Filomat 35:1 (2021), 47–60
https://doi.org/10.2298/FIL2101047M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Composition of Wavelet Transforms and Wave Packet Transform Involving Kontorovich-Lebedev Transform

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Abstract. The main objective of this paper is to study the composition of continuous Kontorovich-Lebedev wavelet transform (KL-wavelet transform) and wave packet transform (WPT) based on the Kontorovich-Lebedev transform (KL-transform). Estimates for KL-wavelet and KL-wavelet transform are obtained, and Plancherel's relation for composition of KL-wavelet transform and WPT-transform are derived. Reconstruction formula for WPT associated to KL-transform is also deduced and at the end Calderon's formula related to KL-transform using its convolution property is obtained.

1. Introduction

Wavelet transform is an integral transform that is a very familiar term for scientists, engineers, researchers working in signal processing, image processing, integral transformations, etc. It has proved remarkable interest in these fields, for instance, see [9, 11, 13, 15, 18, 20, 32, 33] and references therein. Apart from the applications, many researchers have developed mathematical theories [2, 3, 6, 10, 30]. We may also refer Pathak [12] who has studied wavelet transform in various function spaces associated with classical Fourier transform. Further, construction of wavelet transforms by using various kind of integral transforms have been carried out by the authors of the field [11, 13, 17, 18, 20, 30, 31]. In [21, 25], Prasad et al. have constructed and studied key properties of wavelet transform associated with index transforms like Kontorovich-Lebedev transform (KL-transform) and Mehler-Fock transform.
Wavelet transform is categorized into two types; one is a continuous wavelet transform, and another is a discrete wavelet transform. If the variation of scale and position of the signal is smooth, we say that the transform is continuous. If we restrict the scaling and translation parameters to the integral values, then we say that the transform is discrete. The continuous wavelet transform is constructed with the use of a single function φ as a kernel and by scaling and shifting it generates a two parameter family of functions $\varphi_{b,a}$ defined by

$$\varphi_{b,a}(x) = |a|^{-\frac{1}{2}}\varphi\left(\frac{x-b}{a}\right),\ a, b \in \mathbb{R},\ a \neq 0,$$

2010 *Mathematics Subject Classification*. Primary **65T60; 42C40; 44A20.**
Keywords. Wavelet; Wavelet transform; Kontorovich-Lebedev transform; Convolution; Wave packet transform.
Received: 26 February 2020; Revised: 03 January 2021; Accepted: 11 January 2021
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here a and b are scaling and translation parameters respectively.
The continuous wavelet transform of a signal $f \in L^2(\mathbb{R})$ with respect to a wavelet $\varphi \in L^2(\mathbb{R})$ is defined as [2, 3]:

$$W_\varphi f(b,a) = \int_{-\infty}^{\infty} f(x)\overline{\varphi_{b,a}(x)}\, dx.$$

Also by using the Parseval's relation for the Fourier transform, from above equation we have

$$W_\varphi f(b,a) = \frac{a}{2\pi\sqrt{|a|}} \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{\varphi}}(a\omega)e^{ib\omega}\, d\omega.$$

In 1938, M. I. Kontorovich and N. N. Lebedev [7, 8] introduced the KL-transform, and after that, the exposition of the theory have been carried out by Srivastava et al. [28, 29]. KL-transform is a kind of index integral transform, Yakubovich et al. have briefly studied this integral transform in his book [34]. The representation of KL-transform and various relations related to it like translation, convolution, Plancherel's and Parseval's relation etc. have been expressed in many ways [1, 5, 22, 23, 27, 35–39].
Now we consider the class of all measurable functions $L^p(\mathbb{R}_+;\ x^{-1}\,dx)$, of f on $\mathbb{R}_+$ with norm given as:

$$\|f\|_{L^p} = \begin{cases} \left(\int\limits_0^\infty |f(x)|^p\, x^{-1}\, dx\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \text{ess sup}|f(x)|, & p = \infty. \end{cases}$$

In this work we use the KL-transform of a function $\varphi \in L^1(\mathbb{R}_+; x^{-1}dx)$ defined as [1, 27]:

$$(\Re\varphi)(\tau) = \int\limits_0^\infty \kappa_{i\tau}(x)\varphi(x)x^{-1}\, dx,\ \tau \in \mathbb{R}_+. \tag{1}$$

The adjoint KL-transform is given as:

$$(\Re'\psi)(x) = x^{-1}\int\limits_0^\infty \kappa_{i\tau}(x)\psi(\tau)\, d\tau,\ x \in \mathbb{R}_+, \tag{2}$$

where $\kappa_{i\tau}(x)$ is Macdonald function or modified Bessel function of third kind which is represented in terms of Fourier cosine integral as [4, p. 82(21)]:

$$\kappa_{i\tau}(x) = \int\limits_0^\infty e^{-x\cosh t}\cos(\tau t)\, dt,\ x \in \mathbb{R}_+,\ \tau \in \mathbb{R}_+. \tag{3}$$

For positive real numbers x and τ the function $\kappa_{i\tau}(x)$ is a real valued and infinitely differentiable.
From (3), Macdonald function can be estimated as $|\kappa_{i\tau}(x)| \le \kappa_0(x)$.
The inversion formula of KL-transform and its adjoint are given by

$$\varphi(x) = \frac{2}{\pi^2}\int\limits_0^\infty \kappa_{i\tau}(x)\tau\sinh(\pi\tau)(\Re\varphi)(\tau)\, d\tau,\ x \in \mathbb{R}_+, \tag{4}$$

and

$$\psi(\tau) = \frac{2}{\pi^2}\tau\sinh(\pi\tau)\int\limits_0^\infty \kappa_{i\tau}(x)(\Re'\psi)(x)\, dx,\ \tau \in \mathbb{R}_+ \tag{5}$$

respectively.
From [4], convolution structure for KL-transform is given as:

$$\frac{2}{\pi^2}\int_0^\infty \kappa_{i\tau}(x)\kappa_{i\tau}(y)\kappa_{i\tau}(z)\tau\sinh(\pi\tau)\,d\tau = T(x,y,z), \tag{6}$$

where $T(x,y,z)$ is represented as:

$$T(x,y,z) = \frac{1}{2}\exp\Big[\frac{-1}{2xyz}(x^2y^2+y^2z^2+z^2x^2)\Big],\ \ x,\ y,\ z\in\mathbb{R}_+, \tag{7}$$

which is symmetric in all three variables $x,\ y$ and z.
From [34, 37], we have some useful relations

$$(i)\ \int_0^\infty T(x,y,z)z^{-1}dz \ = \ \kappa_0(\sqrt{x^2+y^2}), \tag{8}$$

$$(ii)\ \kappa_0(\sqrt{x^2+y^2}) \ \leq \ \kappa_0(x) \text{ or } \kappa_0(y). \tag{9}$$

Using (1), (4) and (6), an integral representation of the product of modified Bessel function of third kind with different arguments is obtained as:

$$\kappa_{i\tau}(x)\kappa_{i\tau}(y) = \int_0^\infty \kappa_{i\tau}(z)T(x,y,z)\,z^{-1}\,dz = (\mathfrak{K}\,T(x,y,z))\,(\tau).$$

From [21], the convolution operator for KL-transform is given as:

$$(f\sharp g)(x) = \int_0^\infty \mathfrak{T}_x f(y)\,g(y)\,y^{-1}\,dy, \tag{10}$$

where $\mathfrak{T}_x f(y)$ represents the translation operator related to KL-transform and it is defined as:

$$\mathfrak{T}_x f(y) = \int_0^\infty T(x,y,z)f(z)z^{-1}\,dz. \tag{11}$$

The convolution structure plays a vital role in constructing the wavelet transform associated with an integral transform. The Plancherel's and Parseval's identities are also useful relations that play a significant role in obtaining an admissibility condition and reconstruction formula for wavelet transform, the composition of the wavelet transform, and wave packet transforms associated with an integral transform.

The **Plancherel's and Parseval's** relations for KL-transform are defined as [1]:

$$\int_0^\infty f(x)\,\overline{g(x)}x^{-1}dx \ \ = \ \ \frac{2}{\pi^2}\int_0^\infty (\mathfrak{K}f)(\tau)\,\overline{(\mathfrak{K}g)}(\tau)\tau\sinh(\pi\tau)d\tau \tag{12}$$

and

$$\int_0^\infty |f(x)|^2x^{-1}dx = \frac{2}{\pi^2}\int_0^\infty |(\mathfrak{K}f)(\tau)|^2\tau\sinh(\pi\tau)d\tau. \tag{13}$$

For adjoint of KL-transform (2), Plancherel's and Parseval's relation are defined as:

$$\frac{\pi^2}{2}\int_0^\infty f(\tau)\,\overline{g(\tau)}\frac{d\tau}{\tau\sinh(\pi\tau)} \;=\; \int_0^\infty (\mathfrak{K}' f)(x)\,(\overline{\mathfrak{K}' g})(x)\,x\,dx, \tag{14}$$

and

$$\frac{\pi^2}{2}\int_0^\infty |f(\tau)|^2\frac{d\tau}{\tau\sinh(\pi\tau)} \;=\; \int_0^\infty |(\mathfrak{K}' f)(x)|^2\,x\,dx. \tag{15}$$

The paper is organized as follows: Section 1 is introductory in which a brief introduction about wavelet transform and various relations related to the KL-transform are given. In Section 2, estimates for the family of KL-wavelet and KL-wavelet transform are obtained, and the composition of KL-wavelet transform is defined. Further, Plancherel's and Parseval's relations for the composition of KL-Wavelet transform is deduced. Section 3, devoted to a brief introduction about the wave packets, then defined the wave packet transform associated with KL-transform. Moreover, its estimate in Lebesgue space is obtained, and a Lemma has been proved. Section 4 can be viewed for Placherel's and Parseval's relations to the wave packet transform associated with KL-transform is obtained, and a reconstruction formula is derived. Ultimately in Section 5, Calderon's formula related to KL-transform is obtained using its convolution property.

2. Composition of Kontorovich-Lebedev wavelet transformation

From [21], the family of KL-wavelets can be constructed by using translation (11) and dilation $\mathcal{D}_a\varphi(x) = \varphi(ax),\ a > 0$, on $\varphi \in L^2(\mathbb{R}_+;\ x^{-1}dx)$ as:

$$\begin{aligned}\varphi_{b,a}(x) &= \mathfrak{T}_b\mathcal{D}_a\varphi(x) = \mathfrak{T}_b\varphi(ax)\\ &= \int_0^\infty \varphi(az)T(b,x,z)z^{-1}\,dz,\end{aligned} \tag{16}$$

where $b,\ a > 0$ are respectively translation and dilation parameters.

Proposition 2.1. *If $\varphi_{b,a}$ be a KL-wavelet defined as* (16)*, then we have*

$$\left\|\frac{\varphi_{b,a}(x)}{\kappa_0(b)}\right\|_{L^2(\mathbb{R}_+;\ \kappa_0(b)b^{-1}db)} \le \|\varphi(az)\|_{L^2(\mathbb{R}_+;\ \kappa_0(z)z^{-1}dz)}. \tag{17}$$

Proof. (*i*) From (16), we have

$$\begin{aligned}|\,\varphi_{b,a}(x)\,| &\le \int_0^\infty |\,\varphi(az)\,|\,T(b,x,z)z^{-1}dz\\ &= \int_0^\infty |\,\varphi(az)\,|\,(T(b,x,z))^{\frac{1}{2}+\frac{1}{2}}\,z^{-1}dz.\end{aligned}$$

Using Hölder's inequality and (8), we get

$$\begin{aligned}|\varphi_{b,a}(x)| &\le \left(\int_0^\infty |\,\varphi(az)\,|^2\,T(b,x,z)z^{-1}dz\right)^{\frac{1}{2}}\left(\int_0^\infty T(b,x,z)z^{-1}dz\right)^{\frac{1}{2}}\\ &= \left(\int_0^\infty |\,\varphi(az)\,|^2\,T(b,x,z)z^{-1}dz\right)^{\frac{1}{2}}\left(\kappa_0(\sqrt{x^2+b^2})\right)^{\frac{1}{2}}.\end{aligned}$$

Now using (9), we have

$$
\begin{aligned}
|\varphi_{b,a}(x)|^2 &\leq \left(\int_0^\infty |\varphi(az)|^2\, T(b,x,z) z^{-1} dz\right)\left(\kappa_0(\sqrt{x^2+b^2})\right) \\
&\leq \kappa_0(b)\int_0^\infty |\varphi(az)|^2\, T(b,x,z) z^{-1} dz. \qquad (18)
\end{aligned}
$$

Therefore

$$
\int_0^\infty |\varphi_{b,a}(x)|^2\, (\kappa_0(b))^{-1} b^{-1} db \leq \int_0^\infty \left(\int_0^\infty |\varphi(az)|^2\, T(b,x,z) z^{-1} dz\right) b^{-1} db.
$$

Using (8), (9) and Fubini's theorem, we get

$$
\begin{aligned}
\int_0^\infty \left|\frac{\varphi_{b,a}(x)}{\kappa_0(b)}\right|^2 \kappa_0(b) b^{-1} db &\leq \int_0^\infty \kappa_0(\sqrt{x^2+z^2})\, |\varphi(az)|^2\, z^{-1} dz \\
&\leq \int_0^\infty |\varphi(az)|^2\, \kappa_0(z) z^{-1} dz.
\end{aligned}
$$

Hence

$$
\left\|\frac{\varphi_{b,a}(x)}{\kappa_0(b)}\right\|_{L^2(\mathbb{R}_+;\ \kappa_0(b)b^{-1}db)} \leq \|\varphi(az)\|_{L^2(\mathbb{R}_+;\ \kappa_0(z)z^{-1}dz)},
$$

This proves the Proposition. □

From [21, p. 6], for any KL-wavelet $\varphi \in L^2(\mathbb{R}_+;\ x^{-1}dx)$, the continuous KL-wavelet transform associated with KL-transform for a function $f \in L^2(\mathbb{R}_+;\ x^{-1}dx)$ is defined as:

$$
\begin{aligned}
(\mathcal{K}_\varphi f)(b,a) &= \int_0^\infty f(x)\overline{\varphi_{b,a}(x)}\, x^{-1}\, dx = \left\langle f,\ x^{-1}\varphi_{b,a}\right\rangle \qquad (19) \\
&= \int_0^\infty\int_0^\infty f(x)\overline{\varphi(az)} T(b,x,z)\, x^{-1}\, z^{-1}\, dx\, dz. \qquad (20)
\end{aligned}
$$

Another representation of continuous KL-wavelet transform is given by [21, p. 7]

$$
(\mathcal{K}_\varphi f)(b,a) = \frac{2}{\pi^2}\int_0^\infty \kappa_{i\tau}(b)\tau \sinh \pi\tau (\Re f)(\tau)\overline{(\Re\varphi)(a,\tau)} d\tau.
$$

By means of inversion formula for KL-transform, it reduces to

$$
(\Re(\mathcal{K}_\varphi f)(b,a))(\tau) = (\Re f)(\tau)\overline{(\Re\varphi)(a,\tau)}. \qquad (21)
$$

Theorem 2.2. *Let $f \in L^p(\mathbb{R}_+; \kappa_0(z)z^{-1}\ dz\ )$ and $\varphi(a\cdot) \in L^q(\mathbb{R}_+; \kappa_0(z)z^{-1}\ dz)$ with $1 \leq p,q < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$, and $(\mathcal{K}_\varphi f)(b,a)$ be KL-wavelet transform. Then $b \to (\mathcal{K}_\varphi f)(b,a)$ is continuous in the variable b on $\mathbb{R}_+\times\mathbb{R}_+$.*

Proof. Let (b_0, a) be an arbitrary but fixed point in $\mathbb{R}_+ \times \mathbb{R}_+$. Then by Hölder's inequality,

$$\begin{aligned}
|(\mathcal{K}_\varphi f)(b,a) - (\mathcal{K}_\varphi f)(b_0,a)| &\leq \int_0^\infty \int_0^\infty f(x)\overline{\varphi(az)}|T(b,x,z) - T(b_0,x,z)|\, x^{-1}\, z^{-1}\, dx\, dz \\
&\leq \left(\int_0^\infty |f(x)|^p \left(\int_0^\infty |T(b,x,z) - T(b_0,x,z)|\, z^{-1}\, dz\right) x^{-1}\, dx\right)^{\frac{1}{p}} \\
&\times \left(\int_0^\infty |\varphi(az)|^q \left(\int_0^\infty |T(b,x,z) - T(b_0,x,z)|\, x^{-1}\, dx\right) z^{-1}\, dz\right)^{\frac{1}{q}}.
\end{aligned}$$

Using (8) and (9), we have

$$\int_0^\infty |T(b,x,z) - T(b_0,x,z)|\, x^{-1}\, dx \leq 2\kappa_0(z).$$

On the other hand, by dominated convergence theorem and continuity of $T(b,x,z)$ in the variable b, we have

$$\lim_{b \to b_0} |(\mathcal{K}_\varphi f)(b,a) - (\mathcal{K}_\varphi f)(b_0,a)| = 0.$$

This proves that $(\mathcal{K}_\varphi f)(b,a)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$ in the variable b for arbitrarily fixed value of a. □

Theorem 2.3. *If any KL-wavelet $\varphi \in L^2(\mathbb{R}_+;\ z^{-1}dz)$, $f \in L^2(\mathbb{R}_+;\ x^{-1}dx)$ and the continuous KL-wavelet transform is defined as* (19), *then*

$$\|(\mathcal{K}_\varphi f)(b,a)\|_{L^2(\mathbb{R}_+;db)} \leq \frac{\pi}{2}\, \|f(x)\|_{L^2(\mathbb{R}_+;x^{-1}dx)} \|\varphi(az)\|_{L^2(\mathbb{R}_+;z^{-1}dz)}.$$

Proof. Using (19), Hölder's inequality and then (18), we have

$$\begin{aligned}
|(\mathcal{K}_\varphi f)(b,a)|^2 &\leq \left(\int_0^\infty |f(x)|^2 x^{-1} dx\right)\left(\int_0^\infty |\varphi_{b,a}(x)|^2 x^{-1} dx\right) \\
&\leq \kappa_0(b)\left(\int_0^\infty |f(x)|^2 x^{-1} dx\right)\left(\int_0^\infty \int_0^\infty |\varphi(az)|^2 T(b,x,z) z^{-1} x^{-1} dz dx\right)
\end{aligned}$$

Using (8) and (9), we get

$$|(\mathcal{K}_\varphi f)(b,a)|^2 \leq (\kappa_0(b))^2 \left(\int_0^\infty |f(x)|^2 x^{-1} dx\right)\left(\int_0^\infty |\varphi(az)|^2 z^{-1} dz\right).$$

Thus

$$\begin{aligned}
\|(\mathcal{K}_\varphi f)(b,a)\|_{L^2(\mathbb{R}_+;db)} &\leq \left(\int_0^\infty (\kappa_0(b))^2 db\right)^{\frac{1}{2}} \|f(x)\|_{L^2(\mathbb{R}_+;x^{-1}dx)} \|\varphi(az)\|_{L^2(\mathbb{R}_+;z^{-1}dz)}. \\
&\leq \frac{\pi}{2} \|f(x)\|_{L^2(\mathbb{R}_+;x^{-1}dx)} \|\varphi(az)\|_{L^2(\mathbb{R}_+;z^{-1}dz)}.
\end{aligned}$$

Hence the Theorem is proved. □

As integral transforms like Fourier transform, fractional Fourier transform etc. the wavelet transform have been constructed and then moving a step ahead extending this work composition of wavelet transform involving Fourier transform, fractional Fourier transform etc. are derived for instance [12, 19]. This is the motivation of our work to construct composition of KL-wavelet transform.

Composition of KL-wavelet transform:

Let φ and ψ are two KL-wavelets and $f \in L^2(\mathbb{R}_+;\ x^{-1}dx)$ such that $\mathcal{K}_\varphi f$ and $\mathcal{K}_\psi f$ are continuous KL-wavelet transforms. Then the composition of two KL-wavelet transforms is defined by

$$\begin{aligned}(\mathbb{K}f)(b,a,c) &= \mathcal{K}_\varphi(\mathcal{K}_\psi f)\\ &= \frac{2}{\pi^2}\int_0^\infty \kappa_{i\tau}(b)\tau\sinh\pi\tau(\Re(\mathcal{K}_\psi f))(\tau)\overline{(\Re\varphi)(a,\tau)}d\tau. \end{aligned}\tag{22}$$

The admissibility condition for the composition of two KL-wavelet transforms for $\varphi,\ \psi \in L^2(\mathbb{R}_+;x^{-1}dx)$ and the weight function $w(a,c) > 0$ is given by

$$\begin{aligned}C_{\varphi,\psi} &= \int_0^\infty\int_0^\infty (\Re\psi)(c,\tau)\overline{(\Re\psi)(c,\tau)}(\Re\varphi)(a,\tau)\overline{(\Re\varphi)(a,\tau)}w(a,c)a^{-1}c^{-1}dadc \\ &= \int_0^\infty\int_0^\infty (\Re\psi)(c\tau)\overline{(\Re\psi)(c\tau)}(\Re\varphi)(a\tau)\overline{(\Re\varphi)(a\tau)}a^{-1}c^{-1}dadc < \infty\end{aligned}\tag{23}$$

Plancherel's and Parseval's relations for the composition of KL-wavelet transform:

Theorem 2.4. *Let $\varphi,\ \psi \in L^2(\mathbb{R}_+;x^{-1}dx)$ be two KL-wavelets which defines the composition of KL-wavelet transform $(\mathbb{K}f)(b,a,c)$ and $(\mathbb{K}g)(b,a,c)$ defined as* (22)*, for two functions $f,g \in L^2(\mathbb{R}_+;x^{-1}dx)$. Then*

$$\int_0^\infty\int_0^\infty\int_0^\infty (\mathbb{K}f)(b,a,c)\overline{(\mathbb{K}g)(b,a,c)}w(a,c)b^{-1}a^{-1}c^{-1}dbdadc = C_{\varphi,\psi}\int_0^\infty f(x)\overline{g(x)}x^{-1}dx,\tag{24}$$

where $C_{\varphi,\psi}$ is defined as (23) *and $w(a,c) > 0$ be any weight function.*

Proof. By using (22), (21) and (2), we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty (\mathbb{K}f)(b,a,c)\overline{(\mathbb{K}g)(b,a,c)}w(a,c)b^{-1}a^{-1}c^{-1}dbdadc\\ &= \int_0^\infty\int_0^\infty\int_0^\infty \left(\frac{2}{\pi^2}b\right)^2 (\Re'(\tau\sinh\pi\tau(\Re f)(\tau)\overline{(\Re\varphi)(c,\tau)}\ \overline{(\Re\psi)(a,\tau)}))(b)\\ &\times\overline{(\Re'(\tau\sinh\pi\tau(\Re g)(\tau)\overline{(\Re\varphi)(c,\tau)}\ \overline{(\Re\psi)(a,\tau)}))(b)}w(a,c)b^{-1}a^{-1}c^{-1}dbdadc\\ &= \int_0^\infty\int_0^\infty \left(\frac{2}{\pi^2}\right)^2\left(\int_0^\infty (\Re'(\tau\sinh\pi\tau(\Re f)(\tau)\overline{(\Re\varphi)(c,\tau)}\ \overline{(\Re\psi)(a,\tau)}))(b)\right.\\ &\times\left.\overline{(\Re'(\tau\sinh\pi\tau(\Re g)(\tau)\overline{(\Re\varphi)(c,\tau)}\ \overline{(\Re\psi)(a,\tau)}))(b)}b\,db\right)w(a,c)a^{-1}c^{-1}dadc.\end{aligned}$$

By using the Plancherel's relation for adjoint KL-transform (14), we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_0^\infty (\mathbb{K}f)(b,a,c)\overline{(\mathbb{K}g)(b,a,c)}w(a,c)b^{-1}a^{-1}c^{-1}dbdadc\\ &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{2}{\pi^2}\tau\sinh\pi\tau(\Re f)(\tau)\overline{(\Re\varphi)(c,\tau)(\Re\psi)(a,\tau)}\\ &\times\overline{(\Re g)(\tau)}\ (\Re\varphi)(c,\tau)(\Re\psi)(a,\tau)w(a,c)a^{-1}c^{-1}d\tau dadc.\end{aligned}$$

Next by using (23) and (12), we get

$$\int_0^\infty \int_0^\infty \int_0^\infty (\mathbb{K}f)(b,a,c)\overline{(\mathbb{K}g)(b,a,c)}w(a,c)b^{-1}a^{-1}c^{-1}dbdadc = C_{\varphi,\psi}\int_0^\infty f(x)\overline{g(x)}x^{-1}dx.$$

Which completes the proof of the Theorem. □

Remark 2.5. *If $f = g$, then from* (24)*, we can write*

$$\int_0^\infty \int_0^\infty \int_0^\infty |(\mathbb{K}f)(b,a,c)|^2 w(a,c)b^{-1}a^{-1}c^{-1}dbdadc = C_{\varphi,\psi}\int_0^\infty |f(x)|^2 x^{-1}dx.$$

3. Wave packet transform (WPT) associated to KL-transform

A wave packet generally refers to as a wave group. It can also be considered as the product of two waves. Representation of a signal using the wave packet is more significant than the ordinary wavelet. The wave packet transform (WPT) is an integral transform of a signal windowed with a wavelet dilated by any parameter a and translated by b. The WPT associated with integral transforms like Fourier transform, fractional Fourier transform etc., have been previously obtained[14, 16, 26].
Thus following in this similar manner, now we define the wave packet transform (WPT) associated with KL-transform for a function $f \in L^2(\mathbb{R}_+;\ x^{-1}dx)$ as:

$$\mathrm{WP}_\varphi f(\tau,b,a) = \int_0^\infty \kappa_{i\tau}(x)f(x)\overline{\varphi_{b,a}}(x)x^{-1}dx, \tag{25}$$

where $\varphi_{b,a}(x)$ is defined as (16).
It can also be represented in terms of convolution defined in (10) as:

$$\mathrm{WP}_\varphi f(\tau,b,a) = (\varphi(a\cdot)\ \sharp\ \kappa_{i\tau}(\cdot)f(\cdot))\,(b).$$

Theorem 3.1. *If φ be any wave packet and $f \in L^1(\mathbb{R}_+;\ \kappa_0(x)x^{-1}\,dx)$, then*

$$\|WP_\varphi f(\tau,b,a)\|_{L^1(\mathbb{R}_+;\ b^{-1}\ db)} \le \|f\|_{L^1(\mathbb{R}_+;\ \kappa_0(x)x^{-1}\ dx)}\|\varphi(az)\|_{L^1(\mathbb{R}_+;\ \kappa_0(z)z^{-1}\ dz)}.$$

Proof. By using (16), (8) and then (9), we have

$$\begin{aligned}\int_0^\infty |\varphi_{b,a}(x)|b^{-1}db &\le \int_0^\infty |\varphi(az)|\kappa_0(\sqrt{x^2+z^2})z^{-1}dz \\ &\le \int_0^\infty |\varphi(az)|\kappa_0(z)z^{-1}dz.\end{aligned}$$

Thus, we have

$$\|\varphi_{b,a}(x)\|_{L^1(\mathbb{R}_+;\ b^{-1}\ db)} \ \le\ \|\varphi(az)\|_{L^1(\mathbb{R}_+;\ \kappa_0(z)z^{-1}\ dz)}. \tag{26}$$

Therefore, from (25) and (26), we have

$$\|WP_\varphi f(\tau,b,a)\|_{L^1(\mathbb{R}_+;\ b^{-1}\ db)} \le \|f\|_{L^1(\mathbb{R}_+;\ \kappa_0(x)x^{-1}\ dx)}\|\varphi(az)\|_{L^1(\mathbb{R}_+;\ \kappa_0(z)z^{-1}\ dz)}.$$

Hence proved. □

If φ be any wave packet, $\alpha,\ \beta$ be any scalar constants and $f,\ g \in L^2(\mathbb{R}_+;\ x^{-1}\,dx)$ then by using (25), we have

$$\left(\mathrm{WP}_\varphi(\alpha f + \beta g)\right)(\tau, b, a) = \alpha(\mathrm{WP}_\varphi f)(\tau, b, a) + \beta(\mathrm{WP}_\varphi g)(\tau, b, a).$$

This is also known as the linearity property.
From [21, p.7], we have if $\varphi \in L^2(\mathbb{R}_+;\ x^{-1}\,dx)$, then

$$(\Re\varphi_{b,a})(\tau) = \kappa_{i\tau}(b)(\Re\varphi)(a, \tau). \tag{27}$$

Lemma 3.2. *If $f \in L^2(\mathbb{R}_+;\ x^{-1}dx)$ be any function and $\varphi \in L^2(\mathbb{R}_+;\ x^{-1}dx)$ is a wave packet then*

$$(\mathrm{WP}_\varphi f)(\tau, b, a) = \Re^{-1}\left((\Re(\kappa_{i\tau}(x)f(x)))(\tau)\ \overline{(\Re\varphi)(a, \tau)}\right)(b). \tag{28}$$

Proof. By using (25), (12), (27) and then inversion of KL-transform (4), we get

$$\begin{aligned}
(\mathrm{WP}_\varphi f)(\tau, b, a) &= \frac{2}{\pi^2}\int_0^\infty (\Re(\kappa_{i\tau}(x)f(x)))\,(\tau)\overline{(\Re\varphi_{b,a})}(\tau)\ \tau\sinh(\pi\tau)\ d\tau \\
&= \frac{2}{\pi^2}\int_0^\infty (\Re(\kappa_{i\tau}(x)f(x)))\,(\tau)\kappa_{i\tau}(b)\overline{(\Re\varphi)}(a, \tau)\ \tau\sinh(\pi\tau)\ d\tau \\
&= \Re^{-1}\left(\Re(\kappa_{i\tau}(x)f(x)))(\tau)\ \overline{(\Re\varphi)}(a, \tau)\right)(b).
\end{aligned}$$

Thus

$$\left(\Re(\mathrm{WP}_\varphi f)(\tau, b, a)\right)(\tau) = (\Re(\kappa_{i\tau}(x)f(x)))\,(\tau)\ \overline{(\Re\varphi)}(a, \tau).$$

Which proves the Lemma. □

4. Plancherel and Parseval's relation for WPT associated to KL-transform

Theorem 4.1. *Let $\varphi,\ \psi \in L^2(\mathbb{R}_+;\ x^{-1}\,dx)$ are KL-wavelets which define WPT as* (25), $\mathrm{WP}_\varphi f(\tau, b, a)$ *and* $\mathrm{WP}_\psi g(\tau, b, a)$ *for $f, g, \in L^2(\mathbb{R}_+;\ x^{-1}\,dx)$ respectively, then for*

$$C'_{\varphi,\psi} = \int_0^\infty (\Re\varphi)(a, \tau)\ \overline{(\Re\psi)(a, \tau)}a^{-1}da\ <\ \infty. \tag{29}$$

As $C'_{\varphi,\psi}$ is a constant so here we have assumed that $(\Re\varphi)(a, \tau)$ is nothing but $(\Re\varphi)(a\tau)$. Thus, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau, b, a)\ \overline{(\mathrm{WP}_\varphi g)(\tau, b, a)}\ a^{-1}b^{-1}da\ db\ d\tau = C'_{\varphi,\psi}\frac{\pi}{2}\int_0^\infty \kappa_0(2x)f(x)\overline{g(x)}x^{-1}dx. \tag{30}$$

Proof. By using (28) and (4), we get

$$\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\overline{(\mathrm{WP}_\varphi g)(\tau,b,a)}\, b^{-1}\, db$$
$$= \int_0^\infty \Re^{-1}\left((\Re(\kappa_{i\tau}(x)f(x)))(\tau)\,\overline{(\Re\psi)(a,\tau)}\right)(b)\,\overline{\Re^{-1}\left((\Re(\kappa_{i\tau}(x)g(x)))(\tau)\,\overline{(\Re\varphi)(a,\tau)}\right)(b)}\, b^{-1}db$$
$$= \left(\frac{2}{\pi^2}\right)^2 \int_0^\infty \left(\int_0^\infty \kappa_{i\tau}(b)(\Re(\kappa_{i\tau}(x)f(x)))(\tau)\,\overline{(\Re\psi)(a,\tau)}\tau\sinh(\pi\tau)\,d\tau\right)$$
$$\times\left(\int_0^\infty \overline{\kappa_{i\tau}(b)(\Re(\kappa_{i\tau}(x)g(x)))(\tau)\,\overline{(\Re\varphi)(a,\tau)}\tau\sinh(\pi\tau)\,d\tau}\right) b^{-1}\, db.$$

By using (2) and (14), it can be written as

$$\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\overline{(\mathrm{WP}_\varphi g)(\tau,b,a)}\, b^{-1}\, db$$
$$= \left(\frac{2}{\pi^2}\right)^2 \int_0^\infty b\left(\Re'\left(\tau\sinh(\pi\tau)(\Re(\kappa_{i\tau}(x)f(x)))(\tau)\,\overline{(\Re\psi)(a,\tau)}\right)\right)(b)$$
$$\times\overline{\left(\Re'\left(\tau\sinh(\pi\tau)(\Re(\kappa_{i\tau}(x)g(x)))(\tau)\,\overline{(\Re\varphi)(a,\tau)}\right)\right)(b)}\, db$$
$$= \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)(\Re(\kappa_{i\tau}(x)f(x)))(\tau)\,\overline{(\Re\psi)}(a,\tau)\;\overline{(\Re(\kappa_{i\tau}(x)g(x)))(\tau)}\;(\Re\varphi)\,(a,\tau)\,d\tau. \tag{31}$$

Therefore

$$\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\overline{(\mathrm{WP}_\varphi g)(\tau,b,a)}\, a^{-1}b^{-1}\, da\, db$$
$$= \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)\left(\int_0^\infty (\Re\varphi)\,(a,\tau)\overline{(\Re\psi)}(a,\tau)\, a^{-1}\, da\right)(\Re(\kappa_{i\tau}(x)f(x)))(\tau)\overline{(\Re(\kappa_{i\tau}(x)g(x)))(\tau)}\, d\tau\ .$$

Hence by using (12) and (29), we get

$$\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\overline{(\mathrm{WP}_\varphi g)(\tau,b,a)}\, a^{-1}b^{-1}\, da\, db = C'_{\varphi,\psi}\int_0^\infty (\kappa_{i\tau}(x))^2 f(x)\overline{g(x)}x^{-1}dx.$$

On integrating both sides with respect to τ, we get

$$\begin{aligned}\int_0^\infty\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\overline{(\mathrm{WP}_\varphi g)(\tau,b,a)}\, a^{-1}b^{-1}\, da\, db &= C'_{\varphi,\psi}\int_0^\infty\left(\int_0^\infty (\kappa_{i\tau}(x))^2 d\tau\right) f(x)\overline{g(x)}x^{-1}dx.\\ &= C'_{\varphi,\psi}\frac{\pi}{2}\int_0^\infty \kappa_0(2x)f(x)\overline{g(x)}x^{-1}dx.\end{aligned}$$

Which proves the Theorem. □

Remark : Following are its deductions:
1. If $\varphi = \psi$ then, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty (\mathrm{WP}_\varphi f)(\tau,b,a)\,\overline{(\mathrm{WP}_\varphi g)(\tau,b,a)}a^{-1}b^{-1}\,da\,db\,d\tau = C'_\varphi\frac{\pi}{2}\int_0^\infty \kappa_0(2x)f(x)\overline{g(x)}x^{-1}dx.$$

2. If we take $f = g$, then

$$\int_0^\infty\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\,\overline{(\mathrm{WP}_\varphi f)(\tau,b,a)}\,a^{-1}b^{-1}\,da\,db\,d\tau = C'_{\varphi,\psi}\frac{\pi}{2}\int_0^\infty \kappa_0(2x)|f(x)|^2x^{-1}dx.$$

3. If $f = g$ and $\varphi = \psi$, then

$$\int_0^\infty\int_0^\infty\int_0^\infty |(\mathrm{WP}_\varphi f)(\tau,b,a)|^2\,a^{-1}b^{-1}\,da\,db\,d\tau = C'_\varphi\frac{\pi}{2}\int_0^\infty \kappa_0(2x)|f(x)|^2x^{-1}dx..$$

Reconstruction Formula

Theorem 4.2. *If $f \in L^2(\mathbb{R}_+;\ x^{-1}\,dx)$ and $\varphi,\ \psi$ are KL-wavelets then f can be reconstructed by the formula*

$$f(x) = \frac{2}{\pi C'_{\varphi,\psi}\kappa_0(2x)}\int_0^\infty\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\varphi_{b,a}(x)\,b^{-1}a^{-1}\,da\,db\,d\tau,$$

or

$$f_\tau(x) = \frac{1}{C'_{\varphi,\psi}}\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\varphi_{b,a}(x)\,b^{-1}a^{-1}\,da\,db,$$

where $C'_{\varphi,\psi}$ is defined as (29).

Proof. From (30) and (25), we have

$$\begin{aligned}
&\int_0^\infty \kappa_0(2x)f(x)\overline{g(x)}x^{-1}dx\\
&= \frac{2}{\pi C'_{\varphi,\psi}}\int_0^\infty\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\left(\int_0^\infty \kappa_{i\tau}(x)\overline{g(x)}\varphi_{b,a}(x)\,x^{-1}\,dx\right)b^{-1}a^{-1}\,da\,db\,d\tau\\
&= \frac{2}{\pi C'_{\varphi,\psi}}\int_0^\infty\left(\int_0^\infty\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\varphi_{b,a}(x)\,b^{-1}a^{-1}\,da\,db\,d\tau\right)\overline{g(x)}\,x^{-1}\,dx.
\end{aligned}$$

Thus

$$f(x) = \frac{2}{\pi C'_{\varphi,\psi}\kappa_0(2x)}\int_0^\infty\int_0^\infty\int_0^\infty (\mathrm{WP}_\psi f)(\tau,b,a)\varphi_{b,a}(x)\,b^{-1}a^{-1}\,da\,db\,d\tau.$$

Hence the Theorem is proved. □

5. Calderon's Formula

In this section, we obtain Calderón's reproducing identity using the properties of Kontorovich-Lebedv convolutions and Kontorovich-Lebedev transform. The classical Calderón's formula related to Fourier convolution $*$ is defined as

$$f(x) = \int_0^\infty (f * \varphi_a * \psi_a)(x)\frac{da}{a}, \tag{32}$$

where $\varphi_a(x) = \frac{1}{a}\varphi(\frac{x}{a})$ and $\psi_a(x) = \frac{1}{a}\psi(\frac{x}{a})$. The Calderón's reproducing formula is useful tool in pure and applied mathematics, particularly in studying wavelet theory. It is also useful in obtaining reconstruction formula in wavelet transform.

Theorem 5.1. *Let φ and ψ are functions from $L^1(\mathbb{R}_+; x^{-1}dx)$ such that following admissibility condition holds*

$$\int_0^\infty (\Re\varphi)(\tau)(\Re\psi)(\tau)\frac{d\tau}{\tau} = 1, \quad \forall\ \tau \in \mathbb{R}_+. \tag{33}$$

Then the following Calderón's identity holds true

$$f(x) = \int_0^\infty (f \sharp \varphi_a \sharp \psi_a)(x)\frac{da}{a}, \quad \forall\ f \in L^1(\mathbb{R}_+; \kappa_0(x)x^{-1}dx). \tag{34}$$

Proof. Applying Kontorovich-Lebedev transform (1) on right-hand side of (34) and changing the order of integration by means of Fubini's theorem with $(\Re\varphi_a)(\tau) = (\Re\varphi)(a\tau)$ and $(\Re\psi_a)(\tau) = (\Re\psi)(a\tau)$, we get

$$\begin{aligned} \Re\left(\int_0^\infty (f \sharp \varphi_a \sharp \psi_a)(x)\frac{da}{a}\right)(\tau) &= \int_0^\infty (\Re f)(\tau)(\Re\varphi_a)(\tau)(\Re\psi_a)(\tau)\frac{da}{a} \\ &= (\Re f)(\tau)\int_0^\infty (\Re\varphi)(a\tau)(\Re\psi)(a\tau)\frac{da}{a}. \end{aligned}$$

Now substituting $a\tau = \eta$ and invoking (33), and then inverse Kontorovich-Lebedev transform (4), we obtain the result. □

Theorem 5.2. *Suppose $\varphi \in L^1(\mathbb{R}_+; \kappa_0(x)x^{-1}dx)$ is real valued and satisfies*

$$\int_0^\infty [(\Re\varphi)(a\tau)]^2\frac{da}{a} = 1. \tag{35}$$

For $f \in L^1(\mathbb{R}_+; \kappa_0(x)x^{-1}dx) \cap L^2(\mathbb{R}_+; x^{-1}dx)$, suppose that

$$f_N(x) = \int_{\frac{1}{N}}^N (f \sharp \varphi_a \sharp \varphi_a)(x)\frac{da}{a}. \tag{36}$$

Then $\|f - f_N\|_{L^2(\mathbb{R}_+; x^{-1}dx)} \to 0$ as $N \to \infty$.

Proof. Taking KL-transform to both sides of (36) and invoking Fubini's theorem, we have

$$(\Re f_N)(\tau) = (\Re f)(\tau)\int_{\frac{1}{N}}^N [(\Re\varphi)(a\tau)]^2\frac{da}{a}. \tag{37}$$

Now using Minkowski's inequality, [24, Proposition 2.1] and [23, Theorem 2.1], we get

$$\begin{aligned} \|f_N\|_{L^2(\mathbb{R}_+; x^{-1}dx)} &= \int_0^\infty \frac{dx}{x}\left|\int_{\frac{1}{N}}^N (f \sharp \varphi_a \sharp \varphi_a)(x)\frac{da}{a}\right|^2 \\ &\leq \int_{\frac{1}{N}}^N \int_0^\infty \left|(f \sharp \varphi_a \sharp \varphi_a)(x)\right|^2 \frac{dx}{x}\frac{da}{a} \\ &\leq \int_{\frac{1}{N}}^N \|(f \sharp \varphi_a \sharp \varphi_a)\|_{L^2(\mathbb{R}_+; x^{-1}dx)}\frac{da}{a} \\ &\leq 2\|\varphi_a\|_{L^1(\mathbb{R}_+; \kappa_0(x)x^{-1}dx)}\|f\|_{L^2(\mathbb{R}_+; x^{-1}dx)} \lg N. \end{aligned}$$

Therefore, $f_N \in L^2(\mathbb{R}_+; x^{-1}dx)$. Hence by Parseval's formula (12), we get

$$\lim_{N\to\infty} \|f - f_N\|^2_{L^2(\mathbb{R}_+;x^{-1}dx)} = \lim_{N\to\infty} \|(\Re f) - (\Re f_N)\|_{L^2(\mathbb{R}_+;\tau\sinh(\pi\tau)d\tau)}$$

$$= \lim_{N\to\infty} \int_0^\infty \left|(\Re f)(\tau)\left(1 - \int_{\frac{1}{N}}^N [(\Re\varphi)(a\tau)]^2 \frac{da}{a}\right)\right|^2 \tau\sinh(\pi\tau)d\tau = 0.$$

Since $\left|(\Re f)(\tau)\left(1 - \int_{\frac{1}{N}}^N [(\Re\varphi)(a\tau)]^2 \frac{da}{a}\right)\right| \leq (\Re f)(\tau)$, therefore, by the dominated convergence theorem, the result follows. $\square$

Acknowledgement:

The authors express their gratitude to the anonymous reviewer for his constructive and valuable suggestions.
Third author is financially supported by Science and Engineering Research Board, Govt. of India, under grant no. EMR/2016/005141.

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