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Quantum Hermite-Hadamrd-Fejér Type Inequalities for (σ, h)-Convex Functions

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Abstract. The aim of this article is to establish some new quantum analogues of Hermite-Hadamard-Féjer type inequalities involving Riemann type of quantum integrals. In order to obtain the main results of the paper, we use the classes of harmonically convex functions, σ-convex functions and (σ, h)-convex functions.

1. Introduction and Preliminaries

A function $\mathcal{X} : I \subseteq \mathbb{R} \to \mathbb{R}, I$ is an interval, is said to be a convex function on I if

$$\mathcal{X}(\mu x + (1-\mu)y) \leq \mu \mathcal{X}(x) + (1-\mu)\mathcal{X}(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\mu \in [0,1]$. If the reversed inequality in (1.1) holds, then $\mathcal{X}$ is said to be concave. Let $\mathcal{X} : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I and $a, b \in I$ with $a < b$. The inequality

$$\mathcal{X}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \mathcal{X}(x)\mathrm{d}x \leq \frac{\mathcal{X}(a) + \mathcal{X}(b)}{2} \tag{1.2}$$

is well known in the literature as Hermite-Hadamard's inequality [12, 13].
In [10] Fejér established the following Hermite-Hadamard-Fejér inequality which is the weighted generalization of the Hermite-Hadamard inequality.

Theorem 1.1 ([10]). *Let $\mathcal{X} : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function. Then the inequality*

$$\mathcal{X}\left(\frac{a+b}{2}\right) \int_a^b \mathcal{W}(x)\mathrm{d}x \leq \frac{1}{b-a} \int_a^b \mathcal{X}(x)\mathcal{W}(x)\mathrm{d}x \leq \frac{\mathcal{X}(a) + \mathcal{X}(b)}{2} \int_a^b \mathcal{W}(x)\mathrm{d}x \tag{1.3}$$

holds, where $\mathcal{W} : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric with respect to $\frac{a+b}{2}$.

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In [16], İşcan gave the definition of a harmonically convex function:

Definition 1.2 ([16]). *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $\mathcal{X} : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$\mathcal{X}\left(\frac{xy}{\mu x + (1-\mu)y}\right) \leq \mu\mathcal{X}(y) + (1-\mu)\mathcal{X}(x) \tag{1.4}$$

for all $x, y \in I$ and $\mu \in [0,1]$. If the inequality in (1.4) is reversed, then $\mathcal{X}$ is said to be harmonically concave.

Definition 1.3 ([23]). *A function $\mathcal{W} : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$, if $\mathcal{W}(x) = \mathcal{W}\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$ holds for all $x \in [a,b]$.*

Definition 1.4 ([17]). *Let $I \subset (0,\infty)$ be a real interval and $\sigma \in \mathbb{R}\backslash\{0\}$. A function $\mathcal{X} : I \to \mathbb{R}$ is said to be σ-convex, if*

$$\mathcal{X}\left([\mu x^{\sigma} + (1-\mu)y^{\sigma}]^{\frac{1}{\sigma}}\right) \leq \mu\mathcal{X}(x) + (1-\mu)\mathcal{X}(y) \tag{1.5}$$

for all $x, y \in I$ and $\mu \in [0,1]$.

It can be easily seen that for $\sigma = 1$ and $\sigma = -1$, σ-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

Definition 1.5 ([22]). *Let $\sigma \in \mathbb{R}\backslash\{0\}$. A function $\mathcal{W} : [a,b] \subset (0,\infty) \to \mathbb{R}$ is said to be σ-symmetric with respect to $[\frac{a^{\sigma}+b^{\sigma}}{2}]^{\frac{1}{\sigma}}$, if $\mathcal{W}(x) = \mathcal{W}([a^{\sigma} + b^{\sigma} - x^{\sigma}]^{\frac{1}{\sigma}})$ holds for all $x \in [a,b]$.*

Definition 1.6 ([9]). *Let $h : J \to \mathbb{R}$ be a non-negative, non zero-function and $(0,1) \subseteq J$. We say that $\mathcal{X} : I \to \mathbb{R}$ is a (σ,h)-convex function or that $\mathcal{X}$ belongs to the class $ghx(h,\sigma,I)$, if $\mathcal{X}$ is non-negative and*

$$\mathcal{X}\left([\mu x^{\sigma} + (1-\mu)y^{\sigma}]^{\frac{1}{\sigma}}\right) \leq h(\mu)\mathcal{X}(x) + h(1-\mu)\mathcal{X}(y), \tag{1.6}$$

for all $x, y \in I$ and $\mu \in (0,1)$. If the reversed the inequality in (1.7) holds, then $\mathcal{X}$ is said to be (σ,h)-concave or belong to the class $ghv(h,\sigma,I)$,.

Note that if we take $h(\mu) = \mu$ in Definition 1.6, then the class of (σ,h)-convex functions reduces to the class of σ-convex functions. If we take $h(\mu) = \mu^{s}$, then we have the class of Breckner type of σ, s-convex functions, see [25]. For $h(\mu) = 1$, we have the class of (σ,P)-convex functions, see [25]. And if we suppose $h(\mu) = \mu(1-\mu)$, then we have a new class known as (σ,tgs)-convex functions. which is defined as:

Definition 1.7. *We say that $\mathcal{X} : I \to \mathbb{R}$ is a (σ,tgs)-convex function or that $\mathcal{X}$ belongs to the class $ghx(tgs,\sigma,I)$, if $\mathcal{X}$ is non-negative and*

$$\mathcal{X}\left([\mu x^{\sigma} + (1-\mu)y^{\sigma}]^{\frac{1}{\sigma}}\right) \leq \mu(1-\mu)[\mathcal{X}(x) + \mathcal{X}(y)], \tag{1.7}$$

for all $x, y \in I$ and $\mu \in (0,1)$.

Quantum calculus often known as calculus without limits. It is considered as bridge between mathematics and physics. Due to its great many applications in various fields of mathematics such as in number theory, combinatorics, orthogonal polynomials and basic hypergeometric functions etc. It is known that there are two types of q-addition, the Nalli-Ward-Al-Salam q-addition and the Jackson-Hahn-Cigler q-addition. The first one is commutative and associative, while the second one is neither. That is why sometimes more than one q-analogue of a mathematical object exists. The book by Kac and Cheung[19] contains some very useful fundamental knowledge on quantum calculus. For some recent studies pertaining to classical inequalities and their variants interested readers are refereed to [1–6, 8, 11, 14, 15, 20, 21, 26, 28, 29, 31, 32].

The aim of this work is to establish the q-analogues of Hermite-Hadamard-Fejér inequalities for some convex type functions. For this we recall some basic concepts of quantum calculus.
Let $0 < q < 1$, the q-Jackson integral from 0 to b is defined as:

$$\int_0^b \mathcal{X}(x)d_qx = (1-q)b\sum_{n=0}^{\infty}\mathcal{X}(bq^n)q^n \tag{1.8}$$

provided the sum converge absolutely.
The q-Jackson integral in a generic interval $[a,b]$ is given by:

$$\int_a^b \mathcal{X}(x)d_qx = \int_0^b \mathcal{X}(x)d_qx - \int_0^a \mathcal{X}(x)d_qx \tag{1.9}$$

For more details, see [18].
In [27] authors presented a Riemann-type q-integral by:

$$\mathcal{R}_q(\mathcal{X};a,b) = (b-a)(1-q)\sum_{k=0}^{\infty}\mathcal{X}\left(a+(b-a)q^k\right)q^k \tag{1.10}$$

In [30] authors introduced another definition from the Riemman-type q-integral:

$$\begin{aligned}&\frac{2}{b-a}\int_a^b \mathcal{X}(x)\mathrm{d}_q^{\mathcal{R}}x\\ &= (1-q)\sum_{k=0}^{\infty}\left(\mathcal{X}\left(\frac{a+b}{2}+\frac{b-a}{2}q^k\right)+\mathcal{X}\left(\frac{a+b}{2}-\frac{b-a}{2}q^k\right)\right)q^k\\ &= \int_{-1}^{1}\mathcal{X}\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)d_q\mu = \int_{-1}^{1}\mathcal{X}\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)d_q\mu.\end{aligned}$$

Contrary to the q-Jackson integral, if

$$\mathcal{X}(x) \le g(x), x \in [a,b],$$

then

$$\int_a^b \mathcal{X}(x)\mathrm{d}_q^{\mathcal{R}}x \le \int_a^b g(x)\mathrm{d}_q^{\mathcal{R}}x.$$

In [24] authors established q-analogue of Hermite-Hadamard's inequalities via harmonic convex functions.

Theorem 1.8 ([24]). *Let $\mathcal{X} : I \subseteq (0,\infty) \to \mathbb{R}$ be an harmonic convex function and $a,b \in I$ with $a < b$, we have*

$$\mathcal{X}\left(\frac{2ab}{a+b}\right) \le \frac{ab}{q(b-a)}\int_a^b \frac{\mathcal{X}(x)}{x^2}\mathrm{d}_q^{\mathcal{R}}x \le \frac{\mathcal{X}(a)+\mathcal{X}(b)}{2}.$$

And in [7] authors have proved following new result:

Theorem 1.9 ([7]). *Let $\mathcal{X} : [a,b] \to \mathbb{R}$ be a convex function and $\mathcal{W} : [a,b] \to \mathbb{R}$ be a nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$. If H and F are defined on $[0,1]$ by*

$$H(\mu) = \int_a^b \mathcal{X}\left(\mu x + (1-\mu)\frac{a+b}{2}\right)\sigma(x)\mathrm{d}_q^{\mathcal{R}}x$$

and

$$F(\mu)=\frac{1}{2}\int_a^b\left[\mathcal{X}\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}x\right)\sigma\left(\frac{x+a}{2}\right)+\mathcal{X}\left(\frac{1+\mu}{2}b+\frac{1-\mu}{2}x\right)\sigma\left(\frac{x+b}{2}\right),\right]\mathrm{d}_q^{\mathcal{R}}x$$

then H, F are convex and increasing on $[0,1]$ *and for all* $\mu\in[0,1]$

$$\mathcal{X}\left(\frac{a+b}{2}\right)\int_a^b\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x=H(0)\le H(\mu)\le H(1)=\int_a^b\mathcal{X}(x)\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x \tag{1.11}$$

and

$$\int_a^b\mathcal{X}(x)\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x=F(0)\le F(\mu)\le F(1)=\frac{\mathcal{X}(a)+\mathcal{X}(b)}{2}\int_a^b\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x \tag{1.12}$$

The main motivation of this paper is to obtain some new quantum analogues of Hermite–Hadamard–Féjer type inequalities using Riemann type of quantum integrals essentially using the classes of harmonically convex functions, σ-convex functions and (σ,h)-convex functions. To the best of our knowledge these results are quite new and we hope that the ideas of this paper will inspire interested readers working in this field.

2. Main Results

In this section, we discuss our main results. First result of this section is a lemma which will be helpful in obtaining next results of the paper.

Lemma 2.1. *Let* $\mathcal{W}:[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *be a nonnegative, integrable and harmonically symmetric with respect to* $\frac{2ab}{a+b}$*, then* $u:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$*, defined by* $u(x)=\mathcal{W}(\frac{1}{x})$ *is symmetric with respect to* $\frac{a+b}{2ab}$.

Proof. Consider the function $u(x)=\mathcal{W}(\frac{1}{x}),\ x\in\left[\frac{1}{b},\frac{1}{a}\right]$. Since $\mathcal{W}$ is harmonically symmetric with respect to $\frac{2ab}{a+b}$, for all $x\in[a,b]$, we have

$$u\left(\frac{a+b}{ab}-x\right)=\mathcal{W}\left(\frac{ab}{a+b-abx}\right)=\mathcal{W}\Big(\frac{1}{x}\Big)=u(x).$$

The proof is completed. □

Theorem 2.2. *Let* $\mathcal{X}:I\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *be an harmonically convex function and* $a,b\in I$ *with* $a<b$*. If* $\mathcal{X}\in L[a,b]$ *and* $\mathcal{W}:[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *is nonnegative, integrable and harmonically symmetric with respect to* $\frac{2ab}{a+b}$*, then*

$$\mathcal{X}\left(\frac{2ab}{a+b}\right)\int_a^b\frac{\mathcal{W}(x)}{x^2}\mathrm{d}_q^{\mathcal{R}}x\le\int_a^b\frac{\mathcal{X}(x)\mathcal{W}(x)}{x^2}\mathrm{d}_q^{\mathcal{R}}x\le\frac{\mathcal{X}(a)+\mathcal{X}(b)}{2}\int_a^b\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x$$

Proof. Consider the function $g(x)=\mathcal{X}(\frac{1}{x})$ and $u(x)=\mathcal{W}(\frac{1}{x})$.
Since g is convex on $[\frac{1}{b},\frac{1}{a}]$ and by Lemma 2.1 u is symmetric with respect to $\frac{a+b}{2ab}$, then by (1.11) and (1.12), we have

$$g\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}u(x)\mathrm{d}_q^{\mathcal{R}}x\le\int_{\frac{1}{b}}^{\frac{1}{a}}g(x)u(x)\mathrm{d}_q^{\mathcal{R}}x\le\frac{g(\frac{1}{b})+g(\frac{1}{a})}{2}\int_{\frac{1}{b}}^{\frac{1}{a}}u(x)\mathrm{d}_q^{\mathcal{R}}x.$$

This implies

$$\mathcal{X}\left(\frac{2ab}{a+b}\right)\int_a^b\frac{\mathcal{W}(x)}{x^2}\mathrm{d}_q^{\mathcal{R}}x\le\int_a^b\frac{\mathcal{X}(x)\mathcal{W}(x)}{x^2}\mathrm{d}_q^{\mathcal{R}}x\le\frac{\mathcal{X}(a)+\mathcal{X}(b)}{2}\int_a^b\frac{\mathcal{W}(x)}{x^2}\mathrm{d}_q^{\mathcal{R}}x.$$

This completes the proof. □

Remark 2.3. *Putting $\mathcal{W}(x) = 1$ in Theorem 2.2, we obtain Theorem 1.8.*

We now derive quantum analogues of Hermite-Hadamard's inequality via σ-convex functions using Riemann type of quantum integrals.

Theorem 2.4. *Let $\mathcal{X} : I \subset (0,\infty) \to \mathbb{R}$ be a σ-convex function, $\sigma \in \mathbb{R}\backslash\{0\}$, and $a, b \in I$ with $a < b$. If $\mathcal{X} \in L[a,b]$ then, we have:*

$$\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{[\sigma]_q}{b^\sigma - a^\sigma}\int_a^b x^{\sigma-1}\mathcal{X}(x)\mathrm{d}_q^{\mathcal{R}}x \leq \frac{\mathcal{X}(a)+\mathcal{X}(b)}{2}$$

Proof. Since $\mathcal{X}$ is σ-convex function, we have

$$\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{\mathcal{X}(x)+\mathcal{X}(y)}{2}, \tag{2.1}$$

for all $x, y \in [a,b]$.
In (2.1) if we choose $x = \left(\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right)^{\frac{1}{\sigma}}$ and $y = \left(\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right)^{\frac{1}{\sigma}}$, we get

$$\begin{aligned}\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) &\leq \frac{1}{2}\mathcal{X}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right) + \frac{1}{2}\mathcal{X}\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\ &\leq \frac{\mathcal{X}(a)+\mathcal{X}(b)}{2}.\end{aligned}$$

q-integrating both sides with respect to μ on $[-1,1]$, we obtain

$$2\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{2}{b^\sigma - a^\sigma}\int_{a^\sigma}^{b^\sigma} \mathcal{X}(x^{\frac{1}{\sigma}})\mathrm{d}_q^{\mathcal{R}}x \leq \mathcal{X}(a)+\mathcal{X}(b).$$

This implies

$$2\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{2[\sigma]_q}{b^\sigma - a^\sigma}\int_a^b x^{\sigma-1}\mathcal{X}(x)\mathrm{d}_q^{\mathcal{R}}x \leq \mathcal{X}(a)+\mathcal{X}(b).$$

Then completes the proof of Theorem 2.4. □

Remark 2.5. *If we take $\sigma = -1$ in Theorem 2.4, then we recaptures the result for harmonically convex functions and if we take $\sigma = 1$, then we get the result for classical convex functions.*

Theorem 2.6. *Let $\mathcal{X} : I \subset (0,\infty) \to \mathbb{R}$ be a σ-convex function, $\sigma \in \mathbb{R}\backslash\{0\}, a,\ b \in I$ with $a < b$. If $\mathcal{X} \in L[a,b]$ and $\mathcal{W} : [a,b] \to \mathbb{R}$ is nonnegative, integrable and σ-symmetric with respect to $\left(\frac{a^\sigma+b^\sigma}{2}\right)^{\frac{1}{\sigma}}$, then*

$$\begin{aligned}\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)\int_a^b x^{\sigma-1}\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x &\leq \int_a^b x^{\sigma-1}\mathcal{X}(x)\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x\\ &\leq \frac{\mathcal{X}(a)+\mathcal{X}(b)}{2}\int_a^b x^{\sigma-1}\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x.\end{aligned}$$

Proof. According to the definition of σ-convex function and by $\mathcal{W}$ is positive, we have

$$\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)\mathcal{W}(x) \leq \frac{\mathcal{X}(x)+\mathcal{X}(y)}{2}\mathcal{W}(x), \tag{2.2}$$

for all $x, y \in [a, b]$.
In (2.2), if we choose $x = \left(\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right)^{\frac{1}{\sigma}}$ and $y = \left(\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right)^{\frac{1}{\sigma}}$, we get

$$\begin{aligned}
&\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)\mathcal{W}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&\leq \frac{1}{2}\mathcal{X}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\mathcal{W}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&+ \frac{1}{2}\mathcal{X}\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\mathcal{W}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&\leq \frac{1}{2}\left(\frac{1-\mu}{2}\mathcal{X}(a) + \frac{1+\mu}{2}\mathcal{X}(b) + \frac{1+\mu}{2}\mathcal{X}(a) + \frac{1-\mu}{2}\mathcal{X}(b)\right)\mathcal{W}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&= \frac{\mathcal{X}(a) + \mathcal{X}(b)}{2}\mathcal{W}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right).
\end{aligned}$$

Since $\mathcal{W}(.)$ is σ-symmetric function, so, we have

$$\begin{aligned}
&\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)\mathcal{W}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&\leq \frac{1}{2}\mathcal{X}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\mathcal{W}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&+ \frac{1}{2}\mathcal{X}\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\mathcal{W}\left(\left[a^\sigma + b^\sigma - \frac{1-\mu}{2}a^\sigma - \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&= \frac{1}{2}\mathcal{X}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\mathcal{W}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&+ \frac{1}{2}\mathcal{X}\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\mathcal{W}\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&\leq \frac{\mathcal{X}(a) + \mathcal{X}(b)}{2}\mathcal{W}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)
\end{aligned}$$

q-integrating with respect to μ on $[-1, 1]$, we obtain

$$\begin{aligned}
\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)\int_{a^\sigma}^{b^\sigma}\mathcal{W}(x^{\frac{1}{\sigma}})\mathrm{d}_q^{\mathcal{R}}x &\leq \int_{a^\sigma}^{b^\sigma}\mathcal{X}(x^{\frac{1}{\sigma}})\mathcal{W}(x^{\frac{1}{\sigma}})\mathrm{d}_q^{\mathcal{R}}x\\
&\leq \frac{\mathcal{X}(a) + \mathcal{X}(b)}{2}\int_{a^\sigma}^{b^\sigma}\mathcal{W}(x^{\frac{1}{\sigma}})\mathrm{d}_q^{\mathcal{R}}x.
\end{aligned}$$

This implies

$$\begin{aligned}
\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)\int_a^b x^{\sigma-1}\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x &\leq \int_a^b x^{\sigma-1}\mathcal{X}(x)\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x\\
&\leq \frac{\mathcal{X}(a) + \mathcal{X}(b)}{2}\int_a^b x^{\sigma-1}\mathcal{W}(x)\mathrm{d}_q^{\mathcal{R}}x
\end{aligned}$$

The proof is completed. □

Remark 2.7. *If we choose $\mathcal{W}(x) = 1$, then Theorem 2.6 reduces to Theorem 2.4.*

Remark 2.8. *If we take $\sigma = -1$ in Theorem 2.6, then we recaptures the result for harmonically convex functions and if we take $\sigma = 1$, then we get the result for classical convex functions.*

Theorem 2.9. *Let $\mathcal{X} \in ghx(h, \sigma, I) \cap L[a, b]$, for $a, b \in I$ with $a < b$, then we have*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{[\sigma]_q}{b^\sigma - a^\sigma}\int_a^b x^{\sigma-1}\mathcal{X}(x)\mathrm{d}_q^{\mathcal{R}}x \leq (\mathcal{X}(a) + \mathcal{X}(b))\int_0^1 h(\mu)\mathrm{d}_q^{\mathcal{R}}\mu.$$

Proof. Since $\mathcal{X} \in ghx(h, \sigma, I)$ for all $\mu \in [-1, 1]$ we have

$$\begin{aligned}\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) &\leq h\left(\frac{1}{2}\right)\mathcal{X}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\ &\quad + h\left(\frac{1}{2}\right)\mathcal{X}\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\ &\leq h\left(\frac{1}{2}\right)\left(h\left(\frac{1-\mu}{2}\right) + h\left(\frac{1+\mu}{2}\right)\right)(\mathcal{X}(a) + \mathcal{X}(b)).\end{aligned}$$

q-integrating the above inequality over $[-1, 1]$, we obtain

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) &\leq \frac{[\sigma]_q}{b^\sigma - a^\sigma}\int_a^b x^{\sigma-1}\mathcal{X}(x)\mathrm{d}_q^{\mathcal{R}}x\\ &\leq (\mathcal{X}(a) + \mathcal{X}(b))\int_0^1 h(\mu)\mathrm{d}_q^{\mathcal{R}}\mu.\end{aligned}$$

This completes the proof. □

Remark 2.10. *Note that applying Theorem 2.9 for $h(\mu) = \mu$, we obtain Theorem 2.4. If $h(\mu) = \mu^s$, $s \in (0, 1)$ and $\sigma = 1$ in Theorem 2.9 we have Theorem 2.2 in [30].*

If we take $h(\mu) = \mu^s$ in Theorem 2.9, then we have following new result:

Corollary 2.11. *Under the assumptions of Theorem 2.9 if $\mathcal{X} \in ghx(s, \sigma, I) \cap L[a, b]$, that is $\mathcal{X}$ is Breckner type of (s, σ)-convex function, where $s \in (0, 1]$, then we have*

$$\frac{1}{2^{1-s}}\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{[\sigma]_q}{b^\sigma - a^\sigma}\int_a^b x^{\sigma-1}\mathcal{X}(x)\mathrm{d}_q^{\mathcal{R}}x \leq \frac{\mathcal{X}(a) + \mathcal{X}(b)}{[s+1]_q}.$$

If we take $h(\mu) = 1$ in Theorem 2.9, then we have following new result:

Corollary 2.12. *Under the assumptions of Theorem 2.9 if $\mathcal{X} \in ghx(P, \sigma, I) \cap L[a, b]$, that is $\mathcal{X}$ is (P, σ)-convex function, then we have*

$$\frac{1}{2}\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{[\sigma]_q}{b^\sigma - a^\sigma}\int_a^b x^{\sigma-1}\mathcal{X}(x)\mathrm{d}_q^{\mathcal{R}}x \leq \mathcal{X}(a) + \mathcal{X}(b).$$

If we take $h(\mu) = 1$ in Theorem 2.9, then we have following new result:

Corollary 2.13. *Under the assumptions of Theorem 2.9 if $\mathcal{X} \in ghx(tgs, \sigma, I) \cap L[a,b]$, that is $\mathcal{X}$ is (tgs,σ)-convex function, then we have*

$$2\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right) \leq \frac{[\sigma]_q}{b^\sigma - a^\sigma}\int_a^b x^{\sigma-1}\mathcal{X}(x)\mathrm{d}_q^{\mathcal{R}}x \leq \frac{q^2(\mathcal{X}(a)+\mathcal{X}(b))}{(1+q)(1+q+q^2)}.$$

Theorem 2.14. *Let $\mathcal{X} \in ghx(h_1, \sigma, I), g \in ghx(h_2, \sigma, I)$ be functions such that $fg \in L[a,b], a, b \in I$ with $a < b$ and $h_1h_2 \in L[0,1]$, then*

$$\begin{aligned}
&\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)g\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)\\
&\leq \frac{[\sigma]_q}{b^\sigma - a^\sigma}\int_a^b x^{\sigma-1}\left(2\mathcal{X}(x)g(x) + \mathcal{X}(x)g((a^\sigma + b^\sigma - x^\sigma)^{\frac{1}{\sigma}}) + \mathcal{X}([a^\sigma + b^\sigma - x^\sigma]^{\frac{1}{\sigma}})g(x)\right)\mathrm{d}_q^{\mathcal{R}}x\\
&\leq (\mathcal{X}(a^\sigma) + \mathcal{X}(b^\sigma))\left(g(a^\sigma) + g(b^\sigma)\right)\int_0^1 (2h_1(x)h_2(x) + h_1(x)h_2(1-x) + h_1(1-x)h_2(x))\,\mathrm{d}_q^{\mathcal{R}}x.
\end{aligned}$$

Proof. Since $\mathcal{X} \in ghx(h_1, \sigma, I)$ and $g \in ghx(h_2, \sigma, I)$, we have

$$\begin{aligned}
&\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathcal{X}\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)g\left(\left[\frac{a^\sigma + b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)\\
&\leq \left\{\mathcal{X}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right) + \mathcal{X}\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\right\}\\
&\quad \times \left\{g\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right) + g\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\right\}\\
&= \mathcal{X}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)g\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&\quad + \mathcal{X}\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)g\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&\quad + \mathcal{X}\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)g\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&\quad + \mathcal{X}\left(\left[\frac{1+\mu}{2}a^\sigma + \frac{1-\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)g\left(\left[\frac{1-\mu}{2}a^\sigma + \frac{1+\mu}{2}b^\sigma\right]^{\frac{1}{\sigma}}\right)\\
&\leq \left(h_1\left(\frac{1-\mu}{2}\right) + h_1\left(\frac{1+\mu}{2}\right)\right)\left(h_2\left(\frac{1-\mu}{2}\right) + h_2\left(\frac{1+\mu}{2}\right)\right)\\
&\quad \times (\mathcal{X}(a^\sigma) + \mathcal{X}(b^\sigma))\left(g(a^\sigma) + g(b^\sigma)\right)\\
&= \left\{h_1\left(\frac{1-\mu}{2}\right)h_2\left(\frac{1-\mu}{2}\right) + h_1\left(\frac{1+\mu}{2}\right)h_2\left(\frac{1+\mu}{2}\right)\right\}\\
&\quad \times (\mathcal{X}(a^\sigma) + \mathcal{X}(b^\sigma))\left(g(a^\sigma) + g(b^\sigma)\right)\\
&+ \left\{h_1\left(\frac{1-\mu}{2}\right)h_2\left(\frac{1+\mu}{2}\right) + h_1\left(\frac{1+\mu}{2}\right)h_2\left(\frac{1-\mu}{2}\right)\right\}\\
&\quad \times (\mathcal{X}(a^\sigma) + \mathcal{X}(b^\sigma))\left(g(a^\sigma) + g(b^\sigma)\right).
\end{aligned}$$

q-integrating with respect to μ on $[-1,1]$, we obtain

$$\frac{2}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathcal{X}\left(\left[\frac{a^\sigma+b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)g\left(\left[\frac{a^\sigma+b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)$$
$$\leq \frac{4}{b^\sigma-a^\sigma}\int_{a^\sigma}^{b^\sigma}\mathcal{X}(x^{\frac{1}{\sigma}})g(x^{\frac{1}{\sigma}})\mathrm{d}_q^{\mathcal{R}}x+\frac{2}{b^\sigma-a^\sigma}\int_{a^\sigma}^{b^\sigma}\mathcal{X}(x^{\frac{1}{\sigma}})g((a^\sigma+b^\sigma-x)^{\frac{1}{\sigma}})\mathrm{d}_q^{\mathcal{R}}x$$
$$+\frac{2}{b^\sigma-a^\sigma}\int_{a^\sigma}^{b^\sigma}\mathcal{X}((a^\sigma+b^\sigma-x)^{\frac{1}{\sigma}})g(x^{\frac{1}{\sigma}})\mathrm{d}_q^{\mathcal{R}}x$$
$$\leq\left(4\int_0^1 h_1(x)h_2(x)\mathrm{d}_q^{\mathcal{R}}x+2\int_0^1 h_1(x)h_2(1-x)\mathrm{d}_q^{\mathcal{R}}x+2\int_0^1 h_1(1-x)h_2(x)\mathrm{d}_q^{\mathcal{R}}x\right)$$
$$\times\left(\mathcal{X}(a^\sigma)+\mathcal{X}(b^\sigma)\right)\left(g(a^\sigma)+g(b^\sigma)\right).$$

This implies

$$\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\mathcal{X}\left(\left[\frac{a^\sigma+b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)g\left(\left[\frac{a^\sigma+b^\sigma}{2}\right]^{\frac{1}{\sigma}}\right)$$
$$\leq\frac{[\sigma]_q}{b^\sigma-a^\sigma}\int_a^b x^{\sigma-1}\left(2\mathcal{X}(x)g(x)+\mathcal{X}(x)g((a^\sigma+b^\sigma-x^\sigma)^{\frac{1}{\sigma}})+\mathcal{X}((a^\sigma+b^\sigma-x^\sigma)^{\frac{1}{\sigma}})g(x)\right)\mathrm{d}_q^{\mathcal{R}}x$$
$$\leq\left(\mathcal{X}(a^\sigma)+\mathcal{X}(b^\sigma)\right)\left(g(a^\sigma)+g(b^\sigma)\right)\int_0^1\left(2h_1(x)h_2(x)+h_1(x)h_2(1-x)+h_1(1-x)h_2(x)\right)\mathrm{d}_q^{\mathcal{R}}x.$$

This completes the proof. □

3. Conclusion

We have used the concepts of quantum calculus and obtained some new quantum analogues of Hermite-Hadamard-Féjer type of inequalities essentially using the classes of harmonically convex, σ-convex and (σ,h)-convex functions. We have discussed some new and known special cases of the obtained results which shows that our results are quite unifying one as they relate some unrelated results. To the best of our knowledge the results presented in this paper are new and we hope that the ideas will inspire interested readers working in this field. The ideas of this paper can be used to obtain some new quantum analogues of the these inequalities by using the classes of σ-preinvex and (σ,h)-preinvex functions. This is an interesting problem for future research.

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