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Topology of Non-Triangular Metric Spaces and Related Fixed Point Results

Aniruddha Deshmukh[a], **Dhananjay Gopal**[b]

[a]*Applied Mathematics and Humanities Department, S. V. National Institute of Technology, Surat, Gujarat-395007, India*
[b]*Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur, Chhattisgarh-495009, India*

Abstract. Non-triangular metric spaces have been introduced by Khojasteh and Khandani [12] in 2020. The concept of non-triangular metric spaces is fresh and therefore, requires quite some analysis on its topology. In this article, we have made an attempt to study the topology of non-triangular metric spaces by giving a natural definition of *open sets* in this context. The introduction of non-triangular metric spaces has shown that there is no inherent need for the triangle inequality to prove various fixed point results. Keeping this in mind, we give a fixed point result for Suzuki type Z-contractions in the context of non-triangular metric spaces by introducing a new property of maps.

1. Introduction

With the formal introduction of the concept of a metric and Banach's famous fixed point theorem, the study of non-linear analysis got an acceleration. Using the power of metric (or norms), it became easier to find solutions to many problems by converting them to a fixed point problem. In due course of time, many researchers developed theories which enabled us to study mappings more general than the Banach's contraction mappings on metric spaces. Many researchers also tried to generalize the metric structure itself, by looking at the consequences of dropping various conditions.

One of the first ones to do so was Czerwik, who in [5] introduced the notion of a b-metric space. In this new concept, the triangle inequality of metric spaces had been generalized. In 1959 (much before Czerwik gave the concept of b-metric spaces), Musielak and Orcliz had introduced the notion of modular spaces in the context of vector spaces (for details, see [15]). In a sense, even this concept was a generalization of the triangle inequality of norms. Recently, in 2000, Hitzler and Seda in [7] gave the concept of dislocated metric spaces and dislocated topologies, which have found applications in logical programming. Taking motivation from these generalizations of metric spaces, each of which aimed at weakening the triangle inequality in some sense, Jleli and Samet in 2015 gave the concept of a generalized metric space (now, it is known as the JS-metric space). In this generalization, Jleli and Samet used the power of sequences to discard triangle inequality from the definition of metric space, thereby forming a large class under which all previous generalizations such as the b-metric, modular metric and dislocated metric take shelter (see [8] for details). Jleli and Samet also proved some well-known fixed point results in their version of generalized metric space.

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Email addresses: aniruddha480@gmail.com (Aniruddha Deshmukh), gopaldhananjay@yahoo.in (Dhananjay Gopal)

Very recently, in 2020, Khojasteh and Khandani further weakened the conditions imposed by Jleli and Samet. They utilized the fact the in all these generalizations, the limits of sequences were unique (whenever they existed), and defined what they have called non-triangular metric spaces in [12]. In this paper, Khojasteh and Khandani also prove various fixed point results and conclude that the triangle inequality is not needed for the proof, but is only helpful in certain cases.

Alongside this research, many others had been trying to generalize the concept of Banach contractions. The definition of contractive and nonexpansive maps were natural from the definition of contractions, and many fixed point results have been developed in this regard. It was observed by many researchers that given a map T on a metric space (X, d) and two points $x, y \in X$, there were six displacements one could work with:

$$d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx).$$

Indeed, Banach had used the first two to define the class of contraction mappings. In 1968, Kannan introduced another class of mappings (which are now called Kannan contractions) using the first, third and fourth displacements. Later, in 1972, Chatterjea introduced yet another class of mappings (which are now known as Chatterjea contractions) using the first and the last two displacements from the list. In 1973, Hardy and Rogers came up with a larger class of mappings, called the generalized contractions which used all six displacements in the list and could cover all previous generalizations (for details, see [6] and the references therein). Fixed point results regarding these mappings gave a complete picture of how any "contraction" would behave on a metric space. After a long time, in 2008, Suzuki in [18] further generalized the concept of Banach contraction by introducing C-mappings. We call these mappings as *Suzuki type contractions*. With the introduction of Suzuki type contractions, many new fixed point results became available.

In 2015, Khojasteh et al. introduced the concept of a simulation function and used it to generalize Banach contractions (see [13]), which they called Z-contractions. In their paper, they also proved various fixed point results for this class of mappings. Although this class is not a very large generalization of Banach contractions, it gives insights in generalizing other type of mappings. In 2018, taking motivation of Khojasteh et al. ([13]) and Suzuki ([18]), Kumam et al. introduced the notion of Suzuki type Z-contractions and proved fixed point results for this class of mappings (see [14]).

In this article, we take motivation from all the above works and give results for Suzuki type contractions and Suzuki type Z-contractions in the context of a non-triangular metric space. The article is organized as follows: In section 2, we give the notion of JS-metric space and non-triangular metric space a few fixed point result from [12]. This particular result will be the backbone of our analysis of Suzuki type mappings. Section 3 deals with the topology of a non-triangular metric space. In particular, we give a way to define *open sets* in a non-triangular metric space without forcing the *balls* to be open. We will also see that the definition given here is *natural*. In section 4, we give the concept of simulation function and Suzuki type Z-contractions, and then look at Suzuki type Z-contractions in the context of non-triangular metric spaces. Finally, in section 5, we give the conclusion and future scope for research in this direction.

2. Generalizations of the Metric Structure

In this section, we give the two most recent generalizations of metric structure, namely the JS-metric and the non-triangular metric spaces.

Definition 2.1 (JS-metric space). ([8]) Let X be a nonempty set and $\mathcal{D} : X \times X \to [0, \infty]$ be a function. For each $x \in X$, define a set $C(X, \mathcal{D}, x) = \left\{ (x_n)_{n\in\mathbb{N}} \subseteq X \middle| \lim_{n\to\infty} \mathcal{D}(x_n, x) = 0 \right\}$. Then, $\mathcal{D}$ is a generalized metric (or a JS-metric) on X if:

(JS1) For each $x, y \in X$, $\mathcal{D}(x, y) = 0 \Rightarrow x = y$.

(JS2) For each $x, y \in X$, $\mathcal{D}(x, y) = \mathcal{D}(y, x)$.

(JS3) There is some $K > 0$ such that for each $x, y \in X$ and $(x_n)_{n\in\mathbb{N}} \in C(X, \mathcal{D}, x)$, we have

$$d(x, y) \leq K \limsup_{n\to\infty} d(x_n, y).$$

Here, $(X, \mathcal{D})$ is called a generalized metric space or a JS-metric space.

Remark 2.2. It is easy to see that every metric space is a JS-metric space with $K = 1$.

Immediately from the definition of the JS-metric, we have the following definitions about sequences. They are motivated from their usual definitions in the metric structure. However, since we do not have the triangle inequality directly with us, it may be possible that topological properties behave differently in JS-metric spaces.

Definition 2.3 (Convergence). ([8]) Let $(X, \mathcal{D})$ be a JS-metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in X$ if $\lim\limits_{n\to\infty} \mathcal{D}(x_n, x) = 0$.

Remark 2.4. Notice that the definition of convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ to a point $x \in X$ is equivalent to the condition that $(x_n)_{n\in\mathbb{N}} \in C(X, \mathcal{D}, x)$.

Definition 2.5 (Cauchy sequence). ([8]) Let $(X, \mathcal{D})$ be a JS-metric space. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in X is Cauchy if $\lim\limits_{n,m\to\infty} \mathcal{D}(x_n, x_{n+m}) = 0$.

Remark 2.6. Equivalently, we can say that a sequence $(x_n)_{n\in\mathbb{N}}$ in a JS-metric space is Cauchy if for every $\epsilon > 0$, there is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $m \in \mathbb{N}$, we have $\mathcal{D}(x_n, x_{n+m}) < \epsilon$.

Definition 2.7 (Completeness). ([8]) A JS-metric space $(X, \mathcal{D})$ is said to be complete if every Cauchy sequence in X is convergent to some point $x \in X$.

Theorem 2.8. *Let $(X, \mathcal{D})$ be a JS-metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence. If $x_n \to x$ and $x_n \to y$, then $x = y$. In other words, limit of a sequence is unique whenever it exists.*

Proof. Let $x_n \to x$ and $x_n \to y$. By the definition of convergence, $(x_n)_{n\in\mathbb{N}} \in C(X, \mathcal{D}, x)$. Therefore, using (JS3), we get

$$\mathcal{D}(x, y) \leq K \limsup_{n\to\infty} \mathcal{D}(x_n, y) = 0.$$

Hence, $\mathcal{D}(x, y) = 0$ and using (JS1), we get $x = y$. □

Remark 2.9. Theorem 2.8 tells us that for $x, y \in X$, either $C(X, \mathcal{D}, x) = C(X, \mathcal{D}, y)$ (exactly when $x = y$) or $C(X, \mathcal{D}, x) \cap C(X, \mathcal{D}, y) = \emptyset$. However, we do notice that these sets may be empty as well.

Remark 2.10. Theorem 2.8 shall become the core of defining the non-triangular metric space, and will also help us to see that the new definition (which we will state in a while) will be a generalization of JS-metric space.

Now, we move to a generalization of the JS-metric space, which has been called the non-triangular metric space in [12].

Definition 2.11 (Non-triangular metric space). Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a map. Then, d is called a non-triangular metric on X if:

(NT1) For each $x \in X$, we have $d(x, x) = 0$.

(NT2) For each $x, y \in X$, we have $d(x, y) = d(y, x)$.

(NT3) For every $x, y \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lim\limits_{n\to\infty} d(x_n, x) = 0$ and $\lim\limits_{n\to\infty} d(x_n, y) = 0$, we have $x = y$.

Remark 2.12. It is worth to notice that if we take $d(x, y) = 0$ and a constant sequence $(x_n)_{n\in\mathbb{N}}$, as $x_n = x$ for all $n \in \mathbb{N}$, using (NT3) of Definition 2.11, we get $x = y$.

Since earlier, we had remarked that the uniqueness of limits as proved in Theorem 2.8 would give rise to a "generalization" of JS-metric space, we will now prove that this is indeed the case. That is, we shall prove that a JS-metric space is a non-triangular metric space, at least in some cases. Before we begin the proof, however, we would like to note the fact that in JS-metric space, the distance between two points can be ∞. On the other hand the definition of non-triangular metric spaces does not allow this. Therefore, in the proof that follows, we shall assume that the JS-metric is $\mathcal{D} : X \times X \to [0, \infty)$.

Theorem 2.13. *Let $(X, \mathcal{D})$ be a JS-metric such that for each $x \in X$, the set $C(X, \mathcal{D}, x)$ (as in Definition 2.1) is non-empty. Then, $(X, \mathcal{D})$ is a non-triangular metric space.*

Proof. Let $(X, \mathcal{D})$ be a JS-metric space. Then, (NT2) of Definition 2.11 follows directly from (JS2) of Definition 2.1. Now, let $x, y \in X$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $\lim_{n\to\infty} \mathcal{D}(x_n, x) = 0$ and $\lim_{n\to\infty} \mathcal{D}(x_n, y) = 0$. Then, from (JS3) of Definition 2.1, we get

$$\mathcal{D}(x, y) \le K \limsup_{n\to\infty} \mathcal{D}(x_n, y) = 0.$$

Hence, $\mathcal{D}(x, y) = 0$ and from (JS1) of Definition 2.1, we can conclude that $x = y$. Thus, we get (NT3) of Definition 2.11.

Now, we also have from (JS3) if Definition 2.1,

$$\mathcal{D}(x, x) \le K \limsup_{n\to\infty} \mathcal{D}(x_n, x) = 0.$$

Hence, $\mathcal{D}(x, x) = 0$. Hence, we have (NT1) of Definition 2.11 and therefore we conclude that $(X, \mathcal{D})$ is a non-triangular metric space. □

Remark 2.14. The proof technique of Theorem 2.13 tells us that in a JS-metric space, whenever $C(X, \mathcal{D}, x) \neq \emptyset$ for some $x \in X$, then $d(x, x) = 0$. In particular, $C(X, \mathcal{D}, x) \neq \emptyset$ if and only if the constant sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = x$ for all $n \in \mathbb{N}$, belongs to the set.

It is natural to ask whether the converse of the statement is true, i.e., whether every non-triangular metric space is a JS-metric space. The next example shows that it is not true. The example is inspired from [12] and we have made a few changes and added our own comments in it.

Example 2.15. Consider $X = [0, \infty)$ and define $d : X \times X \to [0, \infty)$ as

$$d(x, y) = \begin{cases} \dfrac{x+y}{x+y+1}, & x \neq y, x \neq 0, y \neq 0. \\ 0, & x = y. \\ \dfrac{x}{2}, & y = 0. \\ \dfrac{y}{2}, & x = 0. \end{cases}$$

From the definition of d, it is easy to see that for each $x \in X$, $d(x, x) = 0$ and $d(x, y) = d(y, x)$. That is, d satisfies (NT1) and (NT2).

Now, consider a point $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} d(x_n, x) = 0$. Now, either the real sequence $(d(x_n, x))_{n\in\mathbb{N}}$ is eventually constant (to 0), or it is not. In the case it is eventually constant, then it is clear that $(x_n)_{n\in\mathbb{N}}$ itself is eventually constant to x. However, in the case when $(d(x_n, x))_{n\in\mathbb{N}}$ is not eventually constant, we have the following cases:-

If $x = 0$, then $d(x_n, x) = \dfrac{x_n}{2} \to 0$ if and only if $x_n \to 0$ (in the usual sense). On the other hand, if $x \neq 0$, then $d(x_n, x) = \dfrac{x_n + x}{x_n + x + 1} \to 0$ if and only if $x_n + x \to 0$. However, since $x > 0$, $x_n + x \nrightarrow 0$. That is, $\lim_{n\to\infty} d(x_n, x) = 0$

is possible if and only if either $(x_n)_{n\in\mathbb{N}}$ is eventually constant to x or $x = 0$ and $x_n \to 0$ in the usual sense.

Now, let $x, y \in X$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$. If $(x_n)_{n\in\mathbb{N}}$ is an eventually constant sequence (to either of x or y), then it is easy to see that $x = y$. Therefore, let us consider that $(x_n)_{n\in\mathbb{N}}$ is not eventually constant to x. Then, we know that the only possibility is that $x = 0$ and $x_n \to 0$ in the usual sense. If $y \neq 0$, then we would have $d(x_n, y) = \dfrac{x_n + y}{x_n + y + 1} \to \dfrac{y}{y+1} \neq 0$. This is a contradiction to our hypothesis. Therefore, $x = y = 0$ must hold. In either case, we get $x = y$ and therefore, (X, d) is a non-triangular metric space.

Now, we notice that for each $x \in X$, $C(X, d, x) \neq \emptyset$ because the constant sequence which takes the value x everywhere is in the set. Let for $x, y \in X$, $(x_n)_{n\in\mathbb{N}} \in C(X, d, x)$. Consider $x = 0$ and $(x_n)_{n\in\mathbb{N}}$ as $x_n = \dfrac{1}{n}$, for all $n \in \mathbb{N}$. Then, $x_n \to 0$ in the usual sense. Also, we have for $y \neq 0$,

$$d(x_n, y) = \frac{x_n + y}{x_n + y + 1}.$$

Suppose that (X, d) is a JS-metric space, i.e., let there be a $K > 0$ such that for all $x, y \in X$, we have $d(x, y) \leq K \limsup_{n\to\infty} d(x_n, y)$.

Choose $y = 2K > 0$. Then,

$$d(x, y) = \frac{y}{2} = K.$$

$$\limsup_{n\to\infty} d(x_n, y) = \limsup_{n\to\infty} \frac{x_n + y}{x_n + y + 1} = \frac{y}{y+1} = \frac{2K}{2K+1} = \frac{1}{1 + \frac{1}{2K}}.$$

Hence, we have

$$K \limsup_{n\to\infty} d(x_n, y) = K\left(\frac{1}{1 + \frac{1}{2K}}\right) < K = d(x, y),$$

which is a contradiction! Hence, (X, d) is not a JS-metric space.

Example 2.15 indeed shows us that the non-triangular metric space is a generalization of JS-metric space in some cases. However, in general these two concepts are independent. The next example shows that there are JS-metric spaces which are not non-triangular metric spaces.

Example 2.16. Consider a non-empty set X and define $\mathcal{D} : X \times X \to [0, \infty]$ as for each $x, y \in X$, $\mathcal{D}(x, y) = 1$. Then, (JS1) and (JS2) of Definition 2.1 are satisfied trivially. Also, let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ be any sequence in X. Then, it is easy to see that $\lim_{n\to\infty} \mathcal{D}(x_n, x) = 1 \neq 0$ so that there are no convergent sequences in $(X, \mathcal{D})$. In other words, for each $x \in X$, $C(X, \mathcal{D}, x)$ is an empty set and therefore (JS3) is trivially satisfied. Hence, $(X, \mathcal{D})$ is a JS-metric space. However, $\mathcal{D}(x, x) = 1 \neq 0$ and therefore, $(X, \mathcal{D})$ is not a non-triangular metric space.

Now, we look at some topological definitions of sequences in the context of non-triangular metric spaces.

Definition 2.17 (Convergence). ([12]) Let (X, d) be a non-triangular metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$, if $\lim_{n\to\infty} d(x_n, x) = 0$.

Remark 2.18. Notice that (NT3) of Definition 2.11 guarantees that the limit of a convergent sequence is unique.

Definition 2.19 (Cauchy sequence). ([12]) Let (X, d) be a non-triangular metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $(x_n)_{n\in\mathbb{N}}$ is Cauchy if $\lim_{n\to\infty} \sup\{d(x_n, x_m) \,|\, m \geq n\} = 0$.

Definition 2.20 (Completeness). ([12]) A non-triangular metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent to some point $x \in X$.

Now, we shall state a fixed point result in the context of non-triangular metric spaces. The proof is available in [12].

Theorem 2.21. *Let (X, d) be a non-triangular metric space and $T : X \to X$ be a mappings such that the following holds:*

1. *For any Picard sequence $(x_n)_{n\in\mathbb{N}}$ based on $x_0 \in X$ (i.e., $x_n = T^n x_0$) if $(x_{n_k})_{k\in\mathbb{N}}$ and $(x_{m_k})_{k\in\mathbb{N}}$ are two subsequences such that $\lim\limits_{k\to\infty} d(x_{n_k}, x_{m_k}) = L$ and $\lim\limits_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = L$, where $L \geq 0$, and for all $k \in \mathbb{N}$, we have $d(x_{n_k}, x_{m_k}) > L$, then $L = 0$.*
2. *For any $x_0 \in X$ if there is a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim\limits_{k\to\infty} T^{n_k} x_0 = x$ (for some $x \in X$), then $\lim\limits_{k\to\infty} T(T^{n_k} x_0) = Tx$.*

If $x_0 \in X$ is a point such that the set $O(T, x_0) = \left\{d\left(T^i x_0, T^j x_0\right) | i, j \in \mathbb{N}\right\}$ is bounded in $\mathbb{R}$, then T has a fixed point.

Remark 2.22. Condition (2) of Theorem 2.21 is called *orbital continuity* of the mapping T. Formally, a map $T : X \to X$ is said to be orbitally continuous if for any sequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ and $x_0 \in X$, if $T^{n_k} x_0 \to x$, then $T(T^{n_k} x_0) \to Tx$.

Remark 2.23. The condition that $O(T, x_0)$ is bounded in $\mathbb{R}$, can also be translated as the condition of the Picard sequence $(x_n)_{n\in\mathbb{N}}$ being bounded. In fact, we say that a given sequence $(x_n)_{n\in\mathbb{N}}$ is bounded if the set $\{d(x_n, x_m) | n, m \in \mathbb{N}\}$ is bounded in $\mathbb{R}$.

3. Topology of Non-Triangular Metric Spaces

In the usual situation, given a "distance function" (or a "distance-like function"), one tries to first define "open" balls, and through them, we define open sets. Let us try to do so.

Definition 3.1 (Ball). Let (X, d) be a non-triangular metric space and $x \in X$. Then, a ball centered at x and of radius $r > 0$ is the set

$$B(x, r) = \{y \in X | d(x, y) < r\}.$$

Remark 3.2. We deliberately write "ball" instead of an "*open* ball". It is to be noted that in metric spaces, the triangle inequality makes the set defined in Definition 3.1 open. However, with the triangle inequality no longer at our disposal, we cannot guarantee it directly.

Since the "openness" of balls is not guaranteed in non-triangular spaces, we will take another way and define open sets. Instead of defining open sets through balls, we will take help of convergent sequences. We know that in the metric structure, closedness of a set and convergent sequences are intimately related. We try to establish a similar definition of closed sets in the non-triangular case using convergent sequences.

Definition 3.3 (Closed Set). Let (X, d) be a non-triangular metric space. A set $F \subseteq X$ is closed if for any sequence $(x_n)_{n\in\mathbb{N}}$ in F with $x_n \to x \in X$, we have $x \in F$.

Now, let us see what all properties does our definition of closed sets satisfy.

Theorem 3.4. *Let (X, d) be a non-triangular metric space. Then,*

1. *$\emptyset$ and X are closed.*
2. *If $F_1, F_2 \subseteq X$ are closed, then $F_1 \cup F_2$ is closed. In other words, finite union of closed sets is closed.*

3. *If $\{F_\lambda | \lambda \in \Lambda\}$ is an arbitrary collection of closed sets, then $\bigcap_{\lambda\in\Lambda} F_\lambda$ is closed. In other words, arbitrary intersection of closed sets is closed.*

Proof. 1. Suppose that $\emptyset$ is not closed. Then, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\emptyset$ with $x_n \to x \in X$ but $x \notin \emptyset$. However, such a sequence cannot exist and therefore, $\emptyset$ is closed.

Now, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with $x_n \to x \in X$. Then, it is clear that X is closed.

2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $F_1 \cup F_2$ with $x_n \to x \in X$. If there is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $x_n \in F_1$ (or F_2), then it is evident that $x \in F_1$ (or F_2) and therefore, in their union. If this is not the case, choose a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which completely lies in F_1 (or F_2). Since $x_n \to x$, we also have $x_{n_k} \to x$ and consequently, $x \in F_1 \cup F_2$.

3. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\bigcap_{\lambda\in\Lambda} F_\lambda$ with $x_n \to x \in X$. Then, for all $\lambda \in \Lambda$, we have $(x_n)_{n\in\mathbb{N}}$ in F_λ. Since each F_λ is closed, we have for all $\lambda \in \Lambda$, $x \in F_\lambda$. Hence, $x \in \bigcap_{\lambda\in\Lambda} F_\lambda$, and arbitrary intersection of closed sets is closed. □

The next example tells us that an arbitrary union of closed sets need not be closed.

Example 3.5. Consider the non-triangular metric space (X, d) of Example 2.15. We have seen that the only convergent sequences in (X, d) are those which are either eventually constant or are convergent to 0 in the usual sense. Consider the collection $\left\{\left[\frac{1}{n}, 1\right] : n \in \mathbb{N}\right\}$. Then, it is evident that each of $\left[\frac{1}{n}, 1\right]$ is closed in X since $0 \notin \left[\frac{1}{n}, 1\right]$ and is neither a boundary point (in the usual sense). Now, we have $\bigcup_{n\in\mathbb{N}} \left[\frac{1}{n}, 1\right] = (0, 1]$. We consider the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ in $(0, 1]$. Then, we have $\frac{1}{n} \to 0 \in X$ but $0 \notin (0, 1]$. Therefore, arbitrary union of closed sets need not be closed.

Since the properties of closed sets (as per Definition 3.3) are same as those in an arbitrary topological space, we now have an insight into defining open sets. In the ordinary situation, closed sets are defined as complements of open sets. However, since complementation is an involutary operation on sets, we can do the reverse as well. Taking motivation from here, we define open sets as follows:

Definition 3.6 (Open Set). Let (X, d) be a non-triangular metric space. A set $G \subseteq X$ is open if its complement is closed.

Immediately, we have the following properties of open sets.

Theorem 3.7. *Let (X, d) be a non-triangular metric space. Then:*

1. *$\emptyset$ and X are open.*
2. *If $\{G_\lambda | \lambda \in \Lambda\}$ is a collection of open sets, then $\bigcup_{\lambda\in\Lambda} G_\lambda$ is open. In other words, arbitrary union of open sets is open.*
3. *If G_1, G_2 are open sets in X, then $G_1 \cap G_2$ is open in X. In other words, finite intersection of open sets is open.*

Proof. 1. Since $\emptyset^c = X$ and $X^c = \emptyset$ and both are closed, both of them are also open.

2. Since for each $\lambda \in \Lambda$, G_λ is open, we have for each $\lambda \in \Lambda$, G_λ^c is closed. Therefore, $\left\{G_\lambda^c | \lambda \in \Lambda\right\}$ is a collection of closed sets in X. From Theorem 3.4, we know that $\bigcap_{\lambda\in\Lambda} G_\lambda^c = \left(\bigcup_{\lambda\in\Lambda} G_\lambda\right)^c$ is closed. Hence, $\bigcup_{\lambda\in\Lambda} G_\lambda$ is open.

3. Again, from Theorem 3.4, we know that $G_1^c \cup G_2^c = (G_1 \cap G_2)^c$ is closed. Therefore, $G_1 \cap G_2$ is open. □

The next example tells us that arbitrary intersections of open sets need not be open.

Example 3.8. Consider the space (X, d) as in Example 2.15. Consider the collection $\left\{\left[0, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$. For each $n \in \mathbb{N}$, we have $\left[0, \frac{1}{n}\right)^c = \left[\frac{1}{n}, \infty\right)$, which is closed. Hence, the collection is of open sets. Now, we have $\bigcap_{n\in\mathbb{N}} \left[0, \frac{1}{n}\right) = \{0\}$. However, $\{0\}^c = (0, \infty)$ is not closed so that arbitrary intersection of open sets need not be open.

The most important part in metric topology, that is achieved by the triangle inequality is the openness of balls (as defined in Definition 3.1). An important question at this point is: Can we have a similar result for a non-triangular metric space? In other words, are balls in a non-triangular metric space open sets? The next result tells us that under certain conditions on the non-triangular metric d, we can assure the openness of the balls.

Theorem 3.9. *Let (X, d) be a non-triangular metric space such that for any sequence $(x_n)_{n\in\mathbb{N}}$ with $d(x_n, x) \to 0$, for some $x \in X$, we have for any $y \in X$, $d(x_n, y) \to d(x, y)$. Then, any ball $B(x, r)$ for $r > 0$ and $x \in X$, is an open set.*

Proof. Let $G = B(x, r)$ and $F = G^c$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in F converging to some $x_0 \in X$. Then, for all $n \in \mathbb{N}$, we have $d(x_n, x) \geq r$. Hence, $\lim_{n\to\infty} d(x_n, x) \geq r$. However, from the property of d in the hypothesis, we conclude that $d(x_0, x) \geq r$. That is, $x_0 \notin G$ and thus, F is closed. Consequently, G is open. □

Remark 3.10. The property mentioned in Theorem 3.9 can be seen as a *continuity-like* property of d (in the "product non-triangular metric"). However, we do not explicitly write it as continuity because at such an early stage, we do not know how the product topology will look like.

Remark 3.11. Henceforth, the property of a non-triangular metric, d, as mentioned in Theorem 3.9 will be called Property (C).

Remark 3.12. If (X, d) is a metric space, then $d : X \times X \to [0, \infty)$ is a continuous function in the product topology. Therefore, every metric has Property (C).

The next example shows that there are non-triangular metrics, where the balls need not be open.

Example 3.13. Let $X = [0, \infty)$ and consider $d : X \times X \to [0, \infty)$, defined as

$$d(x, y) = \begin{cases} |x - y|, & x \neq 0, y \neq 0. \\ \frac{1}{x}, & y = 0. \\ \frac{1}{y}, & x = 0. \\ 0, & x = y = 0. \end{cases}$$

Clearly, d satisfies (NT1) and (NT2) of Definition 2.11. Now, let $\lim_{n\to\infty} d(x_n, x) = 0$, for some sequence $(x_n)_{n\in\mathbb{N}}$ in X and $x \in X$. If $x = 0$, then we have two cases: either $(x_n)_{n\in\mathbb{N}}$ is eventually constant to 0, or if not, then $d(x_n, 0) = \frac{1}{x_n} \to 0$. However, this is possible if and only if $(x_n)_{n\in\mathbb{N}}$ is an unbounded sequence. If $x \neq 0$, then it is clear that $(x_n)_{n\in\mathbb{N}}$ cannot be eventually constant to 0. Also, assuming that it is not eventually constant to x, we conclude that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $\lim_{n\to\infty} |x_n - x| = 0$. That is, for $x \neq 0$, $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the usual sense.

Therefore, the only convergent sequences (apart from the eventually constant ones) are the ones which are either unbounded (convergent to 0) or convergent in the usual sense to a non-zero number. Now, let $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y) = 0$. If $x = 0$, then either $(x_n)_{n\in\mathbb{N}}$ is eventually constant to 0, or it is unbounded. If the first case is true, then $(d(x_n, y))_{n\in\mathbb{N}}$ is an eventually constant sequence, and we conclude that $y = 0 = x$. If the sequence $(x_n)_{n\in\mathbb{N}}$ is unbounded and $y \neq 0$, then the sequence $\left(\left|x_n - y\right|\right)_{n\in\mathbb{N}}$ is also unbounded and does not converge to 0. Therefore, for $\lim\limits_{n\to\infty} d(x_n, y) = 0$ to hold, $y = 0 = x$ must also hold. On the other hand, for $x \neq 0$, we have that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in the usual sense. Since, in the usual sense, limits of a sequence are unique, $y = x$ holds. Therefore, (NT3) of Definition 2.11 is also satisfied and we conclude that (X, d) is a non-triangular metric space.

Now, we will show that d does not have Property (C). Consider the sequence $(n)_{n\in\mathbb{N}}$ in X which converges to 0. For $y = 1$, we have $d(0, y) = 1$. However, $d(n, y) = |n - 1|$ is unbounded. In other words, $\lim\limits_{n\to\infty} d(n, 1) \neq d(0, 1)$. This also tells us that (X, d) is not a metric space.

Let us now look at balls in (X, d). By definition, $B(x, r) = \{y \in X | d(x, y) < r\}$. If $x = 0$, then for $y \neq 0$, $d(x, y) = \dfrac{1}{y} < r$ if and only if $y > \dfrac{1}{r}$. Also, $0 \in B(0, r)$. If $x \neq 0$, then $0 \in B(x, r)$ if and only if $d(x, 0) = \dfrac{1}{x} < r$ if and only if $x > \dfrac{1}{r}$. Also, for $y \neq 0$, $y \in B(x, r)$ if and only if $d(x, y) = \left|x - y\right| < r$. Hence, we have

$$B(x, r) = \begin{cases} \{0\} \cup \left(\dfrac{1}{r}, \infty\right), & x = 0. \\ \{0\} \cup ((x - r, x + r) \cap (0, \infty)), & x \neq 0, x > \dfrac{1}{r}. \\ (x - r, x + r) \cap (0, \infty), & x \neq 0, x \leq \dfrac{1}{r}. \end{cases}$$

Hence,

$$B(x, r)^c = \begin{cases} \left(0, \dfrac{1}{r}\right], & x = 0. \\ (0, x - r] \cup [x + r, \infty), & x \neq 0, x > \dfrac{1}{r}. \\ [0, x - r] \cup [x + r, \infty), & x \neq 0, x \leq \dfrac{1}{r}. \end{cases}$$

In particular, consider $B(2, 1) = \{0\} \cup (1, 3)$. Then, $B(2, 1)^c = (0, 1] \cup [3, \infty)$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = n + 2$, for all $n \in \mathbb{N}$. Since $(x_n)_{n\in\mathbb{N}}$ is unbounded, we have $x_n \to 0 \in X$ but $0 \notin B(2, 1)^c$. Hence, $B(2, 1)^c$ is not closed and consequently, $B(2, 1)$ is not open.

Remark 3.14. The Property (C) of Theorem 3.9 is only a sufficient condition for balls to be open in a non-triangular metric space; it is not necessary (as of now).

Another important characterization of open sets, in the case of metrics, was that every point inside an open set had a "personal (open) ball space". Therefore, we would want that a similar result should be true in our case of non-triangular metric.

Theorem 3.15. *Let (X, d) be a non-triangular metric space. A set $G \subseteq X$ is open if and only if for each $x \in G$, there is some $r > 0$ such that $B(x, r) \subseteq G$.*

Proof. Suppose that G is open, but there is some $x \in G$ such that for all $r > 0$, $B(x, r) \nsubseteq G$. Then, in particular, for all $n \in \mathbb{N}$, there would exist some $x_n \in B\left(x, \dfrac{1}{n}\right) \cap G^c$. By the definition of balls, we know that $d(x_n, x) < \dfrac{1}{n}$ so that $\lim\limits_{n\to\infty} d(x_n, x) = 0$. Since G is open, G^c is closed and therefore, $x \in G^c$. However, this is a contradiction! Therefore, if G is open, then for each $x \in G$, there should exist some $r > 0$ such that $B(x, r) \subseteq G$.

Conversely, let G be not open. Then, G^c is not closed. That is, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in G^c such that $x_n \to x \in X$ but $x \notin G^c$ (this means that $x \in G$). Now, since $x_n \to x$, for each $r > 0$, there is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $d(x_n, x) < r$. That is, for each $r > 0$, $B(x, r) \cap G^c \neq \emptyset$. In other words, $B(x, r) \nsubseteq G$. Taking the contrapositive, we obtain our result. $\square$

Remark 3.16. Theorem 3.9 together with Theorem 3.15 tell us that whenever the non-triangular metric has Property (C), every point inside a (open) ball has its "personal ball space".

Example 3.17. From Example 3.13, it is easy to see that $0 \in B(2,1)$ does not have its "personal ball space". That is, for each $r > 0$, $B(0,r) = \{0\} \cup \left(\frac{1}{r}, \infty\right) \not\subseteq \{0\} \cup (1,3) = B(2,1)$. This, again, helps us conclude that $B(2,1)$ is not open in (X,d).

4. Suzuki Type Z-Contractions in Non-Triangular Metric Spaces

In this section, we give some details about simulation functions and Z-contractions as defined by Khojasteh et al. in [13]. We then proceed to look at Suzuki type Z-contractions, as defined in [14], in our setting of a non-triangular metric space. Throughout this section, we give our own comments and remarks about these concepts.

Definition 4.1 (Simulation function). ([13]) Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping. Then, ζ is called a simulation function if:-

(S1) $\zeta(0,0) = 0$.

(S2) For all $t, s > 0$, $\zeta(t,s) < s - t$.

(S3) If $(t_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

The set of all simulation functions is denoted in the literature by Z. Several examples of simulation functions can be found in [13]. We proceed to give the definition of a Z-contraction on a metric space. Indeed, this is a generalization of Banach contractions.

Definition 4.2 (Z-contraction). ([13]) Let (X,d) be a metric space and $T : X \to X$ be a mapping. We say that T is a Z-contraction with respect to some $\zeta \in Z$, if for each $x, y \in X$, we have

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0.$$

The study of fixed point results for Z-contractions is done extensively in [13].

Now, let us look at a generalization of Suzuki contractions using the concept of simulation functions. The study of this new class is done by Kumam et al. in [14].

Definition 4.3 (Suzuki type contraction). Let (X,d) be a metric space and $T : X \to X$ be a map. Then, T is called a Suzuki type contraction, if there is some $\lambda \in [0,1)$ such that for each $x, y \in X$ with $x \neq y$, we have

$$\frac{1}{2} d(x,Tx) < d(x,y) \Rightarrow d(Tx,Ty) \leq \lambda d(x,y).$$

Definition 4.4 (Suzuki type Z-contraction). ([14]) Let (X,d) be a metric space and $T : X \to X$ be a map. Then, T is called a Suzuki type Z-contraction (or an SZ-contraction) with respect to ζ if for each $x, y \in X$ with $x \neq y$, we have

$$\frac{1}{2} d(x,Tx) < d(x,y) \Rightarrow \zeta(d(Tx,Ty), d(x,y)) \geq 0.$$

Remark 4.5. It is evident that that every Suzuki type contraction is an SZ-contraction.

Let us now look at Suzuki type contractions and SZ-contraction on non-triangular metric space, and give fixed point results for the same. The technique of proofs will remain similar to the previous works of [13] and [14]. Before we begin with the results, we require the following technical definitions.

Definition 4.6 (Asymptotic regularity). Let (X, d) be a metric space and $T : X \to X$ be a map. We say that T is asymptotically regular at a point $x_0 \in X$ if $\lim_{n\to\infty} d\left(T^n x_0, T^{n+1} x_0\right) = 0$.

Definition 4.7 (Property K). ([14]) Let (X, d) be a metric space. A map $T : X \to X$ is said to have property K, if for a bounded Picard sequence $(x_n)_{n\in\mathbb{N}}$, there are subsequences $(x_{m_k})_{k\in\mathbb{N}}$ and $(x_{n_k})_{k\in\mathbb{N}}$ with $\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = c > 0$, then $\frac{1}{2} d(x_{m_k-1}, x_{m_k}) < d(x_{m_k-1}, x_{n_k-1})$.

Theorem 4.8. *Let (X, d) be a non-triangular metric space and $T : X \to X$ be a Suzuki type contraction, i.e., let there be $\lambda \in [0, 1)$ such that for all $x, y \in X$ with $x \neq y$, we have*

$$\frac{1}{2} d(x, Tx) < d(x, y) \Rightarrow d(Tx, Ty) \le \lambda d(x, y).$$

Then, T is asymptotically regular at each $x \in X$.

Proof. Let $x \in X$ and $(T^n x)_{n\in\mathbb{N}}$ be the Picard sequence based on x. If for some $n_0 \in \mathbb{N}$, we have $T^{n_0+1}x = T^{n_0}x$, then for all $n \ge n_0$, we have $T^{n+1}x = T^n x$ so that $d\left(T^n x, T^{n+1}x\right) = 0$. Therefore, in this case, $\lim_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = 0$.

Let us now assume that for each $n \in \mathbb{N}$, $T^{n+1}x \neq T^n x$. Then, we know that $\frac{1}{2} d\left(T^n x, T^{n+1}x\right) < d\left(T^n x, T^{n+1}x\right)$. Since T is a Suzuki contraction, we have $d\left(T^{n+1}x, T^{n+2}x\right) \le \lambda d\left(T^n x, T^{n+1}x\right) < d\left(T^n x, T^{n+1}x\right)$. Hence, the real valued sequence $\left(d\left(T^n x, T^{n+1}x\right)\right)_{n\in\mathbb{N}}$ is decreasing and bounded below by 0, hence convergent. Let $\lim_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = r$. Then, using the above inequality, we have

$$r = \lim_{n\to\infty} d\left(T^{n+1}x, T^{n+2}x\right) \le \lim_{n\to\infty} \lambda d\left(T^n x, T^{n+1}x\right) = \lambda r.$$

This is possible exactly when $r = 0$. Hence, T is asymptotically regular at each $x \in X$. □

To get fixed point results for Suzuki contractions in a non-triangular metric space, we require the following property of asymptotically regular maps.

Definition 4.9 (Property D). Let (X, d) be a non-triangular metric space $T : X \to X$ be an map which is asymptotically regular at $x_0 \in X$. We say that T has Property D if for some subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $(T^{n_k}x_0)_{k\in\mathbb{N}}$ is not eventually constant, if $T^{n_k}x_0 \to x$ (for some $x \in X$), then the real valued sequence $\left(d\left(T^{n_k}x_0, T^{n_k+1}x_0\right)\right)_{k\in\mathbb{N}}$ converges to 0 faster than $(T^{n_k}x_0)_{k\in\mathbb{N}}$ converges to x. In other words, $\lim_{n\to\infty} \dfrac{d\left(T^{n_k}x_0, T^{n_{k+1}}x_0\right)}{d\left(T_{n_k}x_0, x\right)} = 0$.

Remark 4.10. Definition 4.9 will be required to prove the orbital continuity of Suzuki type contractions in non-triangular metric spaces. For more details on comparing the rates of convergence of sequences, we refer the reader to [17] and the references therein.

Theorem 4.11. *Let (X, d) be a non-triangular metric space and $T : X \to X$ be a Suzuki type contraction with Property K and Property D. Let $x_0 \in X$ be such that the Picard sequence $(x_n)_{n\in\mathbb{N}}$ based on $x_0 \in X$ is bounded. Then, T has a unique fixed point.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be any Picard sequence and $(y_{n_k})_{k\in\mathbb{N}}$ and $(y_{m_k})_{k\in\mathbb{N}}$ be its subsequences such that $\lim_{k\to\infty} d(y_{n_k}, y_{m_k}) = L$ and $\lim_{k\to\infty} d(y_{n_k-1}, y_{m_k-1}) = L$, where $L \ge 0$. Let, if possible, $L > 0$. Since T has property K, we know that $\frac{1}{2} d(y_{m_k-1}, y_{m_k}) < d(y_{n_k-1}, y_{m_k-1})$. Therefore, we get $d(y_{n_k}, y_{m_k}) \le \lambda d(y_{n_k-1}, y_{m_k-1})$. Hence,

$$L = \lim_{k\to\infty} d(y_{n_k}, y_{m_k}) \le \lim_{k\to\infty} \lambda d(y_{n_k-1}, y_{m_k-1}) = \lambda L < L.$$

For $L > 0$, this is a contradiction and therefore, we have $L = 0$.

Now, let for the Picard sequence $(y_n)_{n\in\mathbb{N}}$ based on y_0, there be a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $y_{n_k} \to y$ for some $y \in X$. Since T is asymptotically regular at each point in X and it also has Property D, we know that $\lim\limits_{k\to\infty} \dfrac{d(y_{n_k}, y_{n_k+1})}{d(y_{n_k}, y)} = 0$. In particular, there is some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, we have $\dfrac{d(y_{n_k}, y_{n_{k+1}})}{d(y_{n_k}, y)} < 2$. Or, what is the same, $\frac{1}{2} d(y_{n_k}, y_{n_{k+1}}) < d(y_{n_k}, y)$. Since T is a Suzuki contraction, we have

$$d(Ty_{n_k}, Ty) \leq \lambda d(y_{n_k}, y),$$

so that

$$\lim_{k\to\infty} d(Ty_{n_k}, Ty) \leq \lim_{k\to\infty} \lambda d(y_{n_k}, y) = 0.$$

Therefore, $\lim\limits_{k\to\infty} d(Ty_{n_k}, Ty) = 0$, i.e., $Ty_{n_k} \to Ty$. Therefore, T is orbitally continuous.

Therefore, using Theorem 2.21, we can conclude that T has a fixed point, say $x^* \in X$. Suppose that there is another fixed point $y^* \in X$ with $x^* \neq y^*$. Then, we know that $0 = \frac{1}{2} d(x^*, Tx^*) < d(x^*, y^*)$. This implies, $d(x^*, y^*) = d(Tx^*, Ty^*) \leq \lambda d(x^*, y^*) < d(x^*, y^*)$. However, this is a contradiction and therefore, the fixed point is unique. □

Now, let us look at Suzuki type Z-contractions in the context of non-triangular metric spaces.

Theorem 4.12. *Let (X, d) be a non-triangular metric space and $T : X \to X$ be an SZ-contraction with respect to ζ, i.e., for all $x, y \in X$ with $x \neq y$, we have*

$$\frac{1}{2} d(x, Tx) < d(x, y) \Rightarrow \zeta(d(Tx, Ty), d(x, y)) \geq 0.$$

Then, T is asymptotically regular at each $x \in X$.

Proof. Let $x \in X$ and consider the Picard sequence $(T^n x)_{n\in\mathbb{N}}$ based on x. If for some $n_0 \in \mathbb{N}$, we have $T^{n_0+1}x = T^{n_0}x$, then for all $n \geq n_0$, we have $T^{n+1}x = T^n x$ so that $d\left(T^n x, T^{n+1}x\right) = 0$. Therefore, in this case $\lim\limits_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = 0$.

Now, let us assume that for all $n \in \mathbb{N}$, $T^{n+1}x \neq T^n x$. Then, we have $\frac{1}{2} d\left(T^n x, T^{n+1}x\right) < d\left(T^n x, T^{n+1}x\right)$. This implies

$$0 \leq \zeta\left(d\left(T^{n+1}x, T^{n+2}x\right), d\left(T^n x, T^{n+1}x\right)\right) < d\left(T^n x, T^{n+1}x\right) - d\left(T^{n+1}x, T^{n+2}x\right).$$

This gives us $d\left(T^{n+1}x, T^{n+2}x\right) < d\left(T^n x, T^{n+1}x\right)$. Hence, the real valued sequence $\left(d\left(T^n x, T^{n+1}x\right)\right)_{n\in\mathbb{N}}$ is decreasing and bounded below by 0, hence convergent. Let $\lim\limits_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = r > 0$. Since ζ is a simulation function and T is an SZ-contraction with respect to ζ, we have

$$0 \leq \limsup_{n\to\infty} \zeta\left(d\left(T^{n+1}x, T^{n+2}x\right), d\left(T^n x, T^{n+1}x\right)\right) < 0.$$

This is a contradiction! Hence, $r = 0$ and T is asymptotically regular at each $x \in X$. □

Similar to Theorem 4.11, we now give the existence and uniqueness of fixed points of SZ-contractions in non-triangular metric spaces. Again, Properties K and D would be of importance in the proof.

Theorem 4.13. *Let (X, d) be a non-triangular metric space and $T : X \to X$ be an SZ-contraction with respect to ζ, possessing property K and property D. Let $x_0 \in X$ be such that the Picard sequence $(x_n)_{n\in\mathbb{N}}$ based on x_0 is bounded. Then, T has a unique fixed point.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be any Picard sequence based at some $y_0 \in X$ and $(y_{n_k})_{k\in\mathbb{N}}$ and $(y_{m_k})_{k\in\mathbb{N}}$ be its subsequences such that $\lim_{k\to\infty} d(y_{n_k}, y_{m_k}) = L$ and $\lim_{k\to\infty} d(y_{n_k-1}, y_{m_k-1}) = L$, where $L \geq 0$. Let, if possible, $L > 0$. Since T has Property K, we have $\frac{1}{2}d(y_{m_k-1}, y_{m_k}) < d(y_{m_k-1}, y_{n_k-1})$. Therefore, $\zeta(d(y_{m_k}, y_{n_k}), d(y_{m_k-1}, y_{n_k-1})) \geq 0$. Since ζ is a simulation function and $L > 0$, we have

$$0 \leq \limsup_{n\to\infty} \zeta(d(y_{m_k}, y_{n_k}), d(y_{m_k-1}, y_{n_k-1})) < 0.$$

This is a contradiction and therefore, $L = 0$.

Now, suppose for some Picard sequence $(y_n)_{n\in\mathbb{N}}$, there is a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $y_{n_k} \to y$ (for some $y \in X$). Since T is asymptotically regular at each point in X and has Property D, $\lim_{k\to\infty} \frac{d(y_{n_k}, y_{n_k+1})}{d(y_{n_k}, y)} = 0$. Therefore, there is some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, we have $\frac{d(y_{n_k}, y_{n_k+1})}{d(y_{n_k}, y)} < 2$, i.e., $\frac{1}{2}d(y_{n_k}, y_{n_k+1}) < d(y_{n_k}, y)$. Hence, $\zeta(d(Ty_{n_k}, Ty), d(y_{n_k}, y)) \geq 0$. If $(y_{n_k})_{k\in\mathbb{N}}$ is eventually constant to y, then there is nothing to prove, since $(Ty_{n_k})_{k\in\mathbb{N}}$ would be eventually constant to Ty. Therefore, we assume that $(y_{n_k})_{k\in\mathbb{N}}$ is not eventually constant and therefore, $d(Ty_{n_k}, Ty) \neq 0$ and $d(y_{n_k}, y) \neq 0$. Using the properties of simulation function, we get $d(Ty_{n_k}, Ty) < d(y_{n_k}, y)$ for all $k \geq k_0$. Therefore, $\lim_{k\to\infty} d(Ty_{n_k}, Ty) = 0$ so that $Ty_{n_k} \to Ty$. Hence, T is orbitally continuous.

Using Theorem 2.21, we conclude that T has a fixed point, say $x^* \in X$. Suppose that $y^* \in X$ is another fixed point of T with $x^* \neq y^*$. Then, we would have $0 = \frac{1}{2}d(x^*, Tx^*) < d(x^*, y^*)$. Since T is an SZ-contraction with respect to ζ, we get

$$0 \leq \zeta(d(Tx^*, Ty*), d(x^*, y^*)) = \zeta(d(x^*, y^*), d(x^*, y^*)) < d(x^*, y^*) - d(x^*, y^*) = 0,$$

which is a contradiction! Hence, the fixed point of T is unique. □

5. Conclusion and Future Scope

In this article we looked at some generalizations of metric spaces and Banach contractions. We have started developing a theory for the topology on non-triangular metric space by defining open sets in a non-conventional way (through closed sets). Indeed, because we do not have triangle inequality at hand, many properties of metric spaces cannot be generalized directly into non-triangular metric spaces. Therefore, it is of importance now to study in detail, various consequences which a "distance-like function" can generate. We also saw fixed point results related to these generalizations. As a future scope, one can try to get more conditions on SZ-contractions so that the Picard sequence gets bounded or conditions (1) or (2) of Theorem 2.21 are satisfied. Also, we can try to get similar fixed point results for various other maps such as Kannan contractions, Chatterjea contractions, Generalized contractions of Hardy and Rogers, etc. The important point of study for now is the topology of non-triangular metric spaces. We raise a open question here:

Question 5.1. *Is Property (C) necessary for balls to be open in a non-triangular metric space? Or can we have a non-triangular metric without Property (C) but all the balls are open?*

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