Almost Convergence of Complex Uncertain Double Sequences

Birojit Das[a], Binod Chandra Tripathy[b], Piyali Debnath[a], Baby Bhattacharya[a]

[a]*Department of Mathematics, NIT Agartala, India, 799046*
[b]*Department of Mathematics, Tripura University, 799022*

Abstract. Convergence of real sequences, as well as complex sequences are studied by B. Liu and X. Chen respectively in uncertain environment. In this treatise, we extend the study of almost convergence by introducing double sequences of complex uncertain variable. Almost convergence with respect to almost surely, mean, measure, distribution and uniformly almost surely are presented and interrelationships among them are studied and depicted in the form of a diagram. We also define almost Cauchy sequence in the same format and establish some results. Conventionally we have, every convergent sequence is a Cauchy sequence and the converse case is not true in general. But taking complex uncertain variable in a double sequence, we find that a complex uncertain double sequence is a almost Cauchy sequence if and only if it is almost convergent. Some suitable examples and counter examples are properly placed to make the paper self sufficient.

1. Introduction

In real life, uncertainty not only appears in real quantities but also in complex quantities. Uncertainty theory is inevitable to quantify the future when no data is available, to asses the future when an emergency like war, flood, earthquake arises or the past when counting precise observations or performing measure is nearly impossible, to model non-jagged concepts (for example, tall, young) and dynamical systems with continuous time noise. Liu [5] introduced the notion of uncertain variable as a function from a measurable space to $\mathbb{R}$ and when $\mathbb{R}$ is replaced by the set of complex numbers, it is called complex uncertain variable due to Peng [18]. In the same work, he also initiated the notion of complex uncertain distribution and expected value for the purpose of measure of a complex uncertain variable. Now-a-days the works on uncertainty theory is being explored in almost every field of mathematics, viz. uncertain logic [6] (Liu), uncertain process [7](Liu), uncertain inference [8](Liu), uncertain calculus [9] (Liu), uncertain risk and reliability analysis [10] (Liu), uncertain graph [17](Gao, Gao), uncertain finance [9](Liu) [15](Chen) and many more.

Convergence of sequences [3, 4, 13] plays a pivotal role in the study of fundamental theory of mathematics and so Liu [5] applied the theory of uncertainty on sequences. He established the properties of convergence of uncertain measure by introducing convergence in measure, in mean, in distribution and in almost surely

of an uncertain sequence. You [11] extended this study to convergence in uniformly almost surely and established the interrelationships with the previous four types of convergences reported by Liu [5]. Guo and Xu [14] presented a necessary and sufficient condition of convergence in mean square for uncertain sequences via Cauchy sequence. Cheng et. al [16] studied the concept of convergence of uncertain sequences taking complex uncertain variable into consideration. Datta and Tripathy [12] studied convergence of complex uncertain double sequences.

Almost convergence of sequences was originated and explored by Zygmund [1] [2]. In this paper, we introduce double sequence of complex uncertain variables and study the almost convergence in uncertain environment. Almost convergence with respect to almost surely, mean, measure, distribution and uniformly almost surely are initiated and interrelationships among them are also established. In addition, we also presented the concept of almost Cauchy sequence in the same structure and proved some results:

A double sequence of complex uncertain variable is almost Cauchy if and only if it is almost convergent.

Before going to the main section we need some basic and preliminary ideas about the existing definitions and results which will play a major role in this study.

2. Preliminaries

2.1. Definition [5]

Let $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:

Axiom 1 (Normality Axiom). $\mathcal{M}\{\Gamma\}=1$;

Axiom 2 (Duality Axiom). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\}=1$, for any $\Lambda \in \mathcal{L}$;

Axiom 3 (Subadditivity Axiom). For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$, we have

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}(\Lambda_j)$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space and each element Λ in $\mathcal{L}$ is called an event. In order to obtain an uncertain measure of compound events, a product uncertain measure is defined by as follows:

Axiom 4 (Product Axiom). Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty space for k = 1,2,3..... The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M} \{\prod_{j=1}^{\infty} \Lambda_j\} = \bigwedge_{j=1}^{\infty} \mathcal{M}(\Lambda_j),$$

where Λ_k are arbitrarily chosen events from Γ_k for k=1,2,3,...... respectively.

2.2. Definition [5]

An uncertain variable ζ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\zeta \in B\}=\{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.

2.3. Definition [5]

The uncertainty distribution Φ of an uncertain variable ζ is defined by $\Phi(x) = \mathcal{M}\{\zeta \leq x\}$, for any real x.

2.4. Definition [5]

The expected value operator of an uncertain variable ζ is defined by $E[\zeta] = \int\limits_{0}^{+\infty} \mathcal{M}\{\zeta \geq r\}dr - \int\limits_{-\infty}^{0} \mathcal{M}\{\zeta \leq r\}dr$, provided that at least one of the two integrals is finite.

2.5. Definition [16]

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent almost surely (a.s.) to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\}=1$ such that

$$\lim_{n\to\infty} \|\zeta_n(\gamma) - \zeta(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$. In that case we write $\zeta_n \to \zeta$, a.s.

2.6. Definition [16]

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent in measure to ζ if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathcal{M}\{\|\zeta_n - \zeta\| \geq \varepsilon\} = 0.$$

2.7. Definition [16]

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent in mean to ζ if

$$\lim_{n\to\infty} E[\|\zeta_n - \zeta\|] = 0.$$

2.8. Definition [16]

Let $\Phi, \Phi_1, \Phi_2, \Phi_3,\ldots\ldots$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, \zeta_3\ldots\ldots$ respectively. Then the complex uncertain sequence $\{\zeta_n\}$ is convergent in distribution to ζ if

$$\lim_{n\to\infty} \Phi_n(c) = \Phi(c),$$

for all $c \in \mathbb{C}$, at which $\Phi(c)$ is continuous.

2.9. Definition [16]

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent uniformly almost surely (u.a.s.) to ζ if there exists a sequence of events $\{E_k\}$, with $\mathcal{M}\{E_k\} \to 0$ such that $\{\zeta_n\}$ converges uniformly to ζ in $\Gamma - E_k$, for any fixed $k \in \mathbb{N}$.

3. Almost Convergence of Complex Uncertain Double Sequences

In this section, we extend the concepts of almost convergence of sequences in a complex uncertain double sequence. We establish the interrelationships among them and originate some related results.

3.1. Definition

The double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable is said to be almost convergent with respect to almost surely to ζ if there exists an event Λ, with $\mathcal{M}(\Lambda) = 1$ and for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\left\| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \right\| < \varepsilon, \quad \forall p, q > n_0 \text{ and uniformly } \forall m, n \in \mathbb{N}.$$

3.2. Definition

The complex uncertain double sequence $\{\zeta_{m,n}\}$ is said to be almost convergent in measure to ζ if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}\left\{ \left\| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \right\| \geq \delta \right\} < \varepsilon, \quad \forall p, q > n_0 \text{ and uniformly } \forall m, n \in \mathbb{N}.$$

3.3. Definition

A double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable is said to be almost convergent in mean to ζ if for any given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$E\left[\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta\|\right] < \varepsilon, \quad \forall p, q > n_0 \text{ and uniformly } \forall m, n \in \mathbb{N}.$$

3.4. Definition

Let Φ and $\Phi_{m,n}$ be the complex uncertainty distributions of the complex uncertain variables $\zeta, \zeta_{m,n}$ respectively. Then the double sequence $\{\zeta_{m,n}\}$ is called almost convergent in distribution if for any preassigned $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \Phi_{m+i,n+j}(z) - \Phi(z)\| < \varepsilon,$$

for all z at which Φ is continuous and for all $p, q > n_0$ and $m, n \in \mathbb{N}$.

3.5. Definition

The double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable is said to be almost convergent with respect to uniformly almost surely to ζ if there exists events $\{E_k'\}$ with $\mathcal{M}\{E_k'\} \to 0$ such that $\{\zeta_{m,n}\}$ almost converges uniformly to ζ in $\Gamma - E_k'$, for every fixed $k \in \mathbb{N}$.

3.6. Theorem

If the double sequence of complex uncertain variable $\{\zeta_{m,n}\}$ is almost convergent in mean to ζ, then $\{\zeta_{m,n}\}$ is almost convergent in measure to ζ.

Proof: From the Markov inequality, we have for any given positive preassigned number ε,

$$\mathcal{M}\left\{\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right\} \leq \frac{E\left[\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\|\right]}{\varepsilon} \to 0 \text{ as } n \to \infty.$$

Thus the double sequence $\{\zeta_{m,n}\}$ is almost convergent with respect to measure to ζ.

3.7. Remark

The converse of the above theorem is not true in general, that is a double sequence $\{\zeta_{m,n}\}$ which is almost convergent in measure, may not be almost convergent in mean therein. We demonstrate our claim in the following example.

3.8. Example

Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\Gamma = \{\gamma_1, \gamma_2, ...\}$ with the uncertain measure $\mathcal{M}$ defined by

$$\mathcal{M}(\Lambda) = \begin{cases} \sup_{\gamma_{m+n} \in \Lambda} \frac{1}{(m+n)^2+5} & if \ \sup_{\gamma_{m+n} \in \Lambda} \frac{1}{(m+n)^2+5} < 0.5; \\ 1 - \sup_{\gamma_{m+n} \in \Lambda^c} \frac{1}{(m+n)^2+5} & if \ \sup_{\gamma_{m+n} \in \Lambda^c} \frac{1}{(m+n)^2+5} < 0.5; \\ 0.5 & otherwise. \end{cases}$$

Define the complex uncertain variable $\zeta_{m,n}$ as follows:

$$\zeta_{m,n}(\gamma) = \begin{cases} ((m+n)^2 + 5)i & if \ \gamma = \gamma_{m+n}; \\ 0 & otherwise. \end{cases}$$

for all $m, n \in \mathbb{N}$ and $\zeta(\gamma) \equiv 0$, $\forall \, \gamma \in \Gamma$.

Now, for any preassigned $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and $m, n \geq n_0$ such that

$$\frac{1}{(m+n)^2+5} < 0.5. \tag{1}$$

Then, $\mathcal{M}\left\{\|\frac{1}{pq}\sum\limits_{i=0}^{p-1}\sum\limits_{j=0}^{q-1}\zeta_{m+i,n+j} - \zeta\| \geq \delta\right\}$

$$= \mathcal{M}\left\{\gamma : \|\frac{1}{pq}\sum\limits_{i=0}^{p-1}\sum\limits_{j=0}^{q-1}\zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| \geq \delta\right\}$$

$$= \mathcal{M}\{\gamma = \gamma_{m+n}\}$$

$$= \sup_{\gamma_{m+n}\in\Lambda}\frac{1}{(m+n)^2+5}, \qquad \left[\text{since by (1)}, \quad \frac{1}{(m+n)^2+5} < 0.5 \Rightarrow \sup\frac{1}{(m+n)^2+5} < 0.5.\right]$$

$$< \varepsilon, \quad \text{as } m, n \to \infty.$$

Thus the complex uncertain double sequence $\{\zeta_{m,n}\}$ almost converges in measure to ζ.

Calculation to the distribution function $\Phi_{m,n}$ for the uncertain variable $\|\zeta_{m,n} - \zeta\| = \|\zeta_{m,n}\|$ gives

$$\Phi_{m,n}(x) = \begin{cases} 0 & if \ x < 0; \\[2mm] 1 - \frac{1}{(m+n)^2+5} & if \ 0 \leq x < (m+n)^2 + 5; \\[2mm] 1 & if \, x \geq (m+n)^2 + 5. \end{cases}$$

Then

$$E\left[\|\frac{1}{pq}\sum\limits_{i=0}^{p-1}\sum\limits_{j=0}^{q-1}\zeta_{m+i,n+j} - \zeta\|\right]$$

$$= \int\limits_0^\infty \{1 - \Phi_{m,n}(x)\}dx - \int\limits_{-\infty}^0 \Phi_{m,n}(x)dx$$

$$= \int\limits_0^{(m+n)^2+5}\left(1 - (1 - \frac{1}{(m+n)^2+5})\right)dx + \int\limits_{(m+n)^2+5}^{+\infty}(1-1)dx - \int\limits_\infty^0 0dx$$

$$= \int\limits_0^{(m+n)^2+5}\frac{1}{(m+n)^2+5}dx = 1.$$

Thus the complex uncertain double sequence $\{\zeta_{m,n}\}$ is not almost convergent in mean to ζ.

3.9. Theorem

Let $\{\zeta_{m,n}\}$ be a double sequence of complex uncertain variable with real part $\xi_{m,n}$ and imaginary part $\eta_{m,n}$ respectively, for $m, n = 1, 2, \ldots\ldots\ldots$ Then $\{\zeta_{m,n}\}$ is almost convergent in measure to $\zeta = \xi + i\eta$ if and only if the uncertain sequences $\{\xi_{m,n}\}$ and $\{\eta_{m,n}\}$ almost converges in measure to ξ and η respectively.

Proof: Let the double sequence of uncertain variables be almost convergent in measure. Then for any $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \mathcal{M}\left\{\|\frac{1}{pq}\sum\limits_{i=0}^{p-1}\sum\limits_{j=0}^{q-1}\xi_{m+i,n+j} - \xi\|\right\} \geq \frac{\varepsilon}{\sqrt{2}}$$

and

$$\lim_{m,n\to\infty} \mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\eta_{m+i,n+j} - \eta\|\right\} \geq \frac{\varepsilon}{\sqrt{2}}.$$

Note that, $\|\zeta_{m,n} - \zeta\| = \sqrt{|\xi_{m,n} - \xi|^2 + |\eta_{m,n} - \eta|^2}$.

Then, $\{\|\zeta_{m,n} - \zeta\| \geq \varepsilon\} \subset \{|\xi_{m,n} - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\} \cup \{|\eta_{m,n} - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\}$.

Now, using subadditivity axiom of uncertain measure, we obtain

$$\mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j} - \zeta\| \geq \varepsilon\right\}$$

$$\leq \mathcal{M}\left\{|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\xi_{m+i,n+j} - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\right\} + \mathcal{M}\left\{|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\eta_{m+i,n+j} - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\right\}.$$

Hence, $\lim_{m,n\to\infty} \mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j} - \zeta\| \geq \varepsilon\right\}$

$$\leq \lim_{m,n\to\infty} \mathcal{M}\left\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}|\xi_{m+i,n+j} - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\right\} + \lim_{m,n\to\infty} \mathcal{M}\left\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}|\eta_{m+i,n+j} - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\right\}$$

= 0, so that

$$\lim_{m,n\to\infty} \mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j} - \zeta\| \geq \varepsilon\right\} = 0.$$

Hence the double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable is almost convergent with respect to measure.

Conversely, let the double sequence $\{\zeta_{m,n}\}$ is almost convergent to ζ in measure. Then

$$\mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j} - \zeta\| \geq \delta\right\} \to 0, \text{ as } m,n \to \infty$$

$$\Rightarrow \mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}(\xi_{m+i,n+j} + i\eta_{m+i,n+j}) - (\xi + i\eta)\| \geq \delta\right\} \to 0 \text{ , as } m,n \to \infty$$

$$\Rightarrow \mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}(\xi_{m+i,n+j} - \xi) + i\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}(\eta_{m+i,n+j}) - \eta)\| \geq \delta\right\} \to 0, \quad \text{as } m,n \to \infty.$$

This implies there exists $\delta' < \frac{\delta}{2}$ such that

$$\mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\xi_{m+i,n+j} - \xi\| \geq \delta'\right\} \to 0, \text{ as } m,n \to \infty \text{ and}$$

$$\mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\eta_{m+i,n+j} - \eta\| \geq \delta'\right\} \to 0, \text{ as } m,n \to \infty$$

Hence the uncertain sequences $\{\xi_{m,n}\}$ and $\{\eta_{m,n}\}$ are almost convergent in measure to ξ and η respectively.

3.10. Theorem

If the real part and imaginary part of a complex uncertain double sequence almost converges in measure, then the double sequence almost converges in distribution also.

Proof: Let $\{\zeta_{m,n}\}$ be a double sequence of complex uncertain variable with $\zeta_{m,n} = \xi_{m,n} + i\eta_{m,n}$ and $\xi_{m,n}, \eta_{m,n}$ almost converges to ξ, η in measure respectively.

Then for any given $\varepsilon > 0$, we have

$$\mathcal{M}\left\{\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\xi_{m+i,n+j} - \xi\| \geq \alpha - a\right\} < \frac{\varepsilon}{2}$$

and

$$\mathcal{M}\left\{\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\eta_{m+i,n+j}-\eta\|\geq\beta-b\right\}<\tfrac{\varepsilon}{2}.$$

Let $z = a + ib$ be a point of continuity of the complex uncertainty distribution Φ. Then for any $\alpha > a, \beta > b$,

$$\{\xi_{m,n}\leq a, \eta_{m,n}\leq b\}$$
$$= \{\xi_{m,n}\leq a, \eta_{m,n}\leq b, \xi\leq\alpha,\eta\leq\beta\}\cup\{\xi_{m,n}\leq a,\eta_{m,n}\leq b,\xi>\alpha,\eta>\beta\}\cup$$
$$\{\xi_{m,n}\leq a,\eta_{m,n}\leq b,\xi>\alpha,\eta\leq\beta\}$$
$$\subseteq\{\xi_{m,n}\leq\alpha,\eta_{m,n}\leq\beta\}\cup\{|\xi_{m,n}-\xi|\geq\alpha-a\}\cup\{|\eta_{m,n}-\eta|\geq\beta-b\}.$$

Applying subadditivity axiom of uncertain measure, we get

$$\Phi_{m,n}(z) = \Phi_{m,n}(a + ib)$$
$$\leq\Phi(\alpha+i\beta)+\mathcal{M}\left\{\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\xi_{m+i,n+j}-\xi\|\geq\alpha-a\right\}+\mathcal{M}\left\{\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\eta_{m+i,n+j}-\eta\|\geq\beta-b\right\}$$
$$<\Phi(\alpha+i\beta)+\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}=\Phi(\alpha+i\beta)+\varepsilon,$$

which implies $\Phi_{m,n}(z) \leq \Phi(\alpha + i\beta) + \varepsilon$.

Hence, $\sup \Phi_{m,n}(z) \leq \Phi(\alpha + i\beta)$, for all $\alpha > a, \beta > b$.

Considering $\alpha \to a, \beta \to b$, we get $\alpha + i\beta \to a + ib$ and so,

$$\lim_{m,n\to\infty}\sup\Phi_{m,n}(z)\leq\Phi(z) \tag{2}$$

Again $\{\xi \leq x, \eta \leq y\}$
$$= \{\xi_{m,n}\leq a,\eta_{m,n}\leq b,\xi\leq x,\eta\leq y\}\cup\{\xi_{m,n}\leq a,\eta_{m,n}>b,\xi\leq x,\eta\leq b\}\cup$$
$$\{\xi_{m,n}>a,\eta_{m,n}\leq b,\xi\leq x,\eta\leq y\}\cup\{\xi_{m,n}>a,\eta_{m,n}>b,\xi\leq x,\eta\leq y\}$$
$$\subseteq\{\xi_{m,n}\leq a,\eta_{m,n}\leq b\}\cup\{|\xi_{m,n}-\xi|\geq a-x\}\cup\{|\eta_{m,n}-\eta|\geq b-y\}.$$

Thus like the previous step, we have

$$\Phi(x+iy)\leq\Phi_{m,n}(a+ib)+\mathcal{M}\left\{\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\xi_{m+i,n+j}-\xi\|\geq a-x\right\}$$
$$+\mathcal{M}\left\{\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\eta_{m+i,n+j}-\eta\|\geq b-y\right\}$$
$$<\Phi_{m,n}(a+ib)+\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}=\Phi_{m,n}(a+ib)+\varepsilon,\text{ for all }x<a,y<b.$$

Hence $\Phi(x + iy) \leq \inf \Phi_{m,n}(z)$ and taking $x \to a, y \to b$, we get

$$\Phi(z)\leq\lim_{m,n\to\infty}\inf\Phi_{m,n}(z) \tag{3}$$

From the equations (2) and (3), we conclude that $\Phi_{m,n}(z) \to \Phi(z)$, as $m, n \to \infty$. Therefore, the complex uncertain double sequence $\{\zeta_{m,n}\}$ almost converges in distribution to $\zeta = \xi + i\eta$.

3.11. Corollary

If a complex uncertain double sequence $\{\zeta_{m,n}\}$ is almost convergent in measure, then it is also almost convergent in distribution.
Proof: Combining the theorems 3.9 and 3.10, the above claim is obvious.

3.12. Corollary

If a complex uncertain double sequence $\{\zeta_{m,n}\}$ is almost convergent in mean, then it is also almost convergent in distribution.
Proof: From the theorem 3.6 and corollary 3.11, it can be easily perceived .

3.13. Remark

Almost convergence with respect to distribution does not necessarily imply almost convergence with respect to measure. The following example is presented to justify the above.

3.14. Example

Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and uncertain measure of the events be as follows:

$$\mathcal{M}\{\gamma_1\} = 0.7, \mathcal{M}\{\gamma_2\} = 0.2, \mathcal{M}\{\gamma_3\} = 0.3.$$
$$\text{Then, } \mathcal{M}\{\gamma_1, \gamma_2\} = 0.7, \mathcal{M}\{\gamma_1, \gamma_3\} = 0.8 \text{ and } \mathcal{M}\{\gamma_2, \gamma_3\} = 0.3.$$

Define the complex uncertain variables $\zeta_{m,n}$ and ζ by

$$\zeta_{m,n}(\gamma) = \begin{cases} i & \text{if } \gamma = \gamma_1 \text{ or } \gamma = \gamma_3; \\ -i, & \text{if } \gamma = \gamma_2; \end{cases}$$

and $\zeta(\gamma) = -\zeta_{m,n}(\gamma)$, for $m, n \in \mathbb{N}$.

If the complex uncertainty distribution of the complex uncertain variables $\zeta_{m,n}$ and ζ are $\Phi_{m,n}(z)$ and $\Phi(z)$ respectively, then

$$\Phi_{m,n}(z) = \Phi(z) = \Phi_{m,n}(a + ib) = \begin{cases} 0 & \text{if } a < 0, b \in (-\infty, \infty); \\ 0, & \text{if } a \geq 0, b < -1; \\ 0.2, & \text{if } a \geq 0, -1 \leq b < 1; \\ 0.7, & \text{if } a \geq 0, 1 \leq b < 2; \\ 1, & \text{if } a \geq 0, 1, b \geq 2. \end{cases}$$

Since the uncertain distribution function of $\zeta_{m,n}$ and ζ are same, therefore $\{\zeta_{m,n}\}$ is almost convergent to ζ in distribution.
Now for a given $\varepsilon > 0$, we have
$$\mathcal{M}\left\{ \|\frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon \right\}$$
$$= \mathcal{M}\left\{ \gamma : \|\frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| \geq \varepsilon \right\} = 1.$$
Thus $\{\zeta_{m,n}\}$ is not almost convergent in measure.

3.15. Remark

Both the concepts of almost convergence of a complex double sequence with respect to almost surely and almost convergence in measure are independent of each other.

3.16. Example

Consider an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. Define the measurable function $\mathcal{M}$ as follows:

$$\mathcal{M}(\Lambda) = \begin{cases} 0 & \text{if } \Lambda = \phi; \\ 1, & \text{if } \Lambda = \Gamma; \\ 0.6, & \text{if } \gamma_1 \in \Lambda; \\ 0.4, & \text{if } \gamma_1 \notin \Lambda. \end{cases}$$

Consider the complex uncertain variables $\zeta_{m,n}$ and ζ, which are defined by

$$\zeta_{m,n}(\gamma) = \begin{cases} i & \text{if } \gamma = \gamma_1; \\ 2i, & \text{if } \gamma = \gamma_2; \\ 3i, & \text{if } \gamma = \gamma_3; \\ 4i, & \text{if } \gamma = \gamma_4; \\ 0, & \text{otherwise.} \end{cases}$$

and $\zeta(\gamma) \equiv 0$, $\forall\, \gamma \in \Gamma$ and for $m, n \in \mathbb{N}$.

Observe that $\zeta_{m,n} \to \zeta$, except only for $\gamma = \gamma_1, \gamma_2, \gamma_3, \gamma_4$ and so $\{\zeta_{m,n}\}$ is almost convergent to ζ with respect to almost surely.

However, for some small $\varepsilon > 0$, we have
$\mathcal{M}\{\|\zeta_{m,n} - \zeta\| \geq \varepsilon\} = \mathcal{M}\{\gamma : \|\zeta_{m,n}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} = 1$. Thus the sequence $\{\zeta_{m,n}\}$ is not almost convergent in measure.

3.17. Example

If the complex uncertainty distribution of the complex uncertain variables $\zeta_{m,n}$ and ζ are $\Phi_{m,n}(z)$ and $\Phi(z)$ respectively in the uncertainty space taken in the above example, then

$$\Phi_{m,n}(z) = \Phi_{m,n}(a + ib) = \begin{cases} 0 & \text{if } a < 0, b \in (-\infty, \infty); \\ 0, & \text{if } a \geq 0, b < 1; \\ 0.6, & \text{if } a \geq 0, 1 \leq b < 1; \\ 0.6, & \text{if } a \geq 0, 2 \leq b < 3; \\ 0.6, & \text{if } a \geq 0, 3 \leq b < 4; \\ 1, & \text{if } a \geq 0, 1, b \geq 4. \end{cases}$$

and

$$\Phi(z) = \begin{cases} 0 & \text{if } a < 0, b \in (-\infty, \infty); \\ 0, & \text{if } a \geq 0, b < 0; \\ 1, & \text{if } a \geq 0, b \geq 0. \end{cases}$$

From the above it is obvious that the complex uncertain double sequence $\{\zeta_{m,n}\}$ is not almost convergent in distribution to ζ.

3.18. Example

Consider an uncertainty space $(\Gamma, L, \mathcal{M})$ with $L = [0, 1]$ and $\mathcal{M}$ to be the Lebesgue measure. We define the uncertain variables $\zeta_{m,n}$ and ζ as follows:

$$\zeta_{m,n}(\gamma) = \begin{cases} i, & \text{if } \dfrac{p}{2^{t_1+t_2}} \leq \gamma \leq \dfrac{1+p}{2^{t_1+t_2}}; \\ \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{and } \zeta(\gamma) \equiv 0, \qquad \forall m = 2^{t_1} + p, n = 2^{t_2} + p \in \mathbb{N}, \text{ where } p, t_1, t_2 \in \mathbb{Z}.$$

Then, $\mathcal{M}\left\{ \|\frac{1}{pq} \sum\limits_{i=0}^{p-1} \sum\limits_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| \geq \delta \right\}$

$= \mathcal{M}\left\{ \gamma : \|\frac{1}{pq} \sum\limits_{i=0}^{p-1} \sum\limits_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| \geq \delta \right\} = \frac{1}{2^{t_1+t_2}}$ and hence

$$\lim_{m,n\to\infty} \mathcal{M}\left\{\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| \geq \delta\right\} = \lim_{t_1,t_2\to\infty} \tfrac{1}{2^{t_1+t_2}} = 0.$$

Thus the complex uncertain double sequence $\{\zeta_{m,n}\}$ almost converges to ζ in measure.

On the other hand, let $\gamma \in [0,1]$. Then there exists an infinite number of closed intervals those are of the form $[\tfrac{p}{2^{t_1+t_2}}, \tfrac{p+1}{2^{t_1+t_2}}]$ containing γ, for different values of p. Therefore, $\zeta_{m,n}$ does not converges to ζ and hence the complex uncertain double sequence $\{\zeta_{m,n}\}$ is not almost convergent to ζ with respect to almost surely.

3.19. Example

From the above example, we note that

$$E\left[\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\|\right] = \tfrac{1}{2^{t_1+t_2}}, \text{ which tends to } 0, \text{ as}$$

$m, n \to \infty$. Therefore, the sequence almost converges to ζ in mean also.

3.20. Remark

A double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable is almost convergent with respect to almost surely doesnot necessarily mean that it is almost convergent with respect to mean. The demonstration of the same is shown in the following example.

3.21. Example

Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \dots\dots\}$ and the uncertainty measure of the events are as follows:

$$\mathcal{M}(\Lambda) = \sum_{\gamma_i,\gamma_j\in\Lambda} \tfrac{1}{2^{(i+j)}}$$

Define the complex uncertain variables $\zeta_{m,n}$ and ζ by

$$\zeta_{m,n}(\gamma) = \begin{cases} 2^{(m+n)}i, & if\ \gamma = \gamma_{m+n}; \\ 0, & otherwise. \end{cases}$$

and $\zeta(\gamma) \equiv 0,\ \forall \gamma \in \Gamma$ and for $m, n \in \mathbb{N}$.

Calculating similarly like the above example, we get $\{\zeta_{m,n}\}$ almost converges to ζ with respect to almost surely.

Now the uncertainty distribution function of the uncertain variable $\zeta_{m,n}$ is given by

$$\Phi_{m,n}(z) = \begin{cases} 0, & if\ x < 0; \\ 1 - \tfrac{1}{2^{(m+n)}}, & if\ 0 \leq x < 2^{(m+n)};\ ,\quad for\ m,n \in \mathbb{N} \\ 1, & otherwise. \end{cases}$$

Now $E\left[\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j} - \zeta\|\right] = \int_0^\infty \{1 - \Phi_{m,n}(x)\}dx - \int_{-\infty}^0 \Phi_{m,n}(x)dx$

$= \int_0^{2^{(m+n)}} \left(1 - (1 - \tfrac{1}{2^{(m+n)}})\right)dx + \int_{2^{(m+n)}}^{+\infty} (1-1)dx - \int_\infty^0 0dx$

$= \int_0^{2^{(m+n)}} \tfrac{1}{2^{(m+n)}}dx = 1.$

Hence the double sequence $\{\zeta_{m,n}\}$ is not almost convergent in mean to ζ.

3.22. Proposition

Let $\{\zeta_{m,n}\}$ be a double sequence of complex uncertain variable. Then it almost converges in almost surely to ζ if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$,

$$\mathcal{M}\left(\bigcap_{N=1}^{\infty} \bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \| \geq \varepsilon \right) = 0.$$

Proof: The definition of almost convergence in almost surely of complex uncertain double sequence leads us to the existence of an event Λ with measure 1 such that

$$\lim_{m,n\to\infty} \| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| = 0.$$

Let ε be a preassigned positive number. Then, there exists $N \in \mathbb{N}$ such that for any event $\gamma \in \Lambda$, we have

$$\mathcal{M}\left(\bigcap_{N=1}^{\infty} \bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \| < \varepsilon \right) = 1.$$

So on applying the duality axiom of uncertain measure, we get

$$\mathcal{M}\left(\bigcap_{N=1}^{\infty} \bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \| \geq \varepsilon \right) = 0.$$

Hence the theorem.

3.23. Proposition

Let $\zeta_{m,n}$ be complex uncertain variables, where $m, n = 1, 2, 3, \ldots\ldots$ Then the double sequence $\{\zeta_{m,n}\}$ almost converges with respect to uniformly almost surely to ζ if and only if for a given $\varepsilon > 0$, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that

$$\mathcal{M}\left(\bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \delta \right) < \varepsilon.$$

Proof: Let the double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable almost converges with respect to almost surely to ζ. Then for any $\varepsilon > 0$, there exists $\delta > 0$ and an event B with uncertain measure less than ν and the sequence $\{\zeta_{m,n}\}$ converges uniformly to ζ on $\Gamma - B$. Thus for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \| < \delta$, for all $m, n \geq N$ and all $\gamma \in \Gamma - B$. Thus we have,

$$\bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \| \geq \delta \right\} \subseteq B.$$

Applying the subadditivity axiom of uncertain measure, we get

$$\mathcal{M}\left(\bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \delta \right\} \right) \leq \mathcal{M}(B) < \nu < \varepsilon.$$

Conversely, let $\mathcal{M}\left(\bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \delta \right\} \right) < \varepsilon.$

We take $\delta > 0$. Then for any $\nu > 0$, $a \geq 1$, there exists a positive integer a_s such that

$$\mathcal{M}\left(\bigcup_{m=a_s}^{\infty} \bigcup_{n=a_s}^{\infty} \left\{ \| \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \tfrac{1}{a} \right\} \right) < \tfrac{\nu}{2^a}.$$

Consider $B = \bigcup_{a=1}^{\infty} \bigcup_{m=a_s}^{\infty} \bigcup_{n=a_s}^{\infty} \left\{ \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \frac{1}{a} \right\}.$

Then, $\mathcal{M}(B) \leq \sum_{a=1}^{\infty} \mathcal{M} \left(\bigcup_{m=a_s}^{\infty} \bigcup_{n=a_s}^{\infty} \left\{ \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \frac{1}{n} \right\} \right) \leq \sum_{a=1}^{\infty} \frac{\nu}{2^a} = \nu.$

Moreover, $\sup_{\gamma \in \Gamma - B} \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| < \frac{1}{m}$, where $m = 1, 2, 3, \ldots\ldots$ and $m, n > a_s$.

Hence the theorem.

3.24. Theorem

Let $\{\zeta_{m,n}\}$ of be an almost convergent with respect to uniformly almost surely to ζ. Then, $\{\zeta_{m,n}\}$ is an almost convergent double sequence in almost surely to ζ.

Proof: Taking Theorem 3.22 in consideration, we have if the sequence $\{\zeta_{m,n}\}$ almost converges with respect to uniformly almost surely to ζ, then

$$\mathcal{M} \left(\bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \delta \right\} \right) < \varepsilon.$$

Now, since $\mathcal{M} \left(\bigcap_{N=1}^{\infty} \bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \delta \right\} \right)$

$\leq \mathcal{M} \left(\bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \delta \right\} \right) < \varepsilon$

Hence $\{\zeta_{m,n}\}$ almost converges in almost surely to ζ, by theorem 3.23.

3.25. Theorem

A complex uncertain double sequence $\{\zeta_{m,n}\}$ which almost converges with respect to uniformly almost surely to ζ is also almost converges in measure therein.

Proof: Let $\{\zeta_{m,n}\}$ almost converges uniformly almost surely to $\zeta > 0$. Then for any $\epsilon > 0$ and $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$\mathcal{M} \left(\bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \zeta \| \geq \delta \right\} \right) < \varepsilon$$

and $\mathcal{M} \left(\| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \| \geq \delta \right) \leq \mathcal{M} \left(\bigcup_{m=N}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \| \geq \delta \right\} \right) < \varepsilon$.

Hence $\{\zeta_{m,n}\}$ almost converges in measure to ζ.

3.26. Corollary

If a complex uncertain double sequence is almost convergent with respect to uniformly almost surely, then it is almost convergent in distribution also.

Proof: It is straightforward from the theorems 3.25 and 3.12.

3.27. Remark

Converse of the theorem 3.25 is not true in general. It is illustrated in the following example.

3.28. Example

In example 3.18, the double sequence is almost convergent in measure and it does not almost converge with respect to almost surely. Thus we conclude, it is not almost convergent with respect to uniformly almost surely.

3.29. Diagram

Suppose:
(1) Almost Convergence with respect to almost surely
(2) Almost convergence in mean
(3)Almost convergence in measure
(4) Almost convergence in distribution
(5) Almost convergence with respect to uniformly almost surely
(6) Almost Cauchy with respect to almost surely
(7) Real part and imaginary part almost convergence in measure
(8) Real part and imaginary part almost convergence in mean

Then the interrelationships among these concepts are depicted in the diagram below:
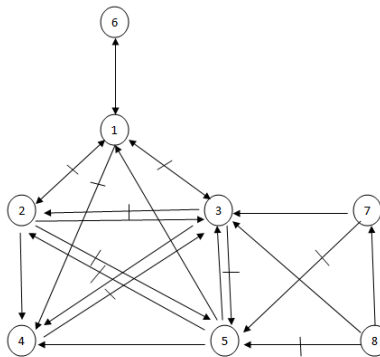
4. Almost Cauchy Double Sequences of Complex Uncertain Variable

In this particular section, we introduce the notion of almost Cauchy complex uncertain double sequence and we see that a complex uncertain double sequence is almost Cauchy if and only if it is almost convergent.

4.1. Definition

The double sequence of complex uncertain variable $\{\zeta_{m,n}\}$ is almost Cauchy with respect to almost surely, if for any preassigned $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ and an event Λ with uncertain measure 1 such that

$$\|\frac{1}{p_1 q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_1+i,n_1+j}(\gamma) - \frac{1}{p_2 q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\zeta_{m_2+i,n_2+j}(\gamma)\| < \varepsilon,$$

for all $p_1, p_2, q_1, q_2 > n_0, \gamma \in \Lambda$ and uniformly for all $m_1, m_2, n_1, n_2 \in \mathbb{N}$.

4.2. Definition

The double sequence of complex uncertain variable $\{\zeta_{m,n}\}$ is almost Cauchy in measure, if for any given $\varepsilon, \delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{M}\left\{\|\frac{1}{p_1 q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_1+i,n_1+j} - \frac{1}{p_2 q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\zeta_{m_2+i,n_2+j} \geq \delta\right\} < \varepsilon,$$

for all $p_1, p_2, q_1, q_2 > n_0$ and uniformly for all $m_1, m_2, n_1, n_2 \in \mathbb{N}$.

4.3. Definition

The double sequence of complex uncertain variable $\{\zeta_{m,n}\}$ is almost Cauchy in mean, if for any given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$E\Big[\|\frac{1}{p_1 q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_1+i,n_1+j} - \frac{1}{p_2 q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\zeta_{m_2+i,n_2+j}\|\Big] < \varepsilon,$$

for all $p_1, p_2, q_1, q_2 > n_0$ and uniformly for all $m_1, m_2, n_1, n_2 \in \mathbb{N}$.

4.4. Definition

Let $\Phi_{m,n}$ be the complex uncertainty distributions of the complex uncertain variable $\zeta_{m,n}$. Then the double sequence $\{\zeta_{m,n}\}$ is called almost Cauchy in distribution if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\frac{1}{p_1 q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\Phi_{m_1+i,n_1+j}(z) - \frac{1}{p_2 q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\Phi_{m_2+i,n_2+j}(z)\| < \varepsilon,$$

for all $p_1, p_2, q_1, q_2 > n_0$, uniformly for all $m_1, m_2, n_1, n_2 \in \mathbb{N}$ and for all z at which the distribution function is continuous.

4.5. Definition

The double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable is said to be almost Cauchy with respect to uniformly almost surely if there exists events $\{E'_k\}$ with $\mathcal{M}\{E'_k\} \to 0$ such that $\{\zeta_{m,n}\}$ is almost Cauchy in $\Gamma - E_k$, for every fixed $k \in \mathbb{N}$.

4.6. Theorem

A complex uncertain double sequence $\{\zeta_{m,n}\}$ is almost convergent with respect to almost surely if and only if it is an almost Cauchy sequence with respect to almost surely.
Proof: Suppose the double sequence $\{\zeta_{m,n}\}$ is almost convergent with respect to almost sure to ζ. Then there exists an event Λ, with $\mathcal{M}(\Lambda) = 1$ and for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| < \frac{\varepsilon}{2}, \text{ for all } p, q > n_0, \gamma \in \Lambda \text{ and uniformly for all } m, n \in \mathbb{N}.$$

Therefore $\|\frac{1}{p_1 q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_1+i,n_1+j}(\gamma) - \frac{1}{p_2 q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\zeta_{m_2+i,n_2+j}(\gamma)\|$

$$\leq \|\frac{1}{p_1 q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_1+i,n_1+j}(\gamma) - \zeta(\gamma)\| + \|\frac{1}{p_2 q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\zeta_{m_2+i,n_2+j}(\gamma) - \zeta(\gamma)\|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \text{for all } p_1, p_2, q_1, q_2 > n_0, \gamma \in \Lambda \text{ and all } m_1, m_2, n_1, n_2 \in \mathbb{N}.$$

Hence, the double sequence of complex uncertain variable $\{\zeta_{m,n}\}$ is almost Cauchy sequence with respect to almost surely.
Conversely, let the double sequence of complex uncertain variable $\{\zeta_{m,n}\}$ is almost Cauchy with respect to almost surely. Then for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ and an event Λ with $\mathcal{M}(\Lambda) = 1$ such that

$$\|\frac{1}{p_1 q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_1+i,n_1+j}(\gamma) - \frac{1}{p_2 q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\zeta_{m_2+i,n_2+j}(\gamma)\| < \frac{\varepsilon}{2},$$

for all $p_1, p_2, q_1, q_2 > N_0, \gamma \in \Lambda$ and all $m_1, m_2, n_1, n_2 \in \mathbb{N}$.

Taking $m_1 = m_2 = m_0$ and $n_1 = n_2 = n_0$, in the above relation, we obtain, $\Big\{\frac{1}{pq}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_0+i,n_0+j}(\gamma)\Big\}$ is a Cauchy

sequence in $\mathbb{C}$, the set of complex numbers and therefore it is convergent to ζ (say), since $\mathbb{C}$ is complete. Then, for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j}(\gamma) - \zeta(\gamma)\| < \tfrac{\varepsilon}{2}, \text{ for all } p,q, > N_1.$$

Then, $\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\|$

$$\leq \|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j}(\gamma) - \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j}(\gamma)\| + \|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j}(\gamma) - \zeta(\gamma)\|$$

$$< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon, \quad \text{for all } p,q > \max(N_0, N_1) \text{ and all } m,n \in \mathbb{N}.$$

So the double sequence is almost convergent with respect to almost surely ζ.

4.7. Theorem

A complex uncertain double sequence $\{\zeta_{m,n}\}$ is almost Cauchy in measure if and only if it is almost convergent in measure.

Proof: Suppose the double sequence $\{\zeta_{m,n}\}$ is almost convergent in measure to ζ. Then for every $\varepsilon, \delta > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\mathcal{M}\left\{\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j} - \zeta\| \geq \delta\right\} < \tfrac{\varepsilon}{2}, \quad \forall p,q > N_0 \text{ and}$$

$\forall m,n \in \mathbb{N}$.

Therefore, $\mathcal{M}\left\{\|\tfrac{1}{p_1 q_1} \sum_{i=0}^{p_1-1} \sum_{j=0}^{q_1-1} \zeta_{m_0+i,n_0+j} - \tfrac{1}{p_2 q_2} \sum_{i=0}^{p_2-1} \sum_{j=0}^{q_2-1} \zeta_{m_0+i,n_0+j}\| \geq \delta\right\}$

$$\leq \mathcal{M}\left\{\|\tfrac{1}{p_1 q_1} \sum_{i=0}^{p_1-1} \sum_{j=0}^{q_1-1} \zeta_{m_0+i,n_0+j} - \zeta\| \geq \delta\right\} + \mathcal{M}\left\{\|\tfrac{1}{p_2 q_2} \sum_{i=0}^{p_2-1} \sum_{j=0}^{q_2-1} \zeta_{m_0+i,n_0+j} - \zeta\| \geq \delta\right\}$$

$$< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon, \quad \forall p_1, p_2, q_1, q_2 > N_0 \text{ and } \forall m_1, m_2, n_1, n_2 \in \mathbb{N}.$$

Hence, the double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable is almost Cauchy in measure.

Conversely, let the complex uncertain double sequence $\{\zeta_{m,n}\}$ is almost Cauchy in measure. Then for every $\varepsilon, \delta > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\mathcal{M}\left\{\|\tfrac{1}{p_1 q_1} \sum_{i=0}^{p_1-1} \sum_{j=0}^{q_1-1} \zeta_{m_0+i,n_0+j} - \tfrac{1}{p_2 q_2} \sum_{i=0}^{p_2-1} \sum_{j=0}^{q_2-1} \zeta_{m_0+i,n_0+j}\| \geq \delta\right\} < \tfrac{\varepsilon}{2},$$

for all $p_1, p_2, q_1, q_2 > N_0$ and uniformly $\forall m_1, m_2, n_1, n_2 \in \mathbb{N}$.

Taking $m_1 = m_2 = m_0$ and $n_1 = n_2 = n_0$ in the above equation, we obtain $\left\{\tfrac{1}{p_1 q_1} \sum_{i=0}^{p_1-1} \sum_{j=0}^{q_1-1} \zeta_{m_0+i,n_0+j}(\gamma)\right\}$ is a Cauchy sequence in $\mathbb{C}$ and therefore it is convergent to ζ (say), since $\mathbb{C}$ is complete.

Then for any given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\mathcal{M}\left\{\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j} - \zeta\| \geq \delta\right\} < \tfrac{\varepsilon}{2}, \quad \forall p,q > N_1.$$

Therefore,

$$\mathcal{M}\left\{\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j} - \tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j}\| \geq \delta\right\} + \mathcal{M}\left\{\|\tfrac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j} - \zeta\| \geq \delta\right\}$$

$$< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon, \quad \forall p,q > \max(N_0, N_1) \text{ and } \forall m,n \in \mathbb{N}.$$

Hence the double sequence is almost convergent in measure.

4.8. Theorem

A complex uncertain double sequence $\{\zeta_{m,n}\}$ is almost Cauchy in mean if and only if it is almost convergent in mean.

Proof: Let $\{\zeta_{m,n}\}$ be almost convergent in mean to ζ. Then, for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$E\Big[\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j}-\zeta\|\Big] < \tfrac{\varepsilon}{2}, \quad \forall p,q > N_0 \text{ and } \forall m_0,n_0 \in \mathbb{N}$$

So, $E\Big[\|\tfrac{1}{p_1q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_1+i,n_1+j}-\tfrac{1}{p_2q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\zeta_{m_2+i,n_2+j}\|\Big]$

$\leq E\Big[\|\tfrac{1}{p_1q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_1+i,n_1+j}-\zeta\|\Big] + E\Big[\|\tfrac{1}{p_2q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\zeta_{m_2+i,n_2+j}-\zeta\|\Big]$

$< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon, \quad \forall p_1,p_2,q_1,q_2 > N_0 \text{ and } \forall m_1,m_2,n_1,n_2 \in \mathbb{N}.$

Hence, $\{\zeta_{m,n}\}$ is an almost Cauchy complex uncertain double sequence in mean.

On the contrary, suppose $\{\zeta_{m,n}\}$ be almost Cauchy in mean. Then for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$E\Big[\|\tfrac{1}{p_1q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_1+i,n_1+j}-\tfrac{1}{p_2q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\zeta_{m_2+i,n_2+j}\|\Big] < \tfrac{\varepsilon}{2},$$

for all $p_1,p_2,q_1,q_2 > N_0$ and uniformly $\forall m_1,m_2,n_1,n_2 \in \mathbb{N}.$

Consider $m_1 = m_2 = m_0$ and $n_1 = n_2 = n_0$. Then, $\Big\{\tfrac{1}{p_1q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\zeta_{m_0+i,n_0+j}\Big\}$ is a Cauchy sequence in $\mathbb{C}$ and

therefore it is convergent to ζ (say), since $\mathbb{C}$ is complete.

Then for any given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$E\Big[\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m_0+i,n_0+j}-\zeta\|\Big] < \tfrac{\varepsilon}{2}, \quad \forall p,q > N_1, \text{ uniformly for all } m_0,n_0 \in \mathbb{N}.$$

Therefore,

$E\Big[\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m+i,n+j}-\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m_0+i,n_0+j}\|\Big] + E\Big[\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\zeta_{m_0+i,n_0+j}-\zeta\|\Big]$

$< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon, \quad \forall p,q > \max(N_0,N_1) \text{ and } \forall m,n \in \mathbb{N}.$ Hence the double sequence is almost convergent in mean.

4.9. Theorem

A complex uncertain double sequence $\{\zeta_{m,n}\}$ is almost Cauchy in distribution if and only if it is almost convergent in distribution.

Proof: Let $\{\zeta_{m,n}\}$ be an almost convergent complex uncertain double sequence in distribution. Let $\Phi_{m,n}$ and Φ be the complex uncertainty distributions of the complex uncertain variables $\zeta_{m,n}$ and ζ respectively. Then, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\tfrac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\Phi_{m_1+i,n_1+j}(z)-\Phi(z)\| < \tfrac{\varepsilon}{2},$$

for all $p,q > n_0$ and for all z at which Φ is continuous.

So, $\|\tfrac{1}{p_1q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\Phi_{m_1+i,n_1+j}(z)-\tfrac{1}{p_2q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\Phi_{m_2+i,n_2+j}(z)\|$

$\leq \|\tfrac{1}{p_1q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\Phi_{m_1+i,n_1+j}(z)-\Phi(z)\| + \|\tfrac{1}{p_2q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\Phi_{m_2+i,n_2+j}-\Phi(z)\|$

$< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon, \quad \forall p_1,p_2,q_1,q_2 > n_0 \text{ and } \forall m_1,m_2,n_1,n_2 \in \mathbb{N}.$

Hence, $\{\zeta_{m,n}\}$ is an almost Cauchy complex uncertain double sequence in distribution.

For the sufficient part, let $\{\zeta_{m,n}\}$ be almost Cauchy in distribution. Then for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\|\tfrac{1}{p_1q_1}\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\Phi_{m_1+i,n_1+j}(z)-\tfrac{1}{p_2q_2}\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\Phi_{m_2+i,n_2+j}(z)\| < \tfrac{\varepsilon}{2},$$

for all $p_1, p_2, q_1, q_2 > N_0$ and uniformly $\forall m_1, m_2, n_1, n_2 \in \mathbb{N}$.

Consider $m_1 = m_2 = m_0$ and $n_1 = n_2 = n_0$. Then, $\left\{ \frac{1}{p_1 q_1} \sum\limits_{i=0}^{p_1-1} \sum\limits_{j=0}^{q_1-1} \Phi_{m_0+i,n_0+j}(z) \right\}$ is a Cauchy sequence in $\mathbb{C}$ and

therefore it is convergent to Φ (say), $\mathbb{C}$ being complete.
Then for any given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \Phi_{m_0+i,n_0+j} - \Phi \| < \frac{\varepsilon}{2}, \quad \forall p, q > N_1.$$

Therefore,
$$\| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \Phi_{m+i,n+j}(z) - \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \Phi_{m_0+i,n_0+j}(z) \| + \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \Phi_{m_0+i,n_0+j}(z) - \Phi(z) \|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall p > \max(N_0, N_1) \text{ and } \forall m, n, m), n_0 \in \mathbb{N}. \text{ Hence the double sequence is almost convergent in}$$
distribution.

4.10. Theorem

A complex uncertain double sequence $\{\zeta_{m,n}\}$ is almost Cauchy with respect to uniformly almost surely if and only if it is almost convergent with respect to uniformly almost surely.

Proof: Suppose the double sequence $\{\zeta_{m,n}\}$ is almost convergent with respect to uniformly almost surely to ζ. Then there exists events $\{E'_k\}$ with $\mathcal{M}\{E'_k\} \to 0$ such that $\{\zeta_{m,n}\}$ is almost uniformly convergent to ζ in $\Gamma - E'_k$, for every fixed $k \in \mathbb{N}$. This implies

$$\| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \| < \frac{\varepsilon}{2}, \quad \forall p, q > n_0, \gamma \in \Gamma - E'_k \text{ and uniformly } \forall m, n \in \mathbb{N}.$$

Therefore $\| \frac{1}{p_1 q_1} \sum\limits_{i=0}^{p_1-1} \sum\limits_{j=0}^{q_1-1} \zeta_{m_1+i,n_1+j}(\gamma) - \frac{1}{p_2 q_2} \sum\limits_{i=0}^{p_2-1} \sum\limits_{j=0}^{q_2-1} \zeta_{m_2+i,n_2+j}(\gamma) \|$

$$\leq \| \frac{1}{p_1 q_1} \sum_{i=0}^{p_1-1} \sum_{j=0}^{q_1-1} \zeta_{m_1+i,n_1+j}(\gamma) - \zeta(\gamma) \| + \| \frac{1}{p_2 q_2} \sum_{i=0}^{p_2-1} \sum_{j=0}^{q_2-1} \zeta_{m_2+i,n_2+j}(\gamma) - \zeta(\gamma) \|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \text{for all } p_1, p_2, q_1, q_2 > N_0, \gamma \in \Gamma - E'_k \text{ and all } m_1, m_2, n_1, n_2 \in \mathbb{N}.$$

Thus, $\{\zeta_{m,n}\}$ is a complex uncertain Cauchy double sequence with respect to uniformly almost surely.
Conversely, let the double sequence of complex uncertain variable $\{\zeta_{m,n}\}$ is almost Cauchy with respect to uniformly almost surely. Then for every $\varepsilon > 0$, there exists events $\{E'_k\}$ with $\mathcal{M}\{E'_k\} \to 0$ such that

$$\| \frac{1}{p_1 q_1} \sum_{i=0}^{p_1-1} \sum_{j=0}^{q_1-1} \zeta_{m_1+i,n_1+j}(\gamma) - \frac{1}{p_2 q_2} \sum_{i=0}^{p_2-1} \sum_{j=0}^{q_2-1} \zeta_{m_2+i,n_2+j}(\gamma) \| < \frac{\varepsilon}{2},$$

for all $p_1, p_2, q_1, q_2 > n_0, \gamma \in \Gamma - E'_k$ and all $m_1, m_2, n_1, n_2 \in \mathbb{N}$.

Taking $m_1 = m_2 = m_0$ and $n_1 = n_2 = n_0$, in the above relation, we have $\left\{ \frac{1}{pq} \sum\limits_{i=0}^{p_1-1} \sum\limits_{j=0}^{q_1-1} \zeta_{m_0+i,n_0+j}(\gamma) \right\}$ is a Cauchy

sequence in $\mathbf{C}$ and so it is convergent to ζ (say), since $\mathbb{C}$ is complete.
Then, for every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j}(\gamma) - \zeta(\gamma) \| < \varepsilon/2, \text{ for all } p, q, > N_1, \ \gamma \in \Gamma - E'_k$$

Then, $\| \frac{1}{pq} \sum\limits_{i=0}^{p-1} \sum\limits_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \zeta(\gamma) \|$

$$\leq \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m+i,n+j}(\gamma) - \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j}(\gamma) \| + \| \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \zeta_{m_0+i,n_0+j}(\gamma) - \zeta(\gamma) \|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{for all } p, q > \max(N_0, N_1), \gamma \in \Gamma - E'_k \text{ and all } m, n, m_0, n_0 \in \mathbb{N}.$$
So the double sequence is almost convergent with respect to uniformly almost surely ζ.

5. Conclusion

In this treatise, several types of almost convergence of complex uncertain double sequence has been studied and interrelationships among them were established. Also, it is proved that a complex uncertain double sequence is almost Cauchy sequence if and only if it is almost convergent with respect to almost surely. These concepts can be generalized and applied for further studies.

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