2}{2}\Big) + \frac{1}{4}A_2 c_1^2\Big)\xi^2 + \ldots, \xi \in \mathbb{U}. \tag{6}$$

From (4) and (6), we know that

$$\Big[1+\frac{\gamma}{1-\gamma}+\frac{1}{2}A_1c_1\xi+\Big(\frac{1}{2}A_1\Big(c_2-\frac{c_1^2}{2}\Big)+\frac{1}{4}A_2c_1^2\Big)\xi^2+\cdots\Big]\times$$
$$(1-\gamma)\Big(\xi+\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j}\xi^2+\frac{D^3p_j(0)(z_0^3)\|z\|}{3!z_j}\xi^3+\cdots\Big)=\xi. \tag{7}$$

Comparing with the coefficient of two sides of the (7) in ξ^2 and ξ^3, we get

$$\frac{1}{2}A_1c_1=-\frac{1}{1-\gamma}\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j} \tag{8}$$

and

$$\frac{1}{2}A_1\Big(c_2-\frac{c_1^2}{2}\Big)+\frac{1}{4}A_2c_1^2=\Big(\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2-\frac{D^3p_j(0)(z_0^3)\|z\|}{3!z_j}. \tag{9}$$

Using the Lemma 2.1, (8) and (9), then

$$|c_2-\frac{1}{2}c_1^2|=\Big|\frac{2}{A_1}\Big(\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2-\frac{2}{A_1}\frac{D^3p_j(0)(z_0^3)\|z\|}{3!z_j}$$
$$-2\frac{A_2}{A_1^3}\frac{1}{(1-\gamma)^2}\Big|\Big(\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2\Big|\leqslant 2-\frac{1}{2}\frac{1}{(1-\gamma)^2}\Big|\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j}\Big|^2\frac{4}{A_1^2}. \tag{10}$$

From (10), it shows that

$$\Big|\Big(\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2-\frac{D^3p_j(0)(z_0^3)\|z\|}{3!z_j}-\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}\Big(\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2\Big|$$
$$\leqslant A_1-\frac{1}{A_1}\frac{1}{(1-\gamma)^2}\Big|\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j}\Big|^2. \tag{11}$$

On the other hand, since $p(z)=(Df(z))^{-1}f(z)$, then

$$z+\frac{D^2f(0)(z^2)}{2!}+\frac{D^3f(0)(z^3)}{3!}+\cdots$$
$$=\Big(I+D^2f(0)(z,.)+\frac{D^3f(0)(z^2,.)}{2!}+\cdots\Big)\times\Big(Dg(0)z+\frac{D^2g(0)(z^2)}{2!}+\frac{D^3g(0)(z^3)}{3!}+\cdots\Big). \tag{12}$$

Comparing with the homogeneous expansion of two sides of the (12), we have

$$Dp(0)z=z,\ \ \frac{D^2p(0)(z^2)}{2!}=-\frac{D^2f(0)(z^2)}{2!} \tag{13}$$

and

$$\frac{D^3f(0)(z^3)}{3!}=\frac{D^3p(0)(z^3)}{3!}+\frac{D^3f(0)(z^3)}{2!}-D^2f(0)\Big(z,\frac{D^2f(0)(z^2)}{2!}\Big). \tag{14}$$

In view of (14), we can obtain

$$F=\Big|\frac{D^3f_j(0)(z_0^3)\|z\|}{3!z_j}-\mu\frac{1}{2}D^2f_j(0)(z_0,\frac{D^2f(0)(z_0^2)}{2!})\frac{\|z\|}{z_j}\Big|$$
$$=\Big|-\frac{1}{2}\frac{D^3p_j(0)(z_0^3)\|z\|}{3!z_j}+\frac{1}{2}D^2f_j(0)(z_0,\frac{D^2f(0)(z_0^2)}{2!})\frac{\|z\|}{z_j}-\mu\Big(\frac{D^2f_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2\Big|. \tag{15}$$

Furthermore, using (3), (13) and (14) in (15), then we have

$$F = \frac{1}{2}\Big| - \frac{D^3 p_j(0)(z_0^3)\|z\|}{3!z_j} + (2-2\mu)\Big(\frac{D^2 f_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2\Big|$$
$$= \frac{1}{2}\Big| - \frac{D^3 p_j(0)(z_0^3)\|z\|}{3!z_j} + \Big(\frac{D^2 f_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2 - \frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}\cdot$$
$$\Big(\frac{D^2 f_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2 + \Big(\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1 - 2\mu\Big)\Big(\frac{D^2 f_j(0)(z_0^2)\|z\|}{2!z_j}\Big)^2\Big|$$
$$\leqslant \frac{1}{2}\Big(A_1 - \frac{1}{A_1}\frac{1}{(1-\gamma)^2}\Big|\frac{D^2 p_j(0)(z_0^2)\|z\|}{2!z_j}\Big|^2 + \Big|\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1 - 2\mu\Big|\Big|\frac{D^2 p_j(0)(z_0^2)\|z\|}{2!z_j}\Big|^2\Big). \tag{16}$$

According to the above inequality (16), we consider the following four cases with using the Lemma 6 in Xu-Liu [28]:

Case 1: If μ satisfies the condition

$$\mu \leqslant \frac{1}{2}\Big[\frac{1}{(1-\gamma)^2}\frac{A_2 - A_1}{A_1^2} + 1\Big],$$

then we have

$$\Big|\frac{D^3 f_j(0)(z_0^3)\|z\|}{3!z_j} - \mu\frac{1}{2}D^2 f_j(0)\Big(z_0, \frac{D^2 f(0)(z_0^2)}{2!}\Big)\frac{\|z\|}{z_j}\Big|$$
$$\leqslant \frac{1}{2}\Big(A_1 + \Big(\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1 - 2\mu - \frac{1}{A_1}\frac{1}{(1-\gamma)^2}\Big)\Big|\frac{D^2 p_j(0)(z_0^2)\|z\|}{2!z_j}\Big|^2\Big)$$
$$\leqslant \frac{1}{2}\Big(A_1 + A_1^2\Big(\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1 - 2\mu - \frac{1}{A_1}\frac{1}{(1-\gamma)^2}\Big)\Big)$$
$$= \frac{1}{2}A_1^2\Big[\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1 - 2\mu + \frac{\gamma^2 - 2\gamma}{(1-\gamma)^2}\frac{1}{A_1}\Big]. \tag{17}$$

Case 2: If μ satisfies the condition

$$\frac{1}{2}\Big[\frac{1}{(1-\gamma)^2}\frac{A_2 - A_1}{A_1^2} + 1\Big] \leqslant \mu \leqslant \frac{1}{2}\Big(\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1\Big),$$

then we have

$$\Big|\frac{D^3 f_j(0)(z_0^3)\|z\|}{3!z_j} - \mu\frac{1}{2}D^2 f_j(0)\Big(z_0, \frac{D^2 f(0)(z_0^2)}{2!}\Big)\frac{\|z\|}{z_j}\Big|$$
$$\leqslant \frac{1}{2}\Big(A_1 + \Big(\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1 - 2\mu - \frac{1}{A_1}\frac{1}{(1-\gamma)^2}\Big)\Big|\frac{D^2 p_j(0)(z_0^2)\|z\|}{2!z_j}\Big|^2\Big) \leqslant \frac{1}{2}A_1. \tag{18}$$

Case 3: If μ satisfies the condition

$$\frac{1}{2}\Big(\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1\Big) \leqslant \mu \leqslant \frac{1}{2}\Big[\frac{1}{(1-\gamma)^2}\frac{A_2 + A_1}{A_1^2} + 1\Big],$$

then we have

$$\Big|\frac{D^3 f_j(0)(z_0^3)\|z\|}{3!z_j} - \mu\frac{1}{2}D^2 f_j(0)\Big(z_0, \frac{D^2 f(0)(z_0^2)}{2!}\Big)\frac{\|z\|}{z_j}\Big|$$
$$\leqslant \frac{1}{2}\Big(A_1 + \Big(2\mu - \frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} - 1 - \frac{1}{A_1}\frac{1}{(1-\gamma)^2}\Big)\Big|\frac{D^2 p_j(0)(z_0^2)\|z\|}{2!z_j}\Big|^2\Big) \leqslant \frac{1}{2}A_1. \tag{19}$$

Case 4: If μ satisfies the condition

$$\mu \geqslant \frac{1}{2}\Big[\frac{1}{(1-\gamma)^2}\frac{A_2+A_1}{A_1^2}+1\Big],$$

then we have

$$\begin{aligned}&\Big|\frac{D^3f_j(0)(z_0^3)\|z\|}{3!z_j}-\mu\frac{1}{2}D^2f_j(0)(z_0,\frac{D^2f(0)(z_0^2)}{2!})\frac{\|z\|}{z_j}\Big|\\&\leqslant\frac{1}{2}\Big(A_1+\Big(2\mu-\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}-1-\frac{1}{A_1}\frac{1}{(1-\gamma)^2}\Big)\Big|\frac{D^2p_j(0)(z_0^2)\|z\|}{2!z_j}\Big|^2\Big)\\&\leqslant\frac{1}{2}\Big(A_1+A_1^2\Big(2\mu-\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}-1-\frac{1}{A_1}\frac{1}{(1-\gamma)^2}\Big)\Big)\\&=\frac{1}{2}A_1^2\Big[2\mu-\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}-1+\frac{\gamma^2-2\gamma}{(1-\gamma)^2}\frac{1}{A_1}\Big].\end{aligned}\tag{20}$$

From (17)-(20), then we have

$$\begin{aligned}&\Big\|\frac{D^3f_j(0)(z_0^3)\|z\|}{3!z_j}-\mu\frac{1}{2}D^2f_j(0)\Big(z_0,\frac{D^2f(0)(z_0^2)}{2!}\frac{\|z\|}{z_j}\Big)\Big\|\\&\leqslant\begin{cases}\frac{1}{2}A_1^2\Big[\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}+1-2\mu+\frac{\gamma^2-2\gamma}{(1-\gamma)^2}\frac{1}{A_1}\Big], & \mu\leqslant\mathfrak{M}_1,\\ \frac{1}{2}A_1, & \mathfrak{M}_1\leqslant\mu\leqslant\mathfrak{M}_2,\\ \frac{1}{2}A_1^2\Big[2\mu-\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}-1+\frac{1}{A_1}\frac{\gamma^2-2\gamma}{(1-\gamma)^2}\Big], & \mu\geqslant\mathfrak{M}_2.\end{cases}\end{aligned}\tag{21}$$

Thus, if $z_0\in\partial_0\mathbb{U}^n$, then for $j=1,2,...,n$, (21) implies that

$$\begin{aligned}&\Big\|\frac{D^3f_j(0)(z_0^3)}{3!}-\mu\frac{1}{2}D^2f_j(0)\Big(z_0,\frac{D^2f(0)(z_0^2)}{2!}\Big)\Big\|\\&\leqslant\begin{cases}\frac{1}{2}A_1^2\Big[\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}+1-2\mu+\frac{\gamma^2-2\gamma}{(1-\gamma)^2}\frac{1}{A_1}\Big], & \mu\leqslant\mathfrak{M}_1,\\ \frac{1}{2}A_1, & \mathfrak{M}_1\leqslant\mu\leqslant\mathfrak{M}_2,\\ \frac{1}{2}A_1^2\Big[2\mu-\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}-1+\frac{1}{A_1}\frac{\gamma^2-2\gamma}{(1-\gamma)^2}\Big], & \mu\geqslant\mathfrak{M}_2.\end{cases}\end{aligned}\tag{22}$$

Also since $\frac{D^3f_j(0)(z^3)}{3!}-\mu\frac{1}{2}D^2f_j(0)\Big(z,\frac{D^2f(0)(z^2)}{2!}\Big)$ are holomorphic functions on $\overline{\mathbb{U}^n}$, in view of the maximum modulus theorem of holomorphic functions on $\mathbb{U}^n$, we get

$$\begin{aligned}&\Big\|\frac{D^3f(0)(z^3)}{3!}-\mu\frac{1}{2}D^2f(0)\Big(z,\frac{D^2f(0)(z^2)}{2!}\Big)\Big\|\\&\leqslant\begin{cases}\frac{1}{2}A_1^2\|z\|^3\Big[\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}+1-2\mu+\frac{\gamma^2-2\gamma}{(1-\gamma)^2}\frac{1}{A_1}\Big], & \mu\leqslant\mathfrak{M}_1,\\ \frac{1}{2}A_1\|z\|^3, & \mathfrak{M}_1\leqslant\mu\leqslant\mathfrak{M}_2,\\ \frac{1}{2}A_1^2\|z\|^3\Big[2\mu-\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}-1+\frac{1}{A_1}\frac{\gamma^2-2\gamma}{(1-\gamma)^2}\Big], & \mu\geqslant\mathfrak{M}_2,\end{cases}\end{aligned}\tag{23}$$

where $z\in\mathbb{U}^n$.

In order to prove that the sharpness, we need to consider the following mappings.

If $\Big|\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2}+1-2\mu\Big|\geqslant\frac{1}{A_1}\frac{1}{(1-\gamma)^2}$, then

$$f(z)=z\exp\int_0^{z_1}(\psi(t)-1)\frac{1}{t}dt, z\in\mathbb{U}^n.\tag{24}$$

It is not difficult to verify that $f \in \mathcal{S}^*_{\psi}(\mathbb{U}^n)$. Taking $z = (r, 0, 0, ..., 0)', 0 < r < 1$, then the first and third equalities in (23) hold true.

If $\left|\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1 - 2\mu\right| \leqslant \frac{1}{A_1}\frac{1}{(1-\gamma)^2}$, then

$$f(z) = z\exp\int_0^{z_1}(\psi(t^2) - 1)\frac{1}{t}dt, z \in \mathbb{U}^n. \tag{25}$$

Also, it is not difficult to verify that $f \in \mathcal{S}^*_{\psi}(\mathbb{U}^n)$. Taking $z = (r, 0, 0, ..., 0)', 0 < r < 1$, then the second equalities in (23) hold true. This completes the proof of Theorem 3.1. □

We can obtain an interesting result for a subclass of $\mathcal{S}^*_{\psi,\gamma}(\mathbb{U}^n)$ by dropping off the condition (3).

Theorem 3.2. *Suppose that $f : \mathbb{U}^n \to \mathbb{C}$, $F(z) = zf(z) \in \mathcal{S}^*_{\psi,\gamma}(\mathbb{U}^n)$, then for $z \in \mathbb{U}^n$, we have*

$$\begin{aligned}&\left\|\frac{D^3F(0)(z^3)}{3!} - \mu\frac{1}{2}D^2F(0)\Big(z, \frac{D^2F(0)(z^2)}{2!}\Big)\right\| \\ &\leqslant \begin{cases}\frac{1}{2}A_1^2\|z\|^3\Big[\frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} + 1 - 2\mu + \frac{\gamma^2-2\gamma}{(1-\gamma)^2}\frac{1}{A_1}\Big], & \mu \leqslant \mathfrak{M}_1,\\ \frac{1}{2}A_1\|z\|^3, & \mathfrak{M}_1 \leqslant \mu \leqslant \mathfrak{M}_2,\\ \frac{1}{2}A_1^2\|z\|^3\Big[2\mu - \frac{A_2}{A_1^2}\frac{1}{(1-\gamma)^2} - 1 + \frac{1}{A_1}\frac{\gamma^2-2\gamma}{(1-\gamma)^2}\Big], & \mu \geqslant \mathfrak{M}_2.\end{cases}\end{aligned} \tag{26}$$

The above estimates are sharp for each real μ.

Proof. We define the function $q_j(\xi)$ as (4). Since $F(z) = zf(z) \in \mathcal{S}^*_{\psi,\gamma}(\mathbb{U}^n)$, then we can deduce that

$$q_j(\zeta)(1-\gamma)f(\zeta z_0) = (1-\gamma)f(\zeta z_0) + Df(\zeta z_0)\zeta z_0. \tag{27}$$

Considering the Taylor series expansions with ζ in (27), then

$$\begin{aligned}&\left(1 + \frac{1}{2}A_1c_1\zeta + \Big(\frac{1}{2}A_1\Big(c_2 - \frac{c_1^2}{2}\Big) + \frac{1}{4}A_2c_1^2\Big)\zeta^2 + ...\right)\cdot(1-\gamma)\mathfrak{P}\\ &= (1-\gamma)\mathfrak{P} + \left(Df(0)(z_0)\zeta + D^2f(0)(z_0^2)\zeta^2 + ...\right),\end{aligned} \tag{28}$$

where

$$\mathfrak{P} = 1 + Df(0)(z_0)\zeta + \frac{D^2f(0)(z_0^2)}{2}\zeta^2 + \cdots.$$

Comparing the homogeneous expansions of two sides in (28), we have

$$\frac{1}{2}A_1c_1(1-\gamma) = Df(0)(z_0) \tag{29}$$

and

$$\frac{1}{2}A_1\Big(c_2 - \frac{c_1^2}{2}\Big) + \frac{1}{4}A_2c_1^2 = \frac{1}{1-\gamma}[D^2f(0)(z_0^2) - (Df(0)(z_0))^2]. \tag{30}$$

On the other hand, from $F(z) = zf(z)$, we note that

$$\frac{D^3F_j(0)(z_0^3)}{3!} = \frac{D^2f(0)(z_0^2)}{2!}\frac{z_j}{\|z\|}, \quad \frac{D^2F_j(0)(z_0^2)}{2!} = Df(0)(z_0)\frac{z_j}{\|z\|}. \tag{31}$$

Thus, together with Theorem 3.1, (29),(30) and (31), it shows that

$$\begin{aligned}
&\Big|\frac{D^3F_j(0)(z_0^3)\|z\|}{3!z_j} - \mu\frac{1}{2}D^2F_j(0)\Big(z_0, \frac{D^2F(0)(z_0^2)}{2!}\Big)\frac{\|z\|}{z_j}\Big|\\
&= \Big|\frac{D^2f(0)(z_0^2)}{2!} - \mu\frac{1}{2}D^2F_j(0)\Big(z_0, Df(0)(z_0)\Big)\frac{\|z\|}{z_j}\Big|\\
&= \Big|\frac{D^2f(0)(z_0^2)}{2!} - \mu Df(z_0)\frac{1}{2}D^2F_j(0)\Big(z_0, z_0)\Big)\frac{\|z\|}{z_j}\Big|\\
&= \Big|\frac{D^2f(0)(z_0^2)}{2!} - \mu(Df(0)(z_0))^2\Big|\\
&= \frac{1}{2}\Big|D^2f(0)(z_0^2) - 2\mu(Df(0)(z_0))^2\Big|\\
&= \frac{1}{2}\Big|(1-\gamma)[\frac{1}{2}A_1\Big(c_2 - \frac{c_1^2}{2}\Big) + \frac{1}{4}A_2c_1^2] + \frac{1}{4}A_1^2c_1^2(1-\gamma)^2 - 2.\mu\frac{1}{4}A_1^2c_1^2(1-\gamma)^2\Big|\\
&= \frac{A_1}{4}(1-\gamma)\Big|c_2 - \frac{c_1^2}{2} + \frac{1}{2}\frac{A_2}{A_1}c_1^2 + \frac{A_1}{2}c_1^2(1-\gamma) - \mu A_1c_1^2(1-\gamma)\Big|\\
&\leqslant \frac{A_1}{4}(1-\gamma)\Big(2 - \frac{1}{2}|c_1|^2 + \frac{1}{2}|c_1|^2\Big|\frac{A_2}{A_1} + A_1(1-\gamma) - 2\mu A_1(1-\gamma)\Big|\Big). \qquad (32)
\end{aligned}$$

The rest of the proof is similar to the case in Theorem 3.1 (see, (16)), we omit it. The proof is completed. □

Theorem 3.3. *Suppose that the function $\psi(\xi) = 1 + A_1\xi + A_2\xi^2 + \cdots + A_n\xi^n + \cdots, (A_1 > 0)$ satisfies the condition as* (1)*, then the following results hold true:*
(I) *If $f \in \mathcal{S}_\psi^*(\mathbb{U}^n)$, $|z_k| = \|z\| = \max_{1\leqslant j\leqslant n}\{|z_j|\}$, $z_0 = \frac{z}{\|z\|}$, $z \in \mathbb{U}^n\backslash\{0\}$ and*

$$\frac{1}{2}D^2f_k(0)\Big(z_0, \frac{D^2f(0)(z_0^2)}{2!}\Big)\frac{z_k}{\|z\|} = \Big(\frac{(D^2f_k(0)(z_0^2)}{2!}\Big)^2,$$

then we have

$$\Big\|\mathcal{L}_1 - \mu\mathcal{N}_1\Big\| \leqslant \frac{1}{2}A_1\max\Big\{1, \Big|\frac{A_2}{A_1}\frac{1}{(1-\gamma)^2} + A_2 - 2\mu A_1\Big| + \frac{\gamma^2-2\gamma}{(1-\gamma)^2}\Big\}\|z\|^3,$$

where

$$\mathcal{L}_1 = \frac{D^3f(0)(z^3)}{3!}, \quad \mathcal{N}_1 = \frac{1}{2}D^2f(0)\Big(z, \frac{D^2f(0)(z^2)}{2!}\Big).$$

The above estimates are sharp for each complex μ.
(II) *Suppose that $f : \mathbb{U}^n \to \mathbb{C}$, $F(z) = zf(z) \in \mathcal{S}_{\psi,\gamma}^*(\mathbb{U}^n)$, then for $z \in \mathbb{U}^n$, we have*

$$\Big\|\mathcal{L}_2 - \mu\mathcal{N}_2\Big\| \leqslant \frac{1}{2}A_1\max\Big\{1, \Big|\frac{A_2}{A_1}\frac{1}{(1-\gamma)^2} + A_2 - 2\mu A_1\Big| + \frac{\gamma^2-2\gamma}{(1-\gamma)^2}\Big\}\|z\|^3,$$

where

$$\mathcal{L}_2 = \frac{D^3F(0)(z^3)}{3!}, \quad \mathcal{N}_2 = \frac{1}{2}D^2F(0)\Big(z, \frac{D^2F(0)(z^2)}{2!}\Big).$$

The above estimates are sharp for each complex μ.

Proof. It is easy to obtain the (I) and (II) by making a straightforward calculation in (16) and (32), respectively. □

Remark 3.4. *(a) We note that* $A_1 = \psi'(0)$ *and* $A_2 = \frac{1}{2}\psi''(0)$. *Thus, when* $\gamma = 0$ *in (II) of Theorem 3.3, the result coincide with the main Theorem proved by Xu-Liu-Liu [31].*
(b) When $\gamma = 0$ *in (I) of Theorem 3.3, the result is the conjecture proposed by Tu-Xiong [25].*
(c) By choosing suitable functions ψ *and real numbers* γ *as Remark 1.2, the solutions on Fekete-Szegö problems for kinds of subclasses of starlike mappings on* $\mathbb{U}^n$ *can be deduced by our main Theorems immediately.*

4. Conclusion

In this paper, by using the subordination techniques, we obtain the sharp coefficients bounds on Fekete-Szegö problem for a certain subclass of starlike mappings, which are defined on the unit polydisk in $\mathbb{C}^n$. Some previous results are improved. Also, the main works give a positive answer to a conjecture proposed by Tu-Xiong [25].

Basic (or q-) series and basic (or q-) polynomials are known to have widespread applications. In a recent survey-cum-expository review article, Srivastava [24] applied a fractional q-calculus operator to define two subclasses of normalized analytic functions with complex order and negative coefficients. With these subclasses, some current developments involving the usages of the basic (or q-) calculus in geometric function theory of complex analysis were investigated. Also, Srivastava [24] exposed the inconsequential nature of the so-called (p, q)-variations of the q-results by inserting an obviously redundant parameter p in the q-results. Subsequently, we might try to consider some Fekete-Szegö problems by using the basic (or q-) calculus.

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