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A Numerical Algorithm Based on Modified Orthogonal Linear Spline for Solving a Coupled Nonlinear Inverse Reaction–Diffusion Problem

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Abstract. In this paper, a modified orthogonal linear spline (OL–spline) is used for the numerical solution of a coupled nonlinear inverse reaction–diffusion problem to determine the unknown boundary conditions. The convergence properties of the new linear combination are obtained. A quasi–linearization technique is utilized to linearize the nonlinear term in the equations. This process produces a linear system of equations which can be solved easily. Using the new inequalities, error estimation and convergence of the proposed method are investigated. Two numerical examples are given to demonstrate the computational efficiency of the method and also the experimental convergence rate of examples are obtained.

1. Introduction

According to Keller [1, 2], two problems are considered inverse to each other if the formulation of them needs complete or partial knowledge of the other. From the definition, it is subjective which can be called the direct problem and which one is the inverse. However, one of the problems has been often studied earlier and, maybe, in more detail. This one has usually named the direct problem, whereas the other denotes to the inverse one.

Mathematical models of the many natural phenomena are formulated using initial and boundary value problems given partial differential equations (PDEs). Inverse problems written by these equations arise in most fields of science and technology [2].

If we find out the behavior of a physical phenomenon, thoroughly; one can represent a conventional mathematical model of this phenomenon consisting of uniqueness, stability, and existence of a solution of the related mathematical problem. Nevertheless, if one of the (functional) parameters explaining this model is to be found from additional boundary/experimental data, then we achieve an inverse problem [3].

In a mathematical perspective, the inverse problems belong to a class of problems called the ill-posed ones. It means that small errors in the measured data which can lead to large oscillations in the estimated values [4, 5]. So, we must search for stable numerical procedures.

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Reaction–diffusion equations (RDEs) have attracted significant attention, partly because of their developments in many fields of science, in physics, chemistry, and biology, partly due to the interesting characteristics and rich diversity of properties of their solutions. The processes of diffusion and reaction play essential roles in lots of systems' dynamics, e.g., in plasma, or semiconductor physics; see [6] and references therein. Reaction–diffusion systems are usually coupled ones of parabolic partial differential equations which consist of pattern formation in morphogenesis, for predator–prey and other ecological systems, for conduction in nerves, for epidemics, for carbon monoxide poisoning, oscillating chemical reactions, pulse splitting, shedding, reactions and competitions in excitable systems and stability issues [7]. This problem has attracted much attention and has been studied by many authors. However, deriving its analytical solution in an explicit form seems to be unlikely except for certain special situations. Therefore, one has to employ the numerical techniques or approximate approaches for getting its solution.
Shirzadi et. al. developed a local integral equation formulation to solve coupled nonlinear reaction–diffusion equations by using moving least square approximation [8]. The nonlinear convection-diffusion-reaction problem in a thin domain with a weak boundary absorption was investigated by Pažanin and Pereira [9]. Miyamoto and Suzuki [10] studied weakly coupled reaction–diffusion systems with rapidly growing nonlinearities and singular initial data. Boundary observers for coupled diffusion–reaction systems were studied by Camacho-Solorio et.al. [11]. Two numerical studies for systems of nonlinear reaction–diffusion equations have been done by Hoff [12] and Liu et. al. [13].
The physical and mathematical importance of these systems is the prediction of the time evolution of the different density distributions (such as population density, mass concentration, neutron flux, temperature) and their relations to the corresponding steady-state distributions [14].
Khater et. al. [15] developed a simple transformation and exact analytical solutions for some nonlinear reaction–diffusion equations. Also, Soliman and Abdou [16] presented the numerical solutions of nonlinear reaction–diffusion equations using the variational iteration method.
An inverse potential problem for a fractional reaction–diffusion equation, and the inverse problem of reconstructing reaction–diffusion systems were studied by Kaltenbacher and Rundell in [17] and [18], respectively. Also, inverse problem for a coupling model of reaction–diffusion and ordinary differential equations systems was investigated by Verdiére et. al. [19].
The Haar wavelets were applied for the inverse solution of the coupled nonlinear reaction–diffusion equations by Foadian et. al. [20].
In this paper, the following inverse problem of coupled nonlinear RDEs is considered and will be solved numerically, using a new high accuracy and easy-to-implement method.

$$u_t(x,t) = \kappa u_{xx}(x,t) + u^2(x,t)v(x,t) - \beta u(x,t),\ 0 < x < 1,\ 0 < t < T, \tag{1}$$

$$v_t(x,t) = \kappa v_{xx}(x,t) - u^2(x,t)v(x,t) + \beta u(x,t),\ 0 < x < 1,\ 0 < t < T, \tag{2}$$

where

$$u(x,0) = f_1(x),\quad v(x,0) = f_2(x),\ 0 \le x \le 1, \tag{3}$$

$$u(1,t) = g_1(t),\quad v(1,t) = g_2(t),\ 0 \le t \le T, \tag{4}$$

$$u(l_1,t) = p_1(t),\quad v(l_2,t) = p_2(t),\ 0 \le t \le T, \tag{5}$$

$$u(0,t) = q_1(t),\quad v(0,t) = q_2(t),\ 0 \le t \le T, \tag{6}$$

where f_1 and f_2 are known as continuous functions, g_1, g_2, p_1, and p_2 are infinitely differentiable known functions, and T represents the final existence time of the time evolution of the problem. The functions of q_1 and q_2 are unknown, which must be identified from additional boundary conditions (5).
The purpose of the research is to present a new method based on orthogonal bases of linear splines in order to solve inverse nonlinear coupled RDEs, announced by (1–6). Moreover, some new features of orthogonal bases for linear splines are established.
This paper is organized as follows: The OL–splines are introduced in Sections 2, and some of its properties are presented in Section 3. Section 4 represent the quasi-linearization method. In Section 5, the numerical method, the estimation of errors, and the convergence of the numerical method are investigated. Some numerical results are presented in Section 6. In the last Section, some concluding remarks are gathered.

2. OL–splines

Univariate splines have been heavily studied in the literature [21, 22]. Here we give just the basics. Let $\mathcal{P}_d$ = space of univariate polynomials of degree at most d. Also, let $\Delta = \{x_i\}_{i=0}^{k}$ with $a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$. we define the space of univariate polynomial splines of smoothness ($r < d$) and degree d with knots Δ as

$$\mathcal{S}_d^r(\Delta) = \left\{ s \in C^r([a,b]) : s\Big|_{(x_i,x_{i+1})} \in \mathcal{P}_d,\ i = 0,1,\ldots,k-1 \right\}.$$

From Curry and Schoenberg Theorem [23], we know that $\mathfrak{n} = \dim \mathcal{S}_d^r(\Delta) = k(d-r)+r+1$. A well-known basis for the space of univariate polynomial splines is the so-called B-splines.
Let $\Delta_e = \{y_i\}_{i=1}^{\mathfrak{n}+d+1}$, be the extended partition of Δ where $a = y_1 = \cdots = y_{d+1}$, $y_{\mathfrak{n}+1} = \cdots = y_{\mathfrak{n}+d+1} = b$, and $y_{d+2} \le y_{d+3} \le \cdots \le y_{\mathfrak{n}}$, wherein $y_j \le x_j \le y_{j+d+1}$, $j = d+2, d+3, \ldots, \mathfrak{n}$. Let

$$Q_i^1(t) = \begin{cases} \frac{1}{y_{i+1}-y_i}, & y_i \le t < y_{i+1}, \\ 0, & \text{o.w.}, \end{cases} \qquad Q_i^m(t) = \begin{cases} \frac{(t-y_i)Q_i^{m-1}(t)+(y_{i+m}-t)Q_{i+1}^{m-1}(t)}{y_{i+m}-y_i}, & y_i \le t < y_{i+m}, \\ 0, & \text{o.w.}, \end{cases}$$

for $2 \le m \le d+1$ and $i = 1, \ldots, \mathfrak{n}+d-m+1$.

Definition 2.1. *Let*

$$B_i^m(t) = (y_{i+m} - y_i) Q_i^m(t), \quad i = 1, 2, \ldots, \mathfrak{n}+d-m+1. \tag{7}$$

We call these the normalized B-splines of order m (or degree $m-1$) associated with the extended partition Δ_e.

The B-splines defined above are not defined at the right-hand endpoint of the domain $[a,b]$. By convention we set their values at b to be their limits as x approaches b from the left. Here are some properties of B-splines [21].

- The B-spline B_i^m vanishes outside of the interval $[y_i, y_{i+m}]$.
- If $y_i < y_{i+m}$ for all i, and $t \in [a,b]$ then $\int_a^b Q_i^m(t)dt = 1/m$ and $\sum_{i=1}^{\mathfrak{n}+d-m+1} B_i^m = 1$.
- $B_1^m(a) = B_{\mathfrak{n}+d-m+1}(b) = 1$.
- $\left\{ B_1^{d+1}, B_2^{d+1}, \ldots, B_{\mathfrak{n}}^{d+1} \right\}$ is a basis for $\mathcal{S}_d^r(\Delta)$.

Now, we introduce an orthogonal basis for the space of univariate polynomial splines.

An orthogonal basis for space of linear splines, named "OL–splines" is introduced in 1993 by Mason et. al. [24]. They have not evaluated the properties of OL–splines. In continuation of their work, the properties of these types of functions are introduced. Furthermore, for simplicity, the B–splines are defined so that all supports belong to a fixed interval.

Let $\Delta_N = (x_k)_{k=-n}^{m}$ be a uniform partition of $[a,b]$ where $N = n+m+1$, $x_{-n} = a$, $x_m = b$, $x_{k+1} - x_k = h$, $k = -n, -n+1, \ldots, m$. Also, let $S_2(\Delta_N)$ be the space of linear splines on Δ_N. The set of $\{L_{-n}, L_{-n+1}, \ldots, L_m\}$ include linear B–splines is a basis for $S_2(\Delta_N)$ where

$$\begin{aligned} L_{-n}(x) &= \frac{1}{h}\begin{cases} x_{-n+1} - x, & x \in [x_{-n}, x_{-n+1}), \\ 0, & \text{o.w.} \end{cases} \\ L_k(x) &= \frac{1}{h}\begin{cases} x_{-n+1} - x, & x \in [x_{-n}, x_{-n+1}), \\ 0, & \text{o.w.} \end{cases} \\ L_m(x) &= \frac{1}{h}\begin{cases} x - x_{m-1}, & x \in (x_{m-1}, x_m], \\ 0, & \text{o.w.} \end{cases} \end{aligned} \tag{8}$$

In order to convert the linear B–splines $\{L_k\}_{k=-n}^{m}$ to a basis of orthogonal linear splines which denoted by $\{P_k\}_{k=-n}^{m}$ and called OL–splines, the following recurrence relation is applied

$$P_{-n} = L_{-n},$$
$$P_i = L_i - a_{i-1}P_{i-1}, i = -n+1, -n+2, \dots, -1, \tag{9}$$
$$P_m = L_m,$$
$$P_j = L_j - a_{j+1}P_{j+1}, j = m-1, m-2, \dots, 1, \tag{10}$$
$$P_0 = L_0 - a_{-1}P_{-1} - a_1P_1. \tag{11}$$

Now, we try to find a_k's influenced by the Gram–Schmidt process. Assuming $f, g \in \mathcal{L}^2[a,b]$, and let $\langle f, g\rangle = \int_a^b f(x)g(x)dx$ be the inner product defined on $\mathcal{L}^2[a,b]$. Also, the induced norm is defined as $\|f\|_2^2 = \langle f, f\rangle$.
Let $k < -1$. Because L_i and P_r, $r \leq i-2$ have disjoint supports,

$$\langle L_i, P_r\rangle = 0, \;\; r \leq i-2, \tag{12}$$

where $i = -n+3, -n+4 \dots, -1$. By multiplying P_{i-1} on the both sides of (9) and integrating on $[a,b]$, we have

$$v_i = a_i\mu_i, \; i = -n, -n+1, \dots, -2, \tag{13}$$

where $\mu_i = \|P_i\|_2^2$, and $v_i = \langle P_i, L_{i+1}\rangle$, $i = -n, -n+1, \dots, -2$.
Note that from (9) and (12), we can conclude that

$$v_i = \langle P_i, L_{i+1}\rangle = \langle L_i, L_{i+1}\rangle = \frac{h}{6}, \; i = -n, -n+1, \dots, -2. \tag{14}$$

Also, by multiplying (9) to itself and integrating on $[a,b]$, enables one to get

$$\mu_i = u_i - a_{i-1}v_{i-1}, \; i = -n+1, -n+2, \dots, -1, \tag{15}$$

where $u_i = \|L_i\|_2^2$. From (8),

$$u_{-n} = u_m = \frac{h}{3},$$
$$u_k = \frac{2h}{3}, \; k = -n+1, -n+2, \dots, m-1. \tag{16}$$

Since, $P_{-n} = L_{-n}$, one obtains $\mu_{-n} = \frac{h}{3}$. Substituting (14) and (16) into (15),

$$\mu_i = \frac{h}{6}(4 - a_{i-1}), \; i = -n+1, -n+2, \dots, -1. \tag{17}$$

Now, (13) and (17) imply that

$$a_{-n} = \frac{1}{2}, \;\; a_i = \frac{1}{4 - a_{i-1}}, \; i = -n+1, -n+2, \dots, -2. \tag{18}$$

The sequence of coefficients (18) is a decreasing one and converges to $a^l = 2 - \sqrt{3}$. Hence,

$$a^l < a_i \leq \frac{1}{2}, \; i = -n, -n+1, \dots -2. \tag{19}$$

Because the coefficients a_i in (9) are less than 1, OL–splines provide a natural normalization that is acceptable for stable numerical computations [24].
Similarly, using (10), we get

$$a_m = \frac{1}{2}, \ \ a_j = \frac{1}{4 - a_{j+1}}, \ j = m-1, m-2, \ldots, 2, \tag{20}$$

and

$$\mu_m = \frac{h}{3}, \ \ \mu_j = \frac{h}{6}(4 - a_{j+1}), \ j = m-1, m-2, \ldots, 1. \tag{21}$$

As well as using (11), we achieved that

$$a_{-1} = \frac{1}{4 - a_{-2}}, \ \ a_1 = \frac{1}{4 - a_2}, \tag{22}$$

and

$$\mu_0 = \frac{2h}{3} - a_{-1}^2\mu_{-1} - a_1^2\mu_1. \tag{23}$$

In Figure 1, the complete system of OL–splines on $[-1, 1]$ for $n = 6$, $m = 5$ is demonstrated.

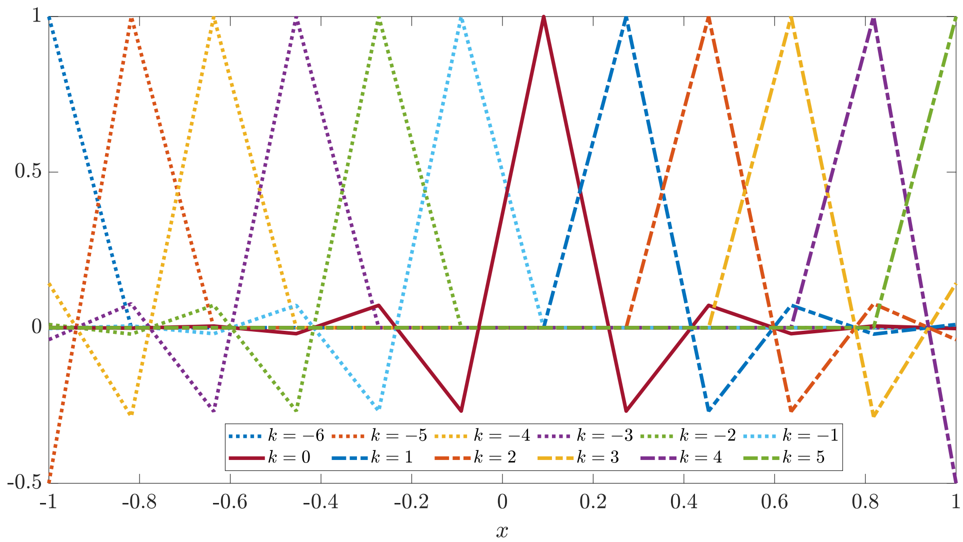

Figure 1: OL–splines on $[-1, 1]$ for $n = 6$, $m = 5$.

3. Some properties of the OL–splines

First some properties of OL–splines are studied. Then, a linear combination of OL–splines is considered as well as, the convergence properties of the linear combination are discussed.

Proposition 3.1. *[25] For $k = -n, -n+1, \ldots, m$, we have*

a) $\|P_k\|_\infty = 1$,
b) $\frac{h}{3} \leq \mu_k \leq \frac{h}{6}\left(2 + \sqrt{3}\right)$,

c) $0 < \sum\limits_{k=-n}^{m} P_k(x) \leq 1, x \in \mathbb{R}$,

d) $0 < \sum\limits_{k=-n}^{m} |P_k(x)| \leq \frac{9}{2}, x \in \mathbb{R}$.

Definition 3.2. *Let $f \in L^2[a,b]$, and $S_N(f) = \sum\limits_{k=-n}^{m} c_k P_k$, where $c_k = \frac{1}{\mu_k}\langle f, P_k\rangle$. Thus, from (9), (10) and (11), $\langle f, P_k\rangle$ are readily determined from following recurrence relation*

$$\begin{aligned}
\langle f, P_{-n}\rangle &= \langle f, L_{-n}\rangle, \ \langle f, P_i\rangle = \langle f, L_i\rangle - a_{i-1}\langle f, P_{i-1}\rangle, \\
\langle f, P_m\rangle &= \langle f, L_m\rangle, \ \left\langle f, P_j\right\rangle = \left\langle f, L_j\right\rangle - a_{j+1}\left\langle f, P_{j+1}\right\rangle, \\
\langle f, P_0\rangle &= \langle f, L_0\rangle - a_{-1}\langle f, P_{-1}\rangle - a_1\langle f, P_1\rangle,
\end{aligned}$$

where $i = -n+1, -n+2, \ldots, -1$, $j = m-1, m-2, \ldots, 1$.

Proposition 3.3. *In Definition 3.2, $\lim\limits_{k\to\infty} c_k = 0$.*

Proof. We know that $\left\|f - S_N(f)\right\|_2^2 = \left\|f\right\|_2^2 - 2\langle f, S_N(f)\rangle + \langle S_N(f), S_N(f)\rangle$. Also,

$$\langle f, S_N(f)\rangle = \int_a^b f(x)\sum_{k=-n}^{m} c_k P_x(x)dx = \sum_{k=-n}^{m} c_k\langle f, P_x\rangle = \sum_{k=-n}^{m} c_k^2\mu_k = \langle S_N(f), S_N(f)\rangle.$$

Therefore, $\left\|f - S_N(f)\right\|_2^2 = \left\|f\right\|_2^2 - \left\|S_N(f)\right\|_2^2$, so $\left\|S_N(f)\right\|_2^2 \leq \left\|f\right\|_2^2$ that is $\sum\limits_{k=-\infty}^{\infty} c_k^2\mu_k \leq \left\|f\right\|_2^2$. Thus, $\lim\limits_{k\to\infty} c_k^2\mu_k = 0$, and $\lim\limits_{k\to\infty} |c_k|\,\|P_k\|_2 = 0$. But Proposition 3.1b implies that $0.5h \leq \|P_k\|_2$ and $0 \leq \lim\limits_{k\to\infty} |c_k|\,0.5h \leq \lim\limits_{k\to\infty} |c_k|\,\|P_k\|_2 = 0$.Thus, $\lim\limits_{k\to\infty} |c_k| = 0$ and $\lim\limits_{k\to\infty} c_k = 0$. □

Proposition 3.4. *[25] Suppose that $f \in C[a,b]$, therefore $S_N(f)$ converges to f uniformly.*

In numerical computations we use vector form of $S_N(f)$ as follows

$$S_N(f)(x) = \sum_{k=-n}^{m} c_k P_k(x) = C_N^T \Pi_N(x), \tag{24}$$

where

$$C_N = (c_{-n}, c_{-n+1}, \ldots, c_m)^T, \tag{25}$$

$$\Pi_N(x) = (P_{-n}(x), P_{-n+1}(x), \ldots, P_m(x))^T. \tag{26}$$

Definition 3.5. *Assuming I_1L, I_1L, I_1P and I_2P are N-square matrices defined by*

$$(I_1L)_{k,l} = \int_{x_{-n}}^{x_l} L_k(t)dt, \ \ (I_2L)_{k,l} = \int_{x_{-n}}^{x_l}\int_{x_{-n}}^{t} L_k(s)dsdt, \ \ (I_1P)_{k,l} = \int_{x_{-n}}^{x_l} P_k(t)dt, \ \ (I_2P)_{k,l} = \int_{x_{-n}}^{x_l}\int_{x_{-n}}^{t} P_k(s)dsdt,$$

where $k, l = -n, -n+1, \dots, m$. *According to the definitions of* L_k*'s in (8),*

$$I_1L = h\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \frac{1}{2} & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}_{N\times N},$$

$$I_2L = h^2\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3}+\frac{1}{2} & \frac{1}{3}+2\frac{1}{2} & \cdots & \frac{1}{3}+(N-3)\frac{1}{2} & \frac{1}{3}+(N-2)\frac{1}{2} \\ 0 & \frac{1}{6} & 1 & 2 & \cdots & N-3 & N-2 \\ 0 & 0 & \frac{1}{6} & 1 & \cdots & N-4 & N-3 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & & \frac{1}{6} & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{6} & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{6} \end{pmatrix}_{N\times N}. \tag{27}$$

Furthermore, (9), (10) and (11) entail

$$\begin{aligned} &(I_\nu P)_{-n,l} = (I_\nu L)_{-n,l}, \ \ (I_\nu P)_{i,l} = (I_\nu L)_{i,l} - a_{i-1}(I_\nu P)_{i-1,l}, \\ &(I_\nu P)_{m,l} = (I_\nu L)_{m,l}, \ \ (I_\nu P)_{j,l} = (I_\nu L)_{j,l} - a_{j+1}(I_\nu P)_{j+1,l}, \\ &(I_\nu P)_{0,l} = (I_\nu L)_{0,l} - a_{-1}(I_\nu P)_{-1,l} - a_1(I_\nu P)_{1,l}, \end{aligned} \tag{28}$$

where $\nu = 1, 2$, $i = -n+1, -n+2, \dots, -1$, $j = m-1, m-2, \dots, 1$. Thus, we can write

$$\int_{x_{-n}}^{x_l} S_N(f)(t)dt = C_N^T I_1^l, \tag{29}$$

$$\int_{x_{-n}}^{x_l}\int_{x_{-n}}^{t} S_N(f)(s)dsdt = C_N^T I_2^l, \tag{30}$$

where I_ν^l is the lth column of matrix $I_\nu P$, that is $I_\nu^l = I_\nu P(:, l)$, $\nu = 1, 2$.

Lemma 3.6. *Let* $IP_k = \int_a^b P_k(x)dx$, *then* $\frac{h}{6}\left(3-\sqrt{3}\right) < IP_k < \frac{h}{6}\left(3+\sqrt{3}\right)$, *where* $k = -n, -n+1, \dots, m$.

Proof. Integrating (9), (10) and (11), in addition using (8), yield

$$\begin{aligned} &IP_{-n} = IP_m = \int_a^b L_{-n}(x)dx = \frac{h}{2}, \\ &IP_i = h - a_{i-1}IP_{i-1}, \ i = -n+1, \dots, -1, \\ &IP_j = h - a_{j+1}IP_{j+1}, \ j = m-1, \dots, 1, \\ &IP_0 = h - a_{-1}IP_{-1} - a_1IP_1. \end{aligned}$$

Thus,

$$\begin{aligned} &h \le IP_k < \frac{1}{6}\left(3+\sqrt{3}\right)h, \ k \ne 0, \\ &\frac{1}{6}\left(3-\sqrt{3}\right)h < IP_0 < \left(\sqrt{3}-1\right)h. \end{aligned}$$

□

Lemma 3.7. *Let* $IP_{l,k} = \int_{x_{-n}}^{x_l} P_k(x)dx$, *then* $-\frac{1}{8}h \le IP_{l,k} < \frac{\sqrt{3}}{2}h$, *where* $k = -n, -n+1, \dots, m$.

Proof. The proof is similar to Lemma 3.6. □

Remark 3.8. *A linear B–splines extension of OL–splines can be obtained from equations (9), (10) and (11), as follows*

$$\begin{aligned} P_i(x) &= L_i(x) + \sum_{v=1}^{n-i} (-1)^v a_{i-1}a_{i-2}\cdots a_{i-v}L_{i-v}(x), \\ P_j(x) &= L_j(x) + \sum_{v=1}^{m-j} (-1)^v a_{j+1}a_{j+2}\cdots a_{j+v}L_{j+v}(x), \\ P_0(x) &= L_0(x) + \sum_{k=1}^{n} (-1)^k a_{-1}a_{-2}\cdots a_{-k}L_{-k}(x) + \sum_{k=1}^{m} (-1)^k a_1a_2\cdots a_kL_k(x), \end{aligned} \tag{31}$$

where $i = -n, -n+1, \dots, -1$, $j = m, m-1, \dots, 1$.

Lemma 3.9. *Let* $I^2P_{l,k} = \int_{x_{-n}}^{x_l}\int_{x_{-n}}^{z} P_k(y)dydz$, *then* $0 \le I^2P_{l,k} < h$, *where* $k = -n, -n+1, \dots, m$.

Proof. According to (27) and (28), also using (31), we can see that $0 \le I^2P_{l,k} < h^2(xN + y - i)$, where $0 < x, y < 1$ and $i = 1, 2, \dots, N-m-1$. Therefore, $0 \le I^2P_{l,k} < h$. □

4. The quasi-linearization method

In this section, a quasi-linearization method is presented to linearize the u^2v term in (1) and (2). The quasi-linearization technique is an application of the Newton–Raphson–Kantrovich approximation method in function space [26–29].
Let $u, v \in C[0,1]\times C[0,T]$ and $h(u,v) = u^2v$. Using two variable Taylor series for h in some open neighborhood around $(u,v) = (u_s, v_s)$, there is $\mathbf{c} = (c_1, c_2)$ where $c_1, c_2 \in C[0,1]\times C[0,T]$, so that

$$h(\mathbf{x}) = h(\mathbf{a}) + (\mathbf{x}-\mathbf{a})\cdot\nabla h(\mathbf{a}) + (\mathbf{x}-\mathbf{a}).H(\mathbf{c}).(\mathbf{x}-\mathbf{a}),$$

where $\mathbf{x}=(u,v)$, $\mathbf{a} = (u_s, v_s)$, and H is the Hessian matrix

$$H(\mathbf{c}) = \begin{pmatrix} h_{c_1c_1}(\mathbf{c}) & h_{c_1c_2}(\mathbf{c}) \\ h_{c_1c_2}(\mathbf{c}) & h_{c_2c_2}(\mathbf{c}) \end{pmatrix}.$$

Therefore,

$$u^2v = 2u_sv_su - 2u_s^2v_s + u_s^2v + c_2(u-u_s)^2 + c_1(u-u_s)(v-v_s). \tag{32}$$

Based on (32), a linear approximation of u^2v is as follows

$$u^2v = 2u_sv_su - 2u_s^2v_s + u_s^2v. \tag{33}$$

5. Analysis of the Method

First, OL-splines are used to propose a numerical method for solving the problem (1–6). Second, an upper bound of error estimation is obtained. Finally, the convergence analysis of the method is investigated.

5.1. Solution Method

In this subsection, the inverse problem (1–6) is solved using S_N as an approximation tool. Let in Section 2 and Section 3, $a = 0$, $b = 1$ and $x_{\nu_1} = l_1$, $x_{\nu_2} = l_2$. Also, we assume that $m = n$, that is $\Delta_N = (x_l)_{l=-n}^{n}$ be a uniform partition of $[0, 1]$, where $N = 2n+1$, $x_{-n} = 0$ and $x_n = 1$ such that $x_{l+1} - x_l = h$, $l = -n, -n+1, \ldots, n-1$. Also, $t_s = s\Delta t$, $s = 0, 1, \ldots, S$ are the equal parts of $[0, T]$ where $\Delta t = \frac{T}{S}$.

To discretize the problem (1–6), the method of [30] is used. We assume that $u_{txx}(x, t)$ can be expanded in terms of OL–splines (24) as

$$u_{txx}(x,t) = \sum_{k=-n}^{n} c_k^s P_k(x) = C_N^T \Pi_N(x), \tag{34}$$

where C_N, $\Pi_N(x)$ are given by (25) and (26). The row vector C_N^T is assumed constant in the subinterval $[t_s, t_{s+1}]$.

By integrating (34) with respect to t from t_s to t, we obtain

$$u_{xx}(x,t) = u_{xx}(x,t_s) + (t - t_s) C_N^T \Pi_N(x). \tag{35}$$

Also, by integrating (34) twice with respect to x from l_1 to x and using (5), gives

$$u_t(x,t) = \dot{p}_1(t) + (x - l_1) u_{tx}(l_1, t) + \sum_{k=-n}^{n} c_k^s \int_{l_1}^{x} \int_{l_1}^{z} P_k(y) dy dz, \tag{36}$$

where . denote the differentiation with respect to t.

By putting $x = 1$ in equation (36) and using (4),

$$u_{tx}(l_1,t) = \frac{1}{1-l_1} \left(\dot{g}_1(t) - \dot{p}_1(t) - \sum_{k=-n}^{n} c_k^s \int_{l_1}^{1} \int_{l_1}^{z} P_k(y) dy dz \right). \tag{37}$$

Substituting equation (37) into equation (36), held

$$u_t(x,t) = \dot{p}_1(t) + \frac{x-l_1}{1-l_1} (\dot{g}_1(t) - \dot{p}_1(t)) + \sum_{k=-n}^{n} c_k^s \left(\int_{l_1}^{x} \int_{l_1}^{z} P_k(y) dy dz - \frac{x-l_1}{1-l_1} \int_{l_1}^{1} \int_{l_1}^{z} P_k(y) dy dz \right). \tag{38}$$

Since,

$$\int_{l_1}^{x} \int_{l_1}^{z} P_k(y) dy dz = \int_{0}^{x} \int_{0}^{z} P_k(y) dy dz - (x - l_1) \int_{0}^{l_1} P_k(y) dy dz - \int_{0}^{l_1} \int_{0}^{z} P_k(y) dy dz,$$

relation (38) can be written as

$$\begin{aligned} u_t(x,t) = \dot{p}_1(t) + \frac{x-l_1}{1-l_1} (\dot{g}_1(t) - \dot{p}_1(t)) + \sum_{k=-n}^{n} \Bigg(c_k^s \Big(\int_{0}^{x} \int_{0}^{z} P_k(y) dy dz - \int_{0}^{l_1} \int_{0}^{z} P_k(y) dy dz \Big) \\ - \frac{x-l_1}{1-l_1} \Big(\int_{0}^{1} \int_{0}^{z} P_k(y) dy dz - \int_{0}^{l_1} \int_{0}^{z} P_k(y) dy dz \Big) \Bigg). \end{aligned} \tag{39}$$

By integrating (39) with respect to t from t_s to t, we obtain

$$\begin{aligned} u(x,t) = {} & u(x,t_s) + p_1(t) - p_1(t_s) + \frac{x-l_1}{1-l_1} (g_1(t) - g_1(t_s) - p_1(t) + p_1(t_s)) \\ & + (t - t_s) \sum_{k=-n}^{n} c_k^s \Big(\int_{0}^{x} \int_{0}^{z} P_k(y) dy dz - \int_{0}^{l_1} \int_{0}^{z} P_k(y) dy dz \\ & - \frac{x-l_1}{1-l_1} \Big(\int_{0}^{1} \int_{0}^{z} P_k(y) dy dz - \int_{0}^{l_1} \int_{0}^{z} P_k(y) dy dz \Big) \Big). \end{aligned} \tag{40}$$

Further, by discretizing (35), (39) and (40), assuming $x \to x_l$, and using (29) and (30), we get

$$u_{xx}(x_l,t) = u_{xx}(x_l,t_s) + (t-t_s)C_N^T\Pi_N(x_l), \tag{41}$$

$$u_t(x_l,t) = \dot{p}_1(t) + \frac{x_l - l_1}{1-l_1}\left(\dot{g}_1(t) - \dot{p}_1(t)\right) + C_N^T\left(I_2^l - I_2^{\nu_1} - \frac{x_l - l_1}{1-l_1}\left(I_2^n - I_2^{\nu_1}\right)\right), \tag{42}$$

$$\begin{aligned} u(x_l,t) = {} & u(x_l,t_s) + p_1(t) - p_1(t_s) + \frac{x_l - l_1}{1-l_1}\left(g_1(t) - g_1(t_s) - p_1(t) + p_1(t_s)\right) \\ & + (t-t_s)C_N^T\left(I_2^l - I_2^{\nu_1} - \frac{x_l - l_1}{1-l_1}\left(I_2^n - I_2^{\nu_1}\right)\right). \end{aligned} \tag{43}$$

Similarly, if we assume that

$$v_{txx}(x,t) = \sum_{k=-n}^{n} d_k^s P_k(x) = D_N^T\Pi_N(x), \tag{44}$$

then we have

$$v_{xx}(x_l,t) = v_{xx}(x_l,t_s) + (t-t_s)D_N^T\Pi_N(x_l), \tag{45}$$

$$v_t(x_l,t) = \dot{p}_2(t) + \frac{x_l - l_2}{1-l_2}\left(\dot{g}_2(t) - \dot{p}_2(t)\right) + D_N^T\left(I_2^l - I_2^{\nu_1} - \frac{x_l - l_2}{1-l_2}\left(I_2^n - I_2^{\nu_1}\right)\right), \tag{46}$$

$$\begin{aligned} v(x_l,t) = {} & v(x_l,t_s) + p_2(t) - p_2(t_s) + \frac{x_l - l_2}{1-l_2}\left(g_2(t) - g_2(t_s) - p_2(t) + p_2(t_s)\right) \\ & + (t-t_s)D_N^T\left(I_2^l - I_2^{\nu_1} - \frac{x_l - l_2}{1-l_2}\left(I_2^n - I_2^{\nu_1}\right)\right). \end{aligned} \tag{47}$$

Also, assuming $t \to t_{s+1}$ in (41), (42), (43), (45), (46) and (47), lead to

$$u_{xx}(x_l,t_{s+1}) = u_{xx}(x_l,t_s) + \Delta t C_N^T\Pi_N(x_l), \tag{48}$$

$$u_t(x_l,t_{s+1}) = \dot{p}_1(t_{s+1}) + \frac{x_l - l_1}{1-l_1}\left(\dot{g}_1(t_{s+1}) - \dot{p}_1(t_{s+1})\right) + C_N^T\left(I_2^l - I_2^{\nu_1} - \frac{x_l - l_1}{1-l_1}\left(I_2^n - I_2^{\nu_1}\right)\right), \tag{49}$$

$$\begin{aligned} u(x_l,t_{s+1}) = {} & u(x_l,t_s) + p_1(t_{s+1}) - p_1(t_s) + \frac{x_l - l_1}{1-l_1}\left(g_1(t_{s+1}) - g_1(t_s) - p_1(t_{s+1}) + p_1(t_s)\right) \\ & + \Delta t C_N^T\left(I_2^l - I_2^{\nu_1} - \frac{x_l - l_1}{1-l_1}\left(I_2^n - I_2^{\nu_1}\right)\right), \end{aligned} \tag{50}$$

$$v_{xx}(x_l,t_{s+1}) = v_{xx}(x_l,t_s) + \Delta t D_N^T\Pi_N(x_l), \tag{51}$$

$$v_t(x_l,t_{s+1}) = \dot{p}_2(t_{s+1}) + \frac{x_l - l_2}{1-l_2}\left(\dot{g}_2(t_{s+1}) - \dot{p}_2(t_{s+1})\right) + D_N^T\left(I_2^l - I_2^{\nu_1} - \frac{x_l - l_2}{1-l_2}\left(I_2^n - I_2^{\nu_1}\right)\right), \tag{52}$$

$$\begin{aligned} v(x_l,t_{s+1}) = {} & v(x_l,t_s) + p_2(t_{s+1}) - p_2(t_s) + \frac{x_l - l_2}{1-l_2}\left(g_2(t_{s+1}) - g_2(t_s) - p_2(t_{s+1}) + p_2(t_s)\right) \\ & + \Delta t D_N^T\left(I_2^l - I_2^{\nu_1} - \frac{x_l - l_2}{1-l_2}\left(I_2^n - I_2^{\nu_1}\right)\right). \end{aligned} \tag{53}$$

Also according to (33), we can write

$$u^2(x_l,t_{s+1})v(x_l,t_{s+1}) \approx 2u(x_l,t_s)v(x_l,t_s)u(x_l,t_{s+1}) - 2u^2(x_l,t_s)v(x_l,t_s) + u^2(x_l,t_s)v(x_l,t_{s+1}). \tag{54}$$

Assuming $x \to x_l$, $t \to t_{s+1}$ in (1) and (2), results in

$$u_t(x_l,t_{s+1}) = \kappa u_{xx}(x_l,t_{s+1}) + u^2(x_l,t_{s+1})v(x_l,t_{s+1}) - \beta u(x_l,t_{s+1}), \tag{55}$$

$$v_t(x_l,t_{s+1}) = \kappa v_{xx}(x_l,t_{s+1}) - u^2(x_l,t_{s+1})v(x_l,t_{s+1}) + \beta u(x_l,t_{s+1}). \tag{56}$$

By substituting (48), (49), (50), (53) and (54) into (55) and substituting (50), (51), (52), (53) and (54) into (56), we obtain

$$\begin{cases} C_N^T z_{1,l,s} - D_N^T z_{2,l,s} = r_{1,l,s}, \\ C_N^T z_{3,l,s} + D_N^T z_{4,l,s} = r_{2,l,s}, \end{cases} \tag{57}$$

where

$$\begin{aligned} z_{1,l,s} &= (1 + \beta\Delta t - 2\Delta t u(x_l,t_s)v(x_l,t_s))\, \mathbb{I}_1^l - \kappa\Delta t \Pi_N(x_l), \quad z_{2,l,s} = \Delta t u^2(x_l,t_s)\mathbb{I}_2^l, \\ z_{3,l,s} &= (-\beta\Delta t + 2\Delta t u(x_l,t_s)v(x_l,t_s))\, \mathbb{I}_1^l, \quad z_{4,l,s} = \Big(1 + \Delta t u^2(x_l,t_s)\Big)\, \mathbb{I}_2^l - \kappa\Delta t \Pi_N(x_l), \\ r_{1,l,s} &= -\dot{p}_1(t_{s+1}) - c_{1,l}\Big(\dot{g}_1(t_{s+1}) - \dot{p}_1(t_{s+1})\Big) + \kappa u_{xx}(x_l,t_s) - 2u^2(x_l,t_s)v(x_l,t_s) - \beta\Big(u(x_l,t_s) + \mathfrak{p}_1^s + c_{1,l}\sigma_1^s\Big) \\ &\quad + 2u(x_l,t_s)v(x_l,t_s)\Big(u(x_l,t_s) + \mathfrak{p}_1^s + c_{1,l}\sigma_1^s\Big) + u^2(x_l,t_s)\Big(v(x_l,t_s) + \mathfrak{p}_2^s + c_{2,l}\sigma_2^s\Big), \\ r_{2,l,s} &= -\dot{p}_2(t_{s+1}) - c_{2,l}\Big(\dot{g}_2(t_{s+1}) - \dot{p}_2(t_{s+1})\Big) + \kappa v_{xx}(x_l,t_s) + 2u^2(x_l,t_s)v(x_l,t_s) + \beta\Big(u(x_l,t_s) + \mathfrak{p}_1^s + c_{1,l}\sigma_1^s\Big) \\ &\quad - 2u(x_l,t_s)v(x_l,t_s)\Big(u(x_l,t_s) + \mathfrak{p}_1^s + c_{1,l}\sigma_1^s\Big) - u^2(x_l,t_s)\Big(v(x_l,t_s) + \mathfrak{p}_2^s + c_{2,l}\sigma_2^s\Big), \end{aligned}$$

and

$$\begin{aligned} &\mathbb{I}_1^l = I_2^l - I_2^{\nu_1} - c_{1,l}(I_2^n - I_2^{\nu_1}), \quad \mathbb{I}_2^l = I_2^l - I_2^{\nu_2} - c_{2,l}(I_2^n - I_2^{\nu_2}), \quad c_{1,l} = \frac{x_l - l_1}{1 - l_1}, \quad c_{2,l} = \frac{x_l - l_2}{1 - l_2}, \\ &\sigma_1^s = g_1(t_{s+1}) - g_1(t_s) - p_1(t_{s+1}) + p_1(t_s), \quad \sigma_2^s = g_2(t_{s+1}) - g_2(t_s) - p_2(t_{s+1}) + p_2(t_s), \\ &\mathfrak{p}_1^s = p_1(t_{s+1}) - p_1(t_s), \quad \mathfrak{p}_2^s = p_2(t_{s+1}) - p_2(t_s). \end{aligned}$$

By organizing (57) respect to $l = -n, -n+1, \dots, n$, we obtain

$$\begin{cases} A^T C_N - B^T D_N = R_1, \\ E^T C_N + F^T D_N = R_2, \end{cases} \tag{58}$$

where

$$\begin{aligned} &A = (z_{1,-n,s}, z_{1,-n+1,s}, \dots, z_{1,n,s}), \qquad B = (z_{2,-n,s}, z_{2,-n+1,s}, \dots, z_{2,n,s}), \\ &E = (z_{3,-n,s}, z_{3,-n+1,s}, \dots, z_{3,n,s}), \qquad F = (z_{4,-n,s}, z_{4,-n+1,s}, \dots, z_{4,n,s}), \\ &R_1 = (r_{1,-n,s}, r_{1,-n+1,s}, \dots, r_{1,n,s})^T, \quad R_2 = (r_{2,-n,s}, r_{2,-n+1,s}, \dots, r_{2,n,s})^T. \end{aligned}$$

Equation (58) can be written in the form of square system of linear equations, as follows

$$\begin{pmatrix} E^T & -F^T \\ G^T & H^T \end{pmatrix} \begin{pmatrix} C_N \\ D_N \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}. \tag{59}$$

Note that for $s = 0$ we use equation (3) as $u_{xx}(x_l,t_0) = f_1''(x_l)$, $u(x_l,t_0) = f_1(x_l)$, $v_{xx}(x_l,t_0) = f_2''(x_l)$ and $v(x_l,t_0) = f_2(x_l)$, otherwise $u_{xx}(x_l,t_s)$, $u(x_l,t_s)$, $v_{xx}(x_l,t_s)$ and $v(x_l,t_s)$ are updated using (48), (50), (51) and (53), respectively.

5.2. Estimation of errors

In this subsection, the upper bounds for error estimations of $u(0,t) = q_1(t)$ and $v(0,t) = q_2(t)$ will be achieved. Let $\tilde{u}(0,t)$ and $\tilde{v}(0,t)$ be approximation values for the exact values of $u(0,t)$ and $v(0,t)$, respectively. Then we have the following proposition

Proposition 5.1. *Suppose that g_1, g_2, p_1, p_2, q_1 and q_2 are continuous functions, then*

$$|u(0,t) - \tilde{u}(0,t)| \leq \frac{\Delta t}{1-l_1}\left(2\gamma_1 + |\Gamma_1|\right),$$
$$|v(0,t) - \tilde{v}(0,t)| \leq \frac{\Delta t}{1-l_2}\left(2\gamma_2 + |\Gamma_2|\right),$$

where $\gamma_1 \leq \max\left\{\|q_1\|_\infty, \|p_1\|_\infty, \|g_1\|_\infty\right\}$, $\gamma_2 \leq \max\left\{\|q_2\|_\infty, \|p_2\|_\infty, \|g_2\|_\infty\right\}$. *In addition,*

$$|\Gamma_1| < (l_1+1)\frac{3\sqrt{3}}{2}\|q_1\|_\infty,$$
$$|\Gamma_2| < (l_2+1)\frac{3\sqrt{3}}{2}\|q_2\|_\infty.$$

Proof. Corresponding (43) and (47), we get

$$\tilde{u}(0,t) = q_1(t_s) + p_1(t) - p_1(t_s) - \frac{l_1}{1-l_1}\left(g_1(t) - g_1(t_s) - p_1(t) + p_1(t_s)\right) + \frac{t-t_s}{1-l_1} C_N^T \left(l_1 I_2^n - I_2^{\nu_1}\right), \tag{60}$$
$$\tilde{v}(0,t) = q_2(t_s) + p_2(t) - p_2(t_s) - \frac{l_2}{1-l_2}\left(g_2(t) - g_2(t_s) - p_2(t) + p_2(t_s)\right) + \frac{t-t_s}{1-l_2} D_N^T \left(l_2 I_2^n - I_2^{\nu_2}\right). \tag{61}$$

Now, assuming $\Gamma_1 = C_N^T\left(l_1 I_2^n - I_2^{\nu_1}\right)$, $\Gamma_2 = D_N^T\left(l_2 I_2^n - I_2^{\nu_2}\right)$, equations (60) and (61) lead to

$$|u(0,t) - \tilde{u}(0,t)| \leq \left|q_1(t) - q_1(t_s)\right| + \frac{1}{1-l_1}\left|p_1(t) - p_1(t_s)\right| + \frac{l_1}{1-l_1}\left|g_1(t) - g_1(t_s)\right| + \frac{t-t_s}{1-l_1}|\Gamma_1|, \tag{62}$$
$$|v(0,t) - \tilde{v}(0,t)| \leq \left|q_2(t) - q_2(t_s)\right| + \frac{1}{1-l_2}\left|p_2(t) - p_2(t_s)\right| + \frac{l_2}{1-l_2}\left|g_2(t) - g_2(t_s)\right| + \frac{t-t_s}{1-l_2}|\Gamma_2|. \tag{63}$$

Using the mean value theorem of derivatives, there are $t_s < \zeta_i < t$, $i = 1, 2, \ldots, 6$, so that

$$q_1(t) - q_1(t_s) = (t-t_s)q_1(\zeta_1), \;\; p_1(t) - p_1(t_s) = (t-t_s)p_1(\zeta_2), \;\; g_1(t) - g_1(t_s) = (t-t_s)g_1(\zeta_3),$$
$$q_2(t) - q_2(t_s) = (t-t_s)q_2(\zeta_4), \;\; p_2(t) - p_2(t_s) = (t-t_s)p_2(\zeta_5), \;\; g_2(t) - g_2(t_s) = (t-t_s)g_2(\zeta_6).$$

Accordingly, (62) and (63) yield

$$|u(0,t) - \tilde{u}(0,t)| \leq (t-t_s)\left(\left|q_1(\zeta_1)\right| + \frac{1}{1-l_1}\left|p_1(\zeta_2)\right| + \frac{l_1}{1-l_1}\left|g_1(\zeta_3)\right| + \frac{1}{1-l_1}|\Gamma_1|\right), \tag{64}$$
$$|v(0,t) - \tilde{v}(0,t)| \leq (t-t_s)\left(\left|q_2(\zeta_4)\right| + \frac{1}{1-l_2}\left|p_2(\zeta_5)\right| + \frac{l_2}{1-l_2}\left|g_2(\zeta_6)\right| + \frac{1}{1-l_2}|\Gamma_2|\right). \tag{65}$$

Let $\gamma_1 = \max\left\{\left|q_1(\zeta_1)\right|, \left|p_1(\zeta_2)\right|, \left|g_1(\zeta_3)\right|\right\}$ and $\gamma_2 = \max\left\{\left|q_2(\zeta_4)\right|, \left|p_2(\zeta_5)\right|, \left|g_2(\zeta_6)\right|\right\}$, then (64) and (65) imply that

$$|u(0,t) - \tilde{u}(0,t)| \leq \frac{t-t_s}{1-l_1}\left(2\gamma_1 + |\Gamma_1|\right) \leq \frac{\Delta t}{1-l_1}\left(2\gamma_1 + |\Gamma_1|\right), \tag{66}$$

$$|v(0,t) - \tilde{v}(0,t)| \leq \frac{t-t_s}{1-l_2}\left(2\gamma_2 + |\Gamma_2|\right) \leq \frac{\Delta t}{1-l_2}\left(2\gamma_2 + |\Gamma_2|\right). \tag{67}$$

Moreover,

$$\Gamma_1 = C_N^T\left(l_1 I_2^n - I_2^{\nu_1}\right) = \sum_{k=-n}^{m} c_k\left(l_1\int_0^1\int_0^z P_k(y)dydz - \int_0^{x_{\nu_1}}\int_0^z P_k(y)dydz\right)$$
$$= l_1\sum_{k=-n}^{m} c_k\int_0^1\int_0^z P_k(y)dydz - \sum_{k=-n}^{m} c_k\int_0^{x_{\nu_1}}\int_0^z P_k(y)dydz.$$

Using Lemma 3.9, one can write

$$|\Gamma_1| < h\,(l_1+1)\sum_{k=-n}^{m}|c_k|. \tag{68}$$

Also, because

$$c_k = \frac{1}{\mu_k}\langle P_k, q_1\rangle = \frac{1}{\mu_k}\int_0^t P_k(t)q_1(t)dt \le \frac{1}{\mu_k}\left|\int_0^t P_k(t)q_1(t)dt\right| \le \frac{\|q_1\|_\infty}{\mu_k}\left|\int_0^t P_k(t)dt\right|,$$

Proposition 3.1b and Lemma 3.7 leading to

$$c_k \le \frac{\|q_1\|_\infty}{\mu_k}\frac{3}{\Delta t}\frac{\sqrt{3}}{2}\Delta t = \frac{3\sqrt{3}}{2}\|q_1\|_\infty.$$

Also, we know that

$$\sum_{k=-n}^{m}|c_k| \le N\max_{-n\le k\le n}|c_k| < N\frac{3\sqrt{3}}{2}\|q_1\|_\infty. \tag{69}$$

By substituting (69) into (68),

$$|\Gamma_1| < h\,(l_1+1)\,N\frac{3\sqrt{3}}{2}\|q_1\|_\infty = (l_1+1)\,\frac{3\sqrt{3}}{2}\|q_1\|_\infty.$$

□

5.3. Convergence analysis

In order to prove convergence of the solution of presented method, we need to show that the maximum errors tend to zero as $h \to 0$, $\Delta t \to 0$.

Theorem 5.2. *Assume that $u_{txx}(x,t)$ and $v_{txx}(x,t)$ are the exact solutions, and $\tilde{u}_{txx}(x,t) = \sum_{k=-n}^{n} c_k^s P_k(x)$, $\tilde{v}_{txx}(x,t) = \sum_{k=-n}^{n} d_k^s P_k(x)$ are the numerical solutions of problem (1–6), which $t \in [t_s, t_{s+1}]$. In addition, let*

$$e_N(x,t) = u_{txx}(x,t) - \tilde{u}_{txx}(x,t), \tag{70}$$
$$\varepsilon_N(x,t) = v_{txx}(x,t) - \tilde{v}_{txx}(x,t), \tag{71}$$

represent errors in (x,t) point. Then

$$\lim_{h\to 0}\|u(x,t)-\tilde{u}(x,t)\|_\infty = 0, \qquad \lim_{h\to 0}\|v(x,t)-\tilde{v}(x,t)\|_\infty = 0,$$
$$\lim_{\Delta t\to 0}\|u(0,t)-\tilde{u}(0,t)\|_\infty = 0, \qquad \lim_{\Delta t\to 0}\|v(0,t)-\tilde{v}(0,t)\|_\infty = 0.$$

Proof. The numerical method presented in Section 5, could be rewritten along with the error terms. By integrating (70) with respect to t from t_s to t, we obtain

$$\tilde{u}_{xx}(x,t) = \tilde{u}_{xx}(x,t_s) + u_{xx}(x,t) - u_{xx}(x,t_s) - (t-t_s)e_N(x,t). \tag{72}$$

Integrating (70) twice with respect to x from l_1 to x , leads to

$$\tilde{u}_t(x,t) = u_t(x,t) + (x-l_1)\left(\tilde{u}_{tx}(l_1,t) - u_{tx}(l_1,t)\right) - \int_{l_1}^{x}\int_{l_1}^{z} e_N(y,t)dydz. \tag{73}$$

Putting $x = 1$ in equation (73), we get

$$\tilde{u}_{tx}(l_1, t) - u_{t^s x}(l_1, t) = \frac{1}{1 - l_1} \int_{l_1}^{1} \int_{l_1}^{z} e_N(y, t) dy dz. \tag{74}$$

Substituting equation (74) into equation (73), yields

$$\tilde{u}_t(x, t) = u_t(x, t) + \frac{x - l_1}{1 - l_1} \int_{l_1}^{1} \int_{l_1}^{z} e_N(y, t) dy dz - \int_{l_1}^{x} \int_{l_1}^{z} e_N(y, t) dy dz. \tag{75}$$

By integrating (75) with respect to t from t_s to t, we get

$$\tilde{u}(x, t) = \tilde{u}(x, t_s) + u(x, t) - u(x, t_s) + (t - t_s) \left(\frac{x - l_1}{1 - l_1} \int_{l_1}^{1} \int_{l_1}^{z} e_N(y, t) dy dz - \int_{l_1}^{x} \int_{l_1}^{z} e_N(y, t) dy dz \right). \tag{76}$$

In the same way,

$$\tilde{v}_{xx}(x, t) = \tilde{v}_{xx}(x, t_s) + v_{xx}(x, t) - v_{xx}(x, t_s) - (t - t_s) \varepsilon_N(x, t), \tag{77}$$

$$\tilde{v}_t(x, t) = v_t(x, t) + \frac{x - l_2}{1 - l_2} \int_{l_2}^{1} \int_{l_2}^{z} \varepsilon_N(y, t) dy dz - \int_{l_2}^{x} \int_{l_2}^{z} \varepsilon_N(y, t) dy dz, \tag{78}$$

$$\tilde{v}(x, t) = \tilde{v}(x, t_s) + v(x, t) - v(x, t_s) + (t - t_s) \left(\frac{x - l_2}{1 - l_2} \int_{l_2}^{1} \int_{l_2}^{z} \varepsilon_N(y, t) dy dz - \int_{l_2}^{x} \int_{l_2}^{z} \varepsilon_N(y, t) dy dz \right). \tag{79}$$

According to the initial and boundary conditions (3), (4) and (5), we set

$$\begin{cases} \tilde{u}(x, 0) = u(x, 0), \\ \tilde{v}(x, 0) = v(x, 0), \end{cases} \quad \begin{cases} \tilde{u}(1, t) = u(1, t), \\ \tilde{v}(1, t) = v(1, t), \end{cases} \quad \begin{cases} \tilde{u}(l_1, t) = u(l_1, t), \\ \tilde{v}(l_2, t) = v(l_2, t). \end{cases} \tag{80}$$

Applying equation (72) successively, we have

$$\begin{aligned} u_{xx}(x, t) - \tilde{u}_{xx}(x, t) &= u_{xx}(x, t_s) - \tilde{u}_{xx}(x, t_s) + (t - t_s) e_N(x, t) \\ &= u_{xx}(x, t_{s-1}) - \tilde{u}_{xx}(x, t_{s-1}) + \Delta t e_N(x, t_s) + (t - t_s) e_N(x, t) \\ &\vdots \\ &= u_{xx}(x, t_0) - \tilde{u}_{xx}(x, t_0) + \Delta t \sum_{j=1}^{s} e_N(x, t_j) + (t - t_s) e_N(x, t). \end{aligned}$$

Now the equation (80) concludes

$$u_{xx}(x, t) - \tilde{u}_{xx}(x, t) = \Delta t \sum_{j=1}^{s} e_N(x, t_j) + (t - t_s) e_N(x, t). \tag{81}$$

In a similar way, from equation (77), we obtain

$$v_{xx}(x, t) - \tilde{v}_{xx}(x, t) = \Delta t \sum_{j=1}^{s} \varepsilon_N(x, t_j) + (t - t_s) \varepsilon_N(x, t). \tag{82}$$

As well as, equations (76) and (80), leads to

$$
\begin{aligned}
u(x,t)-\tilde{u}(x,t) &= u(x,t_s)-\tilde{u}(x,t_s)\\
&\quad -(t-t_s)\left(\frac{x-l_1}{1-l_1}\int_{l_1}^{1}\int_{l_1}^{z}e_N(y,t)dydz-\int_{l_1}^{x}\int_{l_1}^{z}e_N(y,t)dydz\right)\\
&= u(x,t_{s-1})-\tilde{u}(x,t_{s-1})-\Delta t\left(\frac{x-l_1}{1-l_1}\int_{l_1}^{1}\int_{l_1}^{z}e_N(y,t_s)dydz-\int_{l_1}^{x}\int_{l_1}^{z}e_N(y,t_s)dydz\right)\\
&\quad -(t-t_s)\left(\frac{x-l_1}{1-l_1}\int_{l_1}^{1}\int_{l_1}^{z}e_N(y,t)dydz-\int_{l_1}^{x}\int_{l_1}^{z}e_N(y,t)dydz\right)\\
&\ \vdots\\
&= -\Delta t\left(\frac{x-l_1}{1-l_1}\int_{l_1}^{1}\int_{l_1}^{z}\sum_{j=1}^{s}e_N(y,t_j)dydz-\int_{l_1}^{x}\int_{l_1}^{z}\sum_{j=1}^{s}e_N(y,t_j)dydz\right)\\
&\quad -(t-t_s)\left(\frac{x-l_1}{1-l_1}\int_{l_1}^{1}\int_{l_1}^{z}e_N(y,t)dydz-\int_{l_1}^{x}\int_{l_1}^{z}e_N(y,t)dydz\right). \qquad (83)
\end{aligned}
$$

In the same way, from equations (79) and (80), we get

$$
\begin{aligned}
v(x,t)-\tilde{v}(x,t) &= -\Delta t\left(\frac{x-l_2}{1-l_2}\int_{l_2}^{1}\int_{l_2}^{z}\sum_{j=1}^{s}\varepsilon_N(y,t_j)dydz-\int_{l_2}^{x}\int_{l_2}^{z}\sum_{j=1}^{s}\varepsilon_N(y,t_j)dydz\right)\\
&\quad -(t-t_s)\left(\frac{x-l_2}{1-l_2}\int_{l_2}^{1}\int_{l_2}^{z}\varepsilon_N(y,t)dydz-\int_{l_2}^{x}\int_{l_2}^{z}\varepsilon_N(y,t)dydz\right). \qquad (84)
\end{aligned}
$$

Corresponding Proposition 3.4, for a fixed $t \in [t_s, t_{s+1}]$, we can conclude that $u_{txx}(x,t) = \sum_{k=-\infty}^{\infty} c_k^s P_k(x)$ and $v_{txx}(x,t) = \sum_{k=-\infty}^{\infty} d_k^s P_k(x)$. As well as, Proposition 3.4 entails that

$$\lim_{N\to\infty}\|e_N(x,t)\|_\infty = \lim_{h\to 0}\|e_N(x,t)\|_\infty = 0, \qquad (85)$$

$$\lim_{N\to\infty}\|\varepsilon_N(x,t)\|_\infty = \lim_{h\to 0}\|\varepsilon_N(x,t)\|_\infty = 0. \qquad (86)$$

From (83) and (84),

$$\|u(x,t)-\tilde{u}(x,t)\|_\infty \le \left(\frac{1}{2}|x-l_1|(1-l_1)+\frac{(l_1-x)^2}{2}\right)\left(\Delta t\sum_{j=1}^{s}\left\|e_N(x,t_j)\right\|_\infty + (t-t_s)\|e_N(x,t)\|_\infty\right), \qquad (87)$$

$$\|v(x,t)-\tilde{v}(x,t)\|_\infty \le \left(\frac{1}{2}|x-l_2|(1-l_2)+\frac{(l_2-x)^2}{2}\right)\left(\Delta t\sum_{j=1}^{s}\varepsilon\left\|e_N(x,t_j)\right\|_\infty + (t-t_s)\|\varepsilon_N(x,t)\|_\infty\right). \qquad (88)$$

By limiting (87) and (88), using induction and applying (85) and (86), one can conclude that

$$\lim_{h\to 0}\|u(x,t)-\tilde{u}(x,t)\|_\infty = 0, \qquad \lim_{h\to 0}\|v(x,t)-\tilde{v}(x,t)\|_\infty = 0.$$

That is, maximum errors tend to zero as $h \to 0$.
Also, from Proposition 5.1 we know that

$$\lim_{\Delta t\to 0}\|u(0,t)-\tilde{u}(0,t)\|_\infty = 0, \qquad \lim_{\Delta t\to 0}\|v(0,t)-\tilde{v}(0,t)\|_\infty = 0.$$

□

In remaining of this section we try to show that

Theorem 5.3. *Let $e_N(x,t)$ and $\varepsilon_N(x,t)$ be the errors in numerical scheme defined by (70) and (71), then for any positive integer N, it holds that $\|e_N(x,t)\|_\infty = \|\varepsilon_N(x,t)\|_\infty$.*

Proof. From Section 4, we know that

$$\begin{aligned} u^2(x,t)v(x,t) &= 2u(x,t_s)v(x,t_s)u(x,t) - 2u^2(x,t_s)v(x,t_s) + u^2(x,t_s)v(x,t) + c_2(x,t_s)(u(x,t)-u(x,t_s))^2 \\ &\quad + c_1(x,t_s)(u(x,t)-u(x,t_s))(v(x,t)-v(x,t_s)). \end{aligned} \tag{89}$$

From equation (1) and (89), we have

$$\begin{aligned} u_t(x,t) - \kappa u_{xx}(x,t) + \beta u(x,t) &= 2u(x,t_s)v(x,t_s)u(x,t) - 2u^2(x,t_s)v(x,t_s) + u^2(x,t_s)v(x,t) \\ &\quad + c_2(x,t_s)(u(x,t)-u(x,t_s))^2 + c_1(x,t_s)(u(x,t)-u(x,t_s))(v(x,t)-v(x,t_s)). \end{aligned} \tag{90}$$

In numerical solution, we put

$$\tilde{u}_t(x,t) = \kappa\tilde{u}_{xx}(x,t) - \beta\tilde{u}(x,t) + 2\tilde{u}(x,t_s)\tilde{v}(x,t_s)\tilde{u}(x,t) - 2\tilde{u}^2(x,t_s)\tilde{v}(x,t_s) + \tilde{u}^2(x,t_s)\tilde{v}(x,t). \tag{91}$$

Substituting (72), (75), (76) and (79) into (91), we obtain

$$\begin{aligned} u_t(x,t) &= \kappa u_{xx}(x,t) - \beta u(x,t) + \kappa\Big(\tilde{u}_{xx}(x,t_s) - u_{xx}(x,t_s) - (t-t_s)e_N(x,t)\Big) - \beta\Big(\tilde{u}(x,t_s) - u(x,t_s)\Big) \\ &\quad + 2\tilde{u}(x,t_s)\tilde{v}(x,t_s)\Big(\tilde{u}(x,t_s) + u(x,t) - u(x,t_s)\Big) \\ &\quad - 2\tilde{u}^2(x,t_s)\tilde{v}(x,t_s) + \tilde{u}^2(x,t_s)\Big(\tilde{v}(x,t_s) + v(x,t) - v(x,t_s)\Big) \\ &\quad + \tilde{u}^2(x,t_s)(t-t_s)\left(\frac{x-l_2}{1-l_2}\int_{l_2}^1\int_{l_2}^z \varepsilon_N(y,t)dydz - \int_{l_2}^x\int_{l_2}^z \varepsilon_N(y,t)dydz\right) \\ &\quad + \Big(-1-\beta(t-t_s) + 2\tilde{u}(x,t_s)\tilde{v}(x,t_s)(t-t_s)\Big)\left(\frac{x-l_1}{1-l_1}\int_{l_1}^1\int_{l_1}^z e_N(y,t)dydz - \int_{l_1}^x\int_{l_1}^z e_N(y,t)dydz\right). \end{aligned} \tag{92}$$

By substituting (90) in left side of (92), we have

$$\begin{aligned} &2u(x,t_s)v(x,t_s)u(x,t) - 2u^2(x,t_s)v(x,t_s) + u^2(x,t_s)v(x,t) + c_2(x,t_s)\Big(u(x,t)-u(x,t_s)\Big)^2 \\ &\quad + c_1(x,t_s)(u(x,t)-u(x,t_s))(v(x,t)-v(x,t_s)) + \beta\left(\tilde{u}(x,t_s)-u(x,t_s)\right) + 2\tilde{u}^2(x,t_s)\tilde{v}(x,t_s) \\ &\quad - 2\tilde{u}(x,t_s)\tilde{v}(x,t_s)\left(\tilde{u}(x,t_s) + u(x,t) - u(x,t_s)\right) - \tilde{u}^2(x,t_s)\left(\tilde{v}(x,t_s) + v(x,t) - v(x,t_s)\right) \\ &= \kappa\left(\tilde{u}_{xx}(x,t_s) - u_{xx}(x,t_s) - (t-t_s)e_N(x,t)\right) \\ &\quad + \tilde{u}^2(x,t_s)(t-t_s)\left(\frac{x-l_2}{1-l_2}\int_{l_2}^1\int_{l_2}^z \varepsilon_N(y,t)dydz - \int_{l_2}^x\int_{l_2}^z \varepsilon_N(y,t)dydz\right) \\ &\quad + \Big(-1-\beta(t-t_s) + 2\tilde{u}(x,t_s)\tilde{v}(x,t_s)(t-t_s)\Big)\left(\frac{x-l_1}{1-l_1}\int_{l_1}^1\int_{l_1}^z e_N(y,t)dydz - \int_{l_1}^x\int_{l_1}^z e_N(y,t)dydz\right). \end{aligned} \tag{93}$$

Similarly, using equations (2), (89), (76), (77) and (79), one can show that

$$\begin{aligned} &2u(x,t_s)v(x,t_s)u(x,t) - 2u^2(x,t_s)v(x,t_s) + u^2(x,t_s)v(x,t) + c_2(x,t_s)\Big(u(x,t)-u(x,t_s)\Big)^2 \\ &\quad + c_1(x,t_s)(u(x,t)-u(x,t_s))(v(x,t)-v(x,t_s)) + \beta\left(\tilde{u}(x,t_s)-u(x,t_s)\right) + 2\tilde{u}^2(x,t_s)\tilde{v}(x,t_s) \\ &\quad - 2\tilde{u}(x,t_s)\tilde{v}(x,t_s)\left(\tilde{u}(x,t_s) + u(x,t) - u(x,t_s)\right) - \tilde{u}^2(x,t_s)\left(\tilde{v}(x,t_s) + v(x,t) - v(x,t_s)\right) \\ &= -\kappa\left(\tilde{v}_{xx}(x,t_s) - v_{xx}(x,t_s) - (t-t_s)\varepsilon_N(x,t)\right) \\ &\quad - \Big(\beta(t-t_s) - 2\tilde{u}(x,t_s)\tilde{v}(x,t_s)(t-t_s)\Big)\left(\frac{x-l_1}{1-l_1}\int_{l_1}^1\int_{l_1}^z e_N(y,t)dydz - \int_{l_1}^x\int_{l_1}^z e_N(y,t)dydz\right) \\ &\quad - \Big(-1-\tilde{u}^2(x,t_s)(t-t_s)\Big)\left(\frac{x-l_2}{1-l_2}\int_{l_2}^1\int_{l_2}^z \varepsilon_N(y,t)dydz - \int_{l_2}^x\int_{l_2}^z \varepsilon_N(y,t)dydz\right). \end{aligned} \tag{94}$$

Left side of equations (93) and (94) are equal, so we obtain

$$\kappa\Big(\tilde{u}_{xx}(x,t_s) - u_{xx}(x,t_s) - (t-t_s)e_N(x,t)\Big) + \kappa\Big(\tilde{v}_{xx}(x,t_s) - v_{xx}(x,t_s) - (t-t_s)\varepsilon_N(x,t)\Big)$$
$$= \frac{x-l_1}{1-l_1}\int_{l_1}^{1}\int_{l_1}^{z} e_N(y,t)dydz - \int_{l_1}^{x}\int_{l_1}^{z} e_N(y,t)dydz$$
$$+ \frac{x-l_2}{1-l_2}\int_{l_2}^{1}\int_{l_2}^{z} \varepsilon_N(y,t)dydz - \int_{l_2}^{x}\int_{l_2}^{z} \varepsilon_N(y,t)dydz. \tag{95}$$

Applying (81) and (82) in left side of (95), we obtain

$$\kappa\Delta t\sum_{j=1}^{s}\Big(e_N(x,t_j) + \varepsilon_N(x,t_j)\Big) + \kappa(t-t_s)\Big(e_N(x,t) + \varepsilon_N(x,t)\Big)$$
$$= \left(\int_{l_1}^{x}\int_{l_1}^{z} e_N(y,t)dydz - \frac{x-l_1}{1-l_1}\int_{l_1}^{1}\int_{l_1}^{z} e_N(y,t)dydz\right)$$
$$+ \left(\int_{l_2}^{x}\int_{l_2}^{z} \varepsilon_N(y,t)dydz - \frac{x-l_2}{1-l_2}\int_{l_2}^{1}\int_{l_2}^{z} \varepsilon_N(y,t)dydz\right), \tag{96}$$

Second derivative of (96), implies that $e_N(x,t) + \varepsilon_N(x,t) = 0$. That is,

$$\|e_N(x,t)\|_\infty = \|\varepsilon_N(x,t)\|_\infty. \tag{97}$$

□

6. Numerical examples

In this section, we present the numerical results of introduced method on two problems. It is notable that we perform all of the computations by MATLAB R2017a software on a 64-bit PC with 2.40 GHz processor and 4 GB memory.

In numerical examples, we suppose that $u(x,t)$ denote the exact solution and $\tilde{u}(x,t)$ denote the estimated solution. Also, we assume that $T = 20$, $l_1 = l_2 = 0.1$ and $\Delta t = h = 0.01$, so; according to equations (83), (84) and (97), we have $\|u(x,t) - \tilde{u}(x,t)\|_\infty = \|v(x,t) - \tilde{v}(x,t)\|_\infty$. Therefore; the results will be revealed only for $\|u(x,t) - \tilde{u}(x,t)\|_\infty$. The exact solutions to problems are available in [15].

In order to calculate the order of convergence rate of the proposed method p, numerically; suppose that h is a fixed number like $h = 0.01$. We use the following Log ratio formula [31–33]

$$p = \frac{\log\left\|\tilde{u}(0,t)_{\Delta t_1} - u(0,t)\right\|_\infty - \log\left\|\tilde{u}(0,t)_{\Delta t_2} - u(0,t)\right\|_\infty}{\log(\Delta t_1) - \log(\Delta t_2)}. \tag{98}$$

Then, suppose that Δt is a fix number like $\Delta t = 0.01$, and we use the following Log ratio formula

$$p = \frac{\log\left\|\tilde{u}(0,t)_{h_1} - u(0,t)\right\|_\infty - \log\left\|\tilde{u}(0,t)_{h_2} - u(0,t)\right\|_\infty}{\log(h_1) - \log(h_2)}. \tag{99}$$

Example 6.1. *We consider the following coupled nonlinear reaction–diffusion equations*

$$u_t = \frac{3}{4}u_{xx} + u^2 v - \frac{2}{3}u,$$
$$v_t = \frac{3}{4}v_{xx} - u^2 v + \frac{2}{3}u.$$

The exact solution is given by

$$u(x,t) = 1 + \tanh\left(\sqrt{\frac{2}{3}}x + \frac{2}{3}t\right),$$
$$v(x,t) = \frac{4}{3} - \tanh\left(\sqrt{\frac{2}{3}}x + \frac{2}{3}t\right).$$

Figure 2 indicates the relative errors of $u(0,t)$ and $v(0,t)$. The exact and numerical solutions of $u(x,t)$ and $v(x,t)$ are depicted in Figure 3. In Figure 4, the log-log plot is shown for absolute errors of $u(0,t)$ as a function of Δt and $h = 0.01$ is a fixed number. Also, the absolute errors of $u(0,t)$ are shown in Figure 5, as a function of h and $\Delta t = 0.01$ is a fixed number. Tables 1 and 2 present a comparison between the exact and numerical solutions of $u(0,t)$ and $v(0,t)$ for $0 \leq t \leq 1$ and $1 < t \leq 20$, respectively. The order of convergence rate, relative error and condition number of system (59), for fixed $h = 0.01$, which calculated by (98); are tabulated in Table 3. The order of convergence rate, relative error and condition number of system (59), for fixed $\Delta t = 0.01$, which calculated by (99); are presented in Table 4.

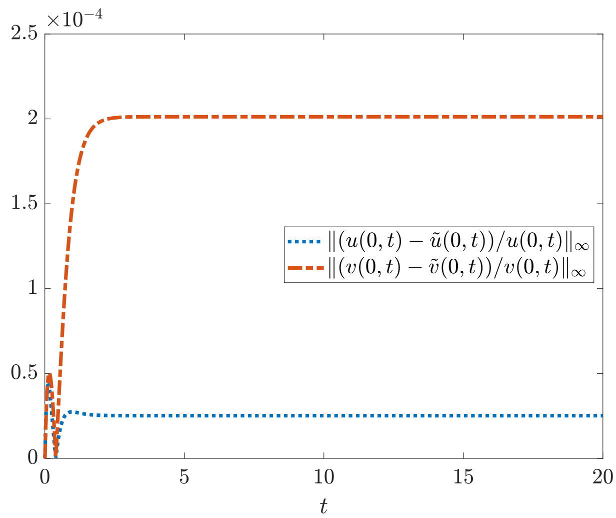

Figure 2: Relative errors of $u(0,t)$ and $v(0,t)$ of Example 6.1.

Table 1: Comparison of the exact and numerical solutions of $u(0,t)$ and $v(0,t)$ for $0 \leq t \leq 1$, in Example 6.1.

t	$u(0,t)$	$\tilde{u}(0,t)$	$v(0,t)$	$\tilde{v}(0,t)$	$\|u(0,t) - \tilde{u}(0,t)\|_\infty$
0.1	1.066568	1.066579	1.266765	1.266754	$1.110119e-05$
0.2	1.132549	1.132566	1.200785	1.200767	$1.740528e-05$
0.3	1.197375	1.197395	1.135958	1.135938	$2.009849e-05$
0.4	1.260520	1.260541	1.072813	1.072793	$2.029087e-05$
0.5	1.321513	1.321531	1.011821	1.011802	$1.871454e-05$
0.6	1.379949	1.379965	0.953384	0.953368	$1.591577e-05$
0.7	1.435502	1.435514	0.897831	0.897819	$1.232883e-05$
0.8	1.487925	1.487933	0.845408	0.845400	$8.300881e-06$
0.9	1.537050	1.537054	0.796284	0.796280	$4.101888e-06$
1.0	1.582783	1.582783	0.750550	0.750550	$6.763994e-08$

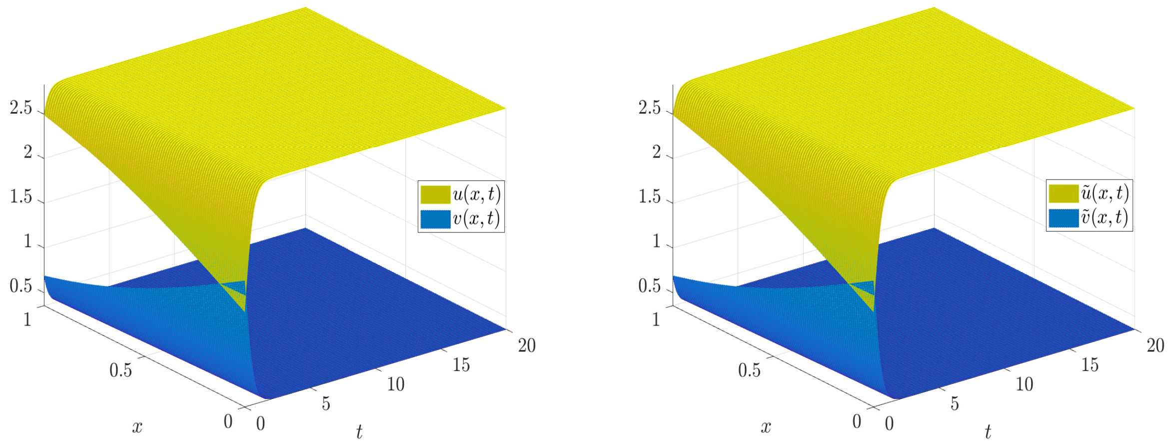

Figure 3: Comparison of the exact (left) and numerical (right) solutions of $u(x,t)$ and $v(x,t)$, in Example 6.1.

Table 2: Comparison of the exact and numerical solutions of $u(0,t)$ and $v(0,t)$ for $1 < t \leq 20$, in Example 6.1.

t	$u(0,t)$	$\tilde{u}(0,t)$	$v(0,t)$	$\tilde{v}(0,t)$	$\|u(0,t) - \tilde{u}(0,t)\|_\infty$
1.5	1.761594	1.761577	0.571739	0.571756	$1.709990e-05$
2.0	1.870062	1.870035	0.463272	0.463298	$2.662342e-05$
2.5	1.931110	1.931078	0.402224	0.402255	$3.131310e-05$
3.0	1.964028	1.963994	0.369306	0.369339	$3.358395e-05$
3.5	1.981368	1.981333	0.351965	0.352000	$3.469965e-05$
4.0	1.990390	1.990355	0.342943	0.342978	$3.525710e-05$
4.5	1.995055	1.995019	0.338279	0.338314	$3.553896e-05$
5.0	1.997458	1.997422	0.335875	0.335911	$3.568248e-05$
6.0	1.999329	1.999294	0.334004	0.334040	$3.579343e-05$
7.0	1.999823	1.999787	0.333510	0.333546	$3.582259e-05$
10.0	1.999997	1.999961	0.333337	0.333372	$3.583283e-05$
15.0	2.000000	1.999964	0.333333	0.333369	$3.583302e-05$
20.0	2.000000	1.999964	0.333333	0.333369	$3.583302e-05$

Table 3: Comparison of the $\left\|u(0,t) - \tilde{u}(0,t)_{\Delta t}\right\|_\infty$ and $(\Delta t)^2$ for different values of Δt for fixed $h = 0.01$, in Example 6.1.

Δt	$\left\|u(0,t) - \tilde{u}(0,t)_{\Delta t}\right\|_\infty$	Condition Number	p
1/4	0.001532	1.9	
1/8	0.000301	2.41	2.34758
1/16	$5.09425e-05$	3.45	2.56282
1/32	$2.72504e-05$	5.72	0.902592
1/64	$2.51498e-05$	11.12	0.115729
1/128	$1.80901e-05$	25.97	0.475351
1/256	$1.31622e-05$	77.65	0.458792
1/512	$1.04111e-05$	321.78	0.33828

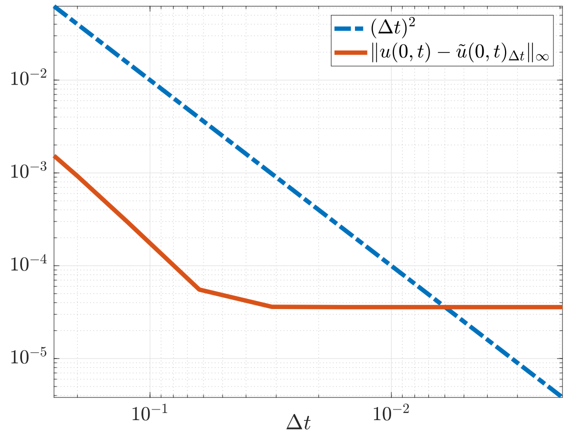

Figure 4: Comparison of the $\left\|u(x,0)-\tilde{u}(x,0)_{\Delta t}\right\|_\infty$ and $(\Delta t)^2$ for different values of Δt for fixed $h = 0.01$, in Example 6.1.

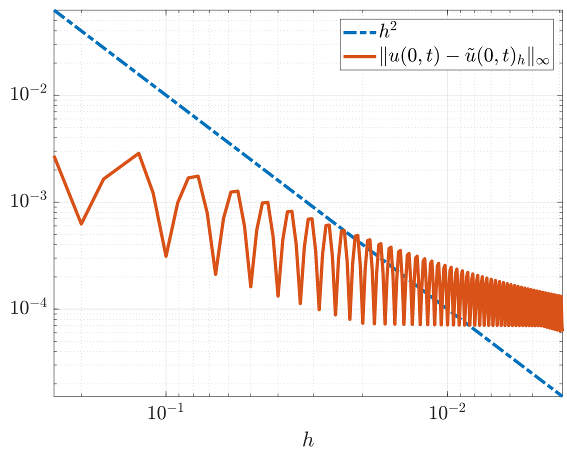

Figure 5: Comparison of the $\left\|u(x,0)-\tilde{u}(x,0)_h\right\|_\infty$ and h^2 for different values of h for fixed $\Delta t = 0.01$, in Example 6.1.

Table 4: Comparison of the $\left\|u(0,t)-\tilde{u}(0,t)_h\right\|_\infty$ and h^2 for different values of Δt for fixed $\Delta t = 0.01$, in Example 6.1.

h	$\left\|u(0,t)-\tilde{u}(0,t)_h\right\|_\infty$	Condition Number	p
1/4	0.081468	8.87	
1/8	0.001372	12.89	5.89188
1/16	0.001407	12.06	−0.036341
1/32	0.000315	23.76	2.12286
1/64	0.000318	22.72	−0.013674
1/128	$9.06976e-05$	17.88	1.80989
1/256	$8.91848e-05$	17.57	0.024265
1/512	$1.94041e-05$	18.26	2.20044

Example 6.2. *Consider the following coupled nonlinear reaction–diffusion equations [20]*

$$u_t = u_{xx} + u^2 v - u,$$
$$v_t = v_{xx} - u^2 v + u,$$

with the exact solution

$$u(x,t) = \sqrt{2}\left(1 + \tanh\left(x + \frac{3}{2}t\right)\right),$$
$$v(x,t) = \sqrt{2}\left(\frac{5}{4} - \tanh\left(x + \frac{3}{2}t\right)\right).$$

The relative errors for $u(0,t)$ and $v(0,t)$ are presented in Figure 6. Figure 7 indicates the exact and numerical solutions of $u(x,t)$ and $v(x,t)$. The log-log plot for absolute errors of $u(0,t)$ is shown in Figure 8, as a function of Δt and $h = 0.01$ is a fixed number. Also, the absolute errors of $u(0,t)$ as a function of h and $\Delta t = 0.01$ is a fixed number, is shown in Figure 9. Comparison between the exact and numerical solutions of $u(0,t)$ and $v(0,t)$ for $0 \le t \le 1$ and $1 < t \le 20$, are presented in Table 5 and 6, respectively. Table 7 indicates the order of convergence rate, relative error and condition number of system (59), for fixed $h = 0.01$, which calculated by (98). Table 8 presents the order of convergence rate, relative error and condition number of system (59), for fixed $\Delta t = 0.01$, which calculated by (99). A comparison between presented method and methods mentioned in [20], for calculating $u(0,t)$ and $v(0,t)$ are indicated, in Table 9 and 10,respectively. In the last Tables the total errors were used [34], where have been calculated using the following formulae

$$E_u = \left(\frac{1}{S-1}\sum_{s=1}^{S}(u(0,t_s) - \tilde{u}(0,t_s))^2\right)^{\frac{1}{2}},$$
$$E_v = \left(\frac{1}{S-1}\sum_{s=1}^{S}(v(0,t_s) - \tilde{v}(0,t_s))^2\right)^{\frac{1}{2}}.$$

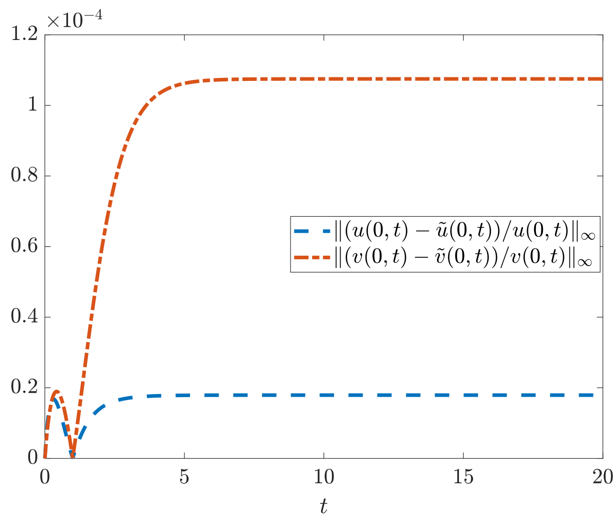

Figure 6: Relative errors of $u(0,t)$ and $v(0,t)$ of Example 6.2.

Table 5: Comparison of the exact and numerical solutions of $u(0,t)$ and $v(0,t)$ for $0 \leq t \leq 1$, in Example 6.2.

t	$u(0,t)$	$\tilde{u}(0,t)$	$v(0,t)$	$\tilde{v}(0,t)$	$\|\|u(0,t) - \tilde{u}(0,t)\|\|_\infty$
0.1	1.624769	1.624838	1.557212	1.557143	$6.883956e-05$
0.2	1.826192	1.826257	1.355789	1.355724	$6.511758e-05$
0.3	2.010869	2.010902	1.171112	1.171078	$3.317332e-05$
0.4	2.173716	2.173714	1.008264	1.008266	$1.930183e-06$
0.5	2.312450	2.312420	0.869531	0.869561	$3.031716e-05$
0.6	2.427212	2.427162	0.754769	0.754819	$4.994164e-05$
0.7	2.519855	2.519793	0.662126	0.662188	$6.208328e-05$
0.8	2.593179	2.593110	0.588801	0.588870	$6.888778e-05$
0.9	2.650312	2.650239	0.531669	0.531741	$7.229039e-05$
1.0	2.694286	2.694213	0.487694	0.487768	$7.370615e-05$

Table 6: Comparison of the exact and numerical solutions of $u(0,t)$ and $v(0,t)$ for $1 < t \leq 20$, in Example 6.2.

t	$u(0,t)$	$\tilde{u}(0,t)$	$v(0,t)$	$\tilde{v}(0,t)$	$\|\|u(0,t) - \tilde{u}(0,t)\|\|_\infty$
1.5	2.797351	2.797279	0.384629	0.384702	$7.269528e-05$
2.0	2.821433	2.821362	0.360547	0.360619	$7.155251e-05$
2.5	2.826864	2.826792	0.355117	0.355188	$7.124628e-05$
3.0	2.828078	2.828007	0.353902	0.353974	$7.117533e-05$
5.0	2.828426	2.828355	0.353554	0.353625	$7.115483e-05$
7.0	2.828427	2.828356	0.353553	0.353625	$7.115478e-05$
10.0	2.828427	2.828356	0.353553	0.353625	$7.115478e-05$
15.0	2.828427	2.828356	0.353553	0.353625	$7.115478e-05$
20.0	2.828427	2.828356	0.353553	0.353625	$7.115478e-05$

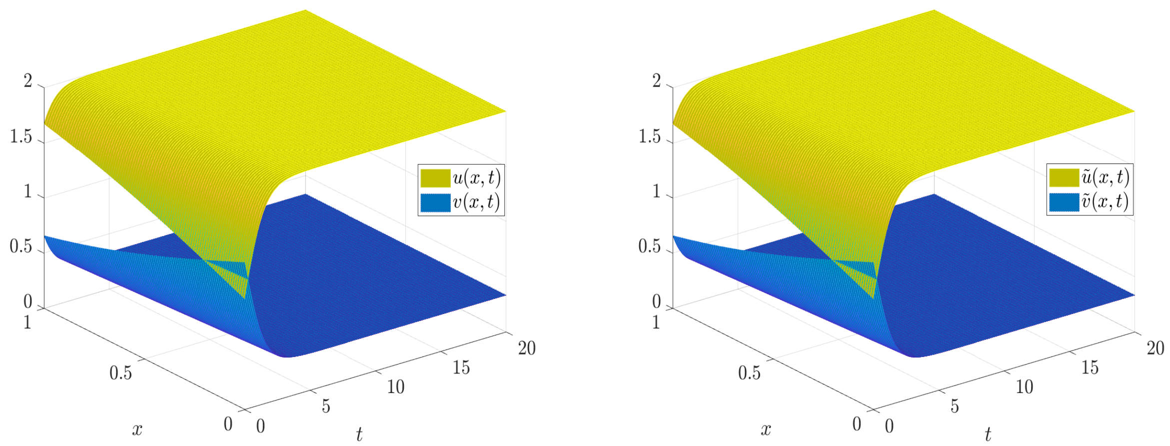

Figure 7: Comparison of the exact (left) and numerical (right) solutions of $u(x,t)$ and $v(x,t)$ in Example 6.2.

Table 7: Comparison of the $\left\|u(0,t)-\tilde{u}(0,t)_{\Delta t}\right\|_{\infty}$ and $(\Delta t)^2$ for different values of Δt for fixed $h = 0.01$, in Example 6.2.

Δt	$\left\|u(0,t)-\tilde{u}(0,t)_{\Delta t}\right\|_{\infty}$	Condition Number	p
1/4	0.015392	2.19	
1/8	0.003599	2.58	2.09651
1/16	0.000683	3.36	2.39764
1/32	0.000151	5.02	2.17734
1/64	$8.47217e-05$	8.83	0.833745
1/128	$7.00614e-05$	18.57	0.274111
1/256	$6.38795e-05$	48.9	0.133267
1/512	$6.09421e-05$	173.87	0.067914

Table 8: Comparison of the $\left\|u(0,t)-\tilde{u}(0,t)_{h}\right\|_{\infty}$ and h^2 for different values of Δt for fixed $\Delta t = 0.01$, in Example 6.2.

h	$\left\|u(0,t)-\tilde{u}(0,t)_{h}\right\|_{\infty}$	Condition Number	p
1/4	0.140951	7.3	
1/8	0.002708	10.47	5.70182
1/16	0.002863	9.97	0.080299
1/32	0.000702	16.59	2.02798
1/64	0.000693	16.01	0.018615
1/128	0.000207	13.45	1.74322
1/256	0.000206	13.25	0.006986
1/512	$6.18851e-05$	13.6	1.73498

Table 9: Approximation of $u(0,t)$ for the present method and numerical methods (Haar, FDM and RBF) proposed in [20] for Example 6.2.

t	$u(0,t)$	$\tilde{u}(0,t)$	Haar	FDM	RBF
0.1	1.624769	1.624838	1.624616	0.935783	1.793949
0.2	1.826192	1.826257	1.826197	1.255080	2.199219
0.3	2.010869	2.010902	2.010769	1.539690	2.097656
0.4	2.173716	2.173714	2.173473	1.798333	2.330078
0.5	2.312450	2.312420	2.312082	2.030951	2.437500
0.6	2.427212	2.427162	2.426757	2.230080	1.986328
0.7	2.519855	2.519793	2.519349	2.427156	2.656250
0.8	2.593179	2.593110	2.592650	2.539897	2.730469
0.9	2.650312	2.650239	2.649777	2.638873	2.715820
1.0	2.694286	2.694213	2.693755	2.713157	2.725586
Total errors	–	$5.50758e-05$	$3.778482e-04$	$3.831014e-01$	$1.948005e-01$

7. Conclusion

In this paper, an orthogonal basis for space of linear splines is introduced, and also; some of its new properties were studied. Besides, a numerical method based on OL–splines is proposed in order to estimate the inverse problems of identifying the unknown boundary condition of a coupled nonlinear RDE. In order to linearize the nonlinear term in the equations, the quasi-linearization method was used. The related system of the linear equations was well-posed and so; there is no need to use methods such as the regularization method. The error estimation and convergence of the method was investigated. Furthermore, the experimental numerical convergence rate is computed, which demonstrates the first-order convergence of the presented method. The results of the research demonstrate the accuracy and

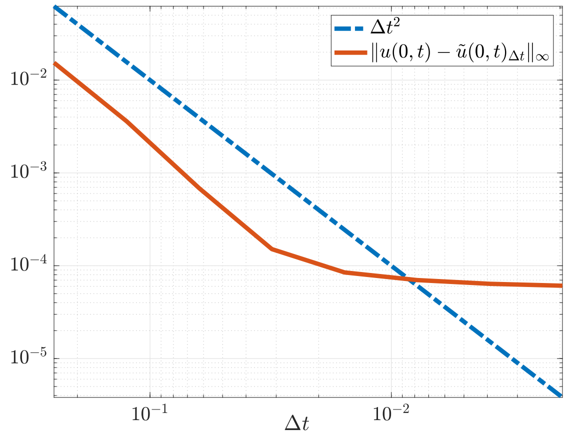

Figure 8: Comparison of the $\left\|u(x,0) - \tilde{u}(x,0)_{\Delta t}\right\|_\infty$ and $(\Delta t)^2$ for different values of Δt for fixed $h = 0.01$, in Example 6.2.

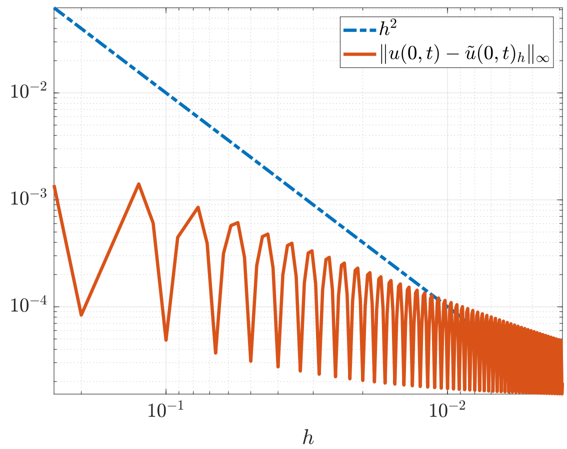

Figure 9: Comparison of the $\left\|u(x,0) - \tilde{u}(x,0)_h\right\|_\infty$ and h^2 for different values of h for fixed $\Delta t = 0.01$, in Example 6.2.

Table 10: Approximation of $v(0,t)$ for the present method and numerical methods proposed in [20] for Example 6.2.

t	$v(0,t)$	$\tilde{v}(0,t)$	Haar	FDM	RBF
0.1	1.557212	1.557143	1.557074	1.786407	1.387463
0.2	1.355789	1.355724	1.355859	1.393373	1.239258
0.3	1.171112	1.171078	1.171382	1.014156	0.996094
0.4	1.008264	1.008266	1.008717	0.690760	0.893555
0.5	0.869531	0.869561	0.870112	0.404529	0.769531
0.6	0.754769	0.754819	0.755419	0.161741	−0.250000
0.7	0.662126	0.662188	0.662800	−0.051236	0.593384
0.8	0.588801	0.588870	0.589469	−0.200171	0.541016
0.9	0.531669	0.531741	0.532316	−0.320760	0.468750
1.0	0.487694	0.487768	0.488313	−0.427614	0.459290
Total errors	–	$5.50758e-05$	$5.115574e-04$	$5.525659e-01$	$1.838014e-01$

applicability of the method.

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