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Some Systems of Tensor Equations Under T-Product and their Applications

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Abstract. In this paper, some systems of tensor equations under t-product are considered. Some practical necessary and sufficient conditions for the existence of a solution to two systems of tensor equations in terms of the Moore-Penrose inverses are given. The general solutions to the systems of tensor equations are presented when they are solvable. An application of the tensor equations in the solvability conditions and general symmetric solution to a system of tensor equations. Some algorithms and numerical examples are provided to illustrate the main results.

1. Introduction

Tensors arise in a wide variety of application areas, including, but not limited to, biology [16], signal processing ([2], [6]), numerical linear algebra [3], image processing [9], data analysis [1], graph theory [18], and elsewhere. For more results and applications of tensor theory, we refer the reader to the recent book [17] and the survey paper [12]. The most common types of tensor multiplications are n-mode, Kronecker, Khatri-Rao, Hadamard, outer, Einstein products and so on.

Among the available tensor multiplications, it is worth to mention the t-product (Definition 2.1) of two tensors. Since Kilmer et al. [11] introduced the concept of t-product in 2011, there have been many papers to discuss the decompositions, generalized inverses and applications of tensors under t-product (e.g., [5], [7], [11], [15]-[20]). For example, Kilmer et al. [10] investigated the necessary theoretical framework for third order tensor computation under t-product. Martin et al. [13] extended the third order tensor SVD and tensor operations to order-p tensors. Jin et al. [8] defined the generalized inverse of order-p tensor under t-product. Miao et al. [14] derived the T-Jordan canonical form and T-Drazin inverse based on the t-product. Zhang and Aeron [21] used t-SVD to consider the problem of recovering third order tensors under random sampling. Nowadays tensor equations and tensor SVD under t-product are widely and heavily used in imaging processing [13], video data completion [21], linear models [8], and so on.

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However, the literature on tensor equations under t-product is limited. Jin et al. [8] gave a solvability condition and general solution to the tensor equation under t-product

$$\mathcal{A} * \mathcal{X} * \mathcal{B} = \mathcal{C}. \tag{1}$$

To the best of our knowledge, there has been little information about the generalization of the tensor equation (1), i.e., the system of tensor equation

$$\begin{cases} \mathcal{A}_1 * \mathcal{X} = \mathcal{C}_1, \\ \mathcal{X} * \mathcal{B}_1 = \mathcal{D}_1, \\ \mathcal{A}_2 * \mathcal{X} * \mathcal{B}_2 = \mathcal{C}_2, \end{cases} \tag{2}$$

where $\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1, \mathcal{A}_2, \mathcal{B}_2$, and $\mathcal{C}_2$ are given tensors, $\mathcal{X}$ is unknown. Motivated by the wide application of tensor equation and t-product and in order to improve the theoretical development of the general solutions to tensor equations, we consider the system of tensor equations (2) under t-product.

The remainder of the paper is organized as follows. In Section 2, we review the definitions of t-product of two tensors, tensor operations, identity tensor, symmetric tensor, Moore-Penrose inverse of tensor under t-product. In Section 3, we derive some necessary and sufficient conditions for the existence of a solution to the system of tensor equations under t-product

$$\begin{cases} \mathcal{A}_1 * \mathcal{X} = \mathcal{C}_1, \\ \mathcal{X} * \mathcal{B}_1 = \mathcal{D}_1. \end{cases} \tag{3}$$

In Section 4, we present some necessary and sufficient conditions for the existence of a solution to the system of tensor equations (2). The general solutions to the systems (3) and (2) are provided in Sections 3 and 4. In Section 5, we derive some solvability conditions and general symmetric solution to the system of tensor equations

$$\begin{cases} \mathcal{A}_1 * \mathcal{X} = \mathcal{C}_1, \\ \mathcal{A}_2 * \mathcal{X} * \mathcal{A}_2^T = \mathcal{C}_2, \end{cases} \quad \mathcal{X} = \mathcal{X}^T. \tag{4}$$

Some algorithms and numerical examples are provided in Sections 4 and 5.

2. Preliminaries

An order N tensor $\mathcal{A} = (a_{i_1 \cdots i_N})_{1 \le i_j \le I_j}$ $(j = 1, \ldots, N)$ is a multidimensional array with $I_1 I_2 \cdots I_N$ entries. Let $\mathbb{R}^{I_1 \times \cdots \times I_N}$ stands for the set of the order N dimension $I_1 \times \cdots \times I_N$ tensors over the real number field $\mathbb{R}$.

For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_p}) \in \mathbb{R}^{n_1 \times \cdots \times n_p}$, the notation $\mathcal{A}_i \in \mathbb{R}^{n_1 \times \cdots \times n_{p-1}}$ denotes the order $p-1$ tensor created by holding the pth index of $\mathcal{A}$ fixed at i. Define unfold$(\cdot)$ to take an $n_1 \times \cdots \times n_p$ tensor and return an $n_1 n_p \times \cdots \times n_{p-1}$ block tensor in the following way:

$$\text{unfold}(\mathcal{A}) = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_{n_p} \end{bmatrix}.$$

The operation fold$(\cdot)$ takes an $n_1 n_p \times n_2 \times \cdots n_{p-1}$ block tensor and returns an $n_1 \times \cdots \times n_p$ tensor. That means

$$\text{fold}(\text{unfold}(\mathcal{A})) = \mathcal{A}.$$

Now we create a tensor in a block circulant pattern, where each block is a tensor whose order is $(p-1)$:

$$\text{circ}(\text{unfold}(\mathcal{A})) = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_{n_p} & \mathcal{A}_{n_p-1} & \cdots & \mathcal{A}_2 \\ \mathcal{A}_2 & \mathcal{A}_1 & \mathcal{A}_{n_p} & \cdots & \mathcal{A}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{n_p} & \mathcal{A}_{n_p-1} & \mathcal{A}_{n_p-2} & \cdots & \mathcal{A}_1 \end{bmatrix},$$

which is an $n_1 n_p \times \cdots \times n_{p-1}$ tensor. For example, let $\mathcal{A}$ be a $n_1 \times n_2 \times n_3$ tensor. Fixing the third index of $\mathcal{A}$, one can get n_3 matrices $A_i \in \mathbb{R}^{n_1 \times n_2}$ matrices, $i = 1, \ldots, n_3$, and

$$\text{unfold}(\mathcal{A}) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n_3} \end{bmatrix}, \quad \text{circ}(\text{unfold}(\mathcal{A})) = \begin{bmatrix} A_1 & A_{n_3} & A_{n_3-1} & \cdots & A_2 \\ A_2 & A_1 & A_{n_3} & \cdots & A_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n_3} & A_{n_3-1} & A_{n_3-2} & \cdots & A_1 \end{bmatrix}.$$

The definition of the t-product of two tensors is given as follows.

Definition 2.1 (t-product of two tensors)**.** *[13] Let $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \cdots \times n_p}$ and $\mathcal{B} \in \mathbb{R}^{n_2 \times l \times n_3 \cdots \times n_p}$ be given. Then the t-product $\mathcal{A} * \mathcal{B}$ is the order-p ($p \geq 3$) tensor defined recursively as*

$$\mathcal{A} * \mathcal{B} = fold(circ(unfold(\mathcal{A})) * unfold(\mathcal{B})) \tag{5}$$

of size $n_1 \times l \times n_3 \times \cdots \times n_p$.

Martin et al., [13] presented an algorithm to compute the t-product of two tensors by using the Fourier transform. Some basic properties of the t-product are given.

Proposition 2.2. *[8] If $\mathcal{A}, \mathcal{B}, C$ are tensors of adequate size, then*
(a) $\mathcal{A} * (\mathcal{B} + C) = \mathcal{A} * \mathcal{B} + \mathcal{A} * C$;
(b) $(\mathcal{A} + \mathcal{B}) * C = \mathcal{A} * C + \mathcal{B} * C$;
(c) $(\mathcal{A} * \mathcal{B}) * C = \mathcal{A} * (\mathcal{B} * C)$.

For more properties of t-product, we refer the reader to the recent papers [11] and [13]. The definition of identity tensor is given as follows.

Definition 2.3 (Identity tensor)**.** *[13] The $n \times n \times n_3 \times \cdots \times n_p$ order-p ($p \geq 3$) identity tensor $\mathcal{I}$ is the tensor such that $\mathcal{I}_1$ is the $n \times n \times n_3 \times \cdots \times n_{p-1}$ order-$(p-1)$ identity tensor and $\mathcal{I}_j$, $j = 2, 3, \ldots, n_p$ is the $n \times n \times n_3 \times \cdots \times n_{p-1}$ order-$(p-1)$ zero tensor.*

We can also give the definition of the transpose of tensors.

Definition 2.4 (Transpose)**.** *[13] Let $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \cdots \times n_p}$, then the transpose of $\mathcal{A}$, which is denoted by $\mathcal{A}^T$, is the $n_2 \times n_1 \cdots \times n_p$ tensor obtained by tensor transposing each $\mathcal{A}_i$, for $i = 1, 2, \ldots, n_p$ and then reversing the order of the $\mathcal{A}_i$ from 2 through n_p, i.e.,*

$$\mathcal{A}^T = fold\left(\begin{bmatrix} \mathcal{A}_1^T \\ \mathcal{A}_{n_p}^T \\ \mathcal{A}_{n_p-1}^T \\ \vdots \\ \mathcal{A}_2^T \end{bmatrix}\right). \tag{6}$$

It is easy to check that $(\mathcal{A} * \mathcal{B})^T = \mathcal{B}^T * \mathcal{A}^T$. The definition of symmetric tensor now follows.

Definition 2.5 (Symmetric tensor)**.** *[13] Let $\mathcal{A} \in \mathbb{R}^{n \times n \cdots \times n_p}$. We say that $\mathcal{A}$ is symmetric if $\mathcal{A}^T = \mathcal{A}$.*

The definition of the Moore-Penrose inverse of the tensor under t-product was first given in [8].

Definition 2.6 (Moore-Penrose inverse)**.** *[8] Let $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \cdots \times n_p}$. The tensor $\mathcal{X} \in \mathbb{R}^{n_2 \times n_1 \times n_3 \cdots \times n_p}$ satisfying the following four tensor equations*
(1) $\mathcal{A} * \mathcal{X} * \mathcal{A} = \mathcal{A}$;
(2) $\mathcal{X} * \mathcal{A} * \mathcal{X} = \mathcal{X}$;
(3) $(\mathcal{A} * \mathcal{X})^T = \mathcal{A} * \mathcal{X}$;
(4) $(\mathcal{X} * \mathcal{A})^T = \mathcal{X} * \mathcal{A}$,
is called the Moore-Penrose inverse of the tensor $\mathcal{A}$, and is denoted by $\mathcal{A}^\dagger$.

The Moore-Penrose inverse of an arbitrary tensor $\mathcal{A}$ exists and is unique [8]. The algorithm for finding the Moore-Penrose inverse of a tensor $\mathcal{A}$ was presented in [8]. For more properties of Moore-Penrose inverse of tensor under t-product, we refer the reader to the recent paper [8].

It is easy to obtain the following results.

Proposition 2.7. *For the tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \cdots \times n_p}$, the symbols $\mathcal{L}_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{A}}$ stand for*

$$\mathcal{L}_{\mathcal{A}} = \mathcal{I} - \mathcal{A}^{\dagger} * \mathcal{A}, \quad \mathcal{R}_{\mathcal{A}} = \mathcal{I} - \mathcal{A} * \mathcal{A}^{\dagger}. \tag{7}$$

Then

$$\mathcal{L}_{\mathcal{A}}^{T} = \mathcal{L}_{\mathcal{A}}, \quad \mathcal{R}_{\mathcal{A}}^{T} = \mathcal{R}_{\mathcal{A}}. \tag{8}$$

3. The general solution to the system (3)

In this section, we consider the system of tensor equations under t-product

$$\begin{cases} \mathcal{A}_1 * \mathcal{X} = \mathcal{C}_1, \\ \mathcal{X} * \mathcal{B}_1 = \mathcal{D}_1, \end{cases} \tag{9}$$

where

$$\mathcal{A}_1 \in \mathbb{R}^{t_1 \times l \times n_3 \times \cdots \times n_p},\ \mathcal{C}_1 \in \mathbb{R}^{t_1 \times m \times n_3 \times \cdots \times n_p},\ \mathcal{B}_1 \in \mathbb{R}^{m \times q_1 \times n_3 \times \cdots \times n_p},\ \mathcal{D}_1 \in \mathbb{R}^{l \times q_1 \times n_3 \times \cdots \times n_p},$$

and $\mathcal{X} \in \mathbb{R}^{l \times m \times n_3 \times \cdots \times n_p}$ is unknown. The following theorem gives some solvability conditions and general solution to the system (9).

Theorem 3.1. *Let $\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1$, and $\mathcal{D}_1$ be given tensors. The system of tensor equations (9) is consistent if and only if*

$$\mathcal{R}_{\mathcal{A}_1} * \mathcal{C}_1 = 0,\ \mathcal{D}_1 * \mathcal{L}_{\mathcal{B}_1} = 0,\ \mathcal{A}_1 * \mathcal{D}_1 = \mathcal{C}_1 * \mathcal{B}_1. \tag{10}$$

In this case, the general solution to (9) can be expressed as

$$\mathcal{X} = \mathcal{A}_1^{\dagger} * \mathcal{C}_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^{\dagger} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{Y} * \mathcal{R}_{\mathcal{B}_1}, \tag{11}$$

where $\mathcal{Y}$ is an arbitrary tensor with suitable order, $\mathcal{L}_{\mathcal{A}_1}$ and $\mathcal{R}_{\mathcal{B}_1}$ are defined as (7).

Proof. "only if"-part. Assume that the system of tensor equations (9) has a solution $\mathcal{X}_0$. Since $\mathcal{R}_{\mathcal{A}_1} * \mathcal{A}_1 = 0$ and $\mathcal{B}_1 * \mathcal{L}_{\mathcal{B}_1} = 0$, we have

$$\mathcal{R}_{\mathcal{A}_1} * \mathcal{C}_1 = \mathcal{R}_{\mathcal{A}_1} * \mathcal{A}_1 * \mathcal{X}_0 = 0 \tag{12}$$

and

$$\mathcal{D}_1 * \mathcal{L}_{\mathcal{B}_1} = \mathcal{X}_0 * \mathcal{B}_1 * \mathcal{L}_{\mathcal{B}_1} = 0. \tag{13}$$

It is easy to verify that

$$\mathcal{A}_1 * \mathcal{D}_1 = \mathcal{A}_1 * \mathcal{X}_0 * \mathcal{B}_1 = \mathcal{C}_1 * \mathcal{B}_1. \tag{14}$$

"if"-part. We prove that $\mathcal{X}$ having the form of (11) is a solution to the system (9) under the hypotheses (10). Substituting (11) into the system (9) yields

$$\begin{aligned} &\mathcal{A}_1 * [\mathcal{A}_1^{\dagger} * \mathcal{C}_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^{\dagger} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{Y} * \mathcal{R}_{\mathcal{B}_1}] \\ =&\mathcal{A}_1 \mathcal{A}_1^{\dagger} * \mathcal{C}_1 \\ =&\mathcal{C}_1 \end{aligned}$$

and

$$\begin{aligned}
&[\mathcal{A}_1^\dagger * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{L}_{\mathcal{A}_1} * \mathcal{Y} * \mathcal{R}_{\mathcal{B}_1}] * \mathcal{B}_1\\
=&\mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_1\\
=&\mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 + \mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_1 - \mathcal{A}_1^\dagger * \mathcal{A}_1 * \mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_1\\
=&\mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 + \mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_1 - \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_1\\
=&\mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 + \mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_1 - \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1\\
=&\mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_1\\
=&\mathcal{D}_1.
\end{aligned}$$

Now we show that if the system (9) is consistent, i.e. (10) holds, then an arbitrary solution $\mathcal{X}_0$ can be expressed by (11). Let

$$\mathcal{Y} = \mathcal{X}_0. \tag{15}$$

Then by

$$\begin{aligned}
\mathcal{X} =&\mathcal{A}_1^\dagger * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{L}_{\mathcal{A}_1} * \mathcal{X}_0 * \mathcal{R}_{\mathcal{B}_1}\\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{D}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * \mathcal{A}_1 * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{X}_0 * \mathcal{R}_{\mathcal{B}_1} - \mathcal{A}_1^\dagger * \mathcal{A}_1 * \mathcal{X}_0 * \mathcal{R}_{\mathcal{B}_1}\\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{D}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 * \mathcal{B}_1^\dagger + \mathcal{X}_0 - \mathcal{X}_0 * \mathcal{B}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * C_1 * \mathcal{R}_{\mathcal{B}_1}\\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{D}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 * \mathcal{B}_1^\dagger + \mathcal{X}_0 - \mathcal{D}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * C_1 + \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 * \mathcal{B}_1^\dagger\\
=&\mathcal{X}_0.
\end{aligned}$$

□

4. The general solution to the system (2)

In this section, we consider the following system of tensor equations

$$\begin{cases} \mathcal{A}_1 * \mathcal{X} = C_1, \\ \mathcal{X} * \mathcal{B}_1 = \mathcal{D}_1, \\ \mathcal{A}_2 * \mathcal{X} * \mathcal{B}_2 = C_2, \end{cases} \tag{16}$$

where

$$\mathcal{A}_1 \in \mathbb{R}^{t_1\times l\times n_3\times\cdots\times n_p},\ C_1 \in \mathbb{R}^{t_1\times m\times n_3\times\cdots\times n_p},\ \mathcal{B}_1 \in \mathbb{R}^{m\times q_1\times n_3\times\cdots\times n_p},\ \mathcal{D}_1 \in \mathbb{R}^{l\times q_1\times n_3\times\cdots\times n_p},$$

$$\mathcal{A}_2 \in \mathbb{R}^{t_2\times l\times n_3\times\cdots\times n_p},\ \mathcal{B}_2 \in \mathbb{R}^{m\times q_2\times n_3\times\cdots\times n_p},\ C_2 \in \mathbb{R}^{t_2\times q_2\times n_3\times\cdots\times n_p},$$

and $\mathcal{X} \in \mathbb{R}^{l\times m\times n_3\times\cdots\times n_p}$ is unknown. In the following theorem, we give some necessary and sufficient conditions for the solvability of system (16), and present the general solution to (16) if it is solvable.

Theorem 4.1. *Let $\mathcal{A}_1, \mathcal{B}_1, C_1, \mathcal{D}_1, \mathcal{A}_2, \mathcal{B}_2$ and C_2 be given tensors. Denote*

$$\mathcal{A}_3 = \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1},\ \mathcal{B}_3 = \mathcal{R}_{\mathcal{B}_1} * \mathcal{B}_2,\ C_3 = C_2 - \mathcal{A}_2 * \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_2 - \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_2. \tag{17}$$

The system of tensor equations (16) is consistent if and only if

$$\mathcal{R}_{\mathcal{A}_1} * C_1 = 0,\ \mathcal{D}_1 * \mathcal{L}_{\mathcal{B}_1} = 0,\ \mathcal{A}_1 * \mathcal{D}_1 = C_1 * \mathcal{B}_1,\ \mathcal{R}_{\mathcal{A}_3} * C_3 = 0,\ C_3 * \mathcal{L}_{\mathcal{B}_3} = 0. \tag{18}$$

In this case, the general solution to (16) can be expressed as

$$\begin{aligned}
\mathcal{X} =&\mathcal{A}_1^\dagger * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{L}_{\mathcal{A}_1} * \mathcal{A}_3^\dagger * C_3 * \mathcal{B}_3^\dagger * \mathcal{R}_{\mathcal{B}_1}\\
&+ \mathcal{L}_{\mathcal{A}_1} * \mathcal{L}_{\mathcal{A}_3} * \mathcal{Z}_1 * \mathcal{R}_{\mathcal{B}_1} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{Z}_2 * \mathcal{R}_{\mathcal{B}_3} * \mathcal{R}_{\mathcal{B}_1},
\end{aligned} \tag{19}$$

where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are arbitrary tensors with suitable order, $\mathcal{L}_{\mathcal{A}_1}, \mathcal{L}_{\mathcal{A}_3}, \mathcal{R}_{\mathcal{B}_1}$ and $\mathcal{R}_{\mathcal{B}_3}$ are defined as (7).

Proof. "only if"-part. We separate the system (16) into two parts

$$\begin{cases} \mathcal{A}_1 * \mathcal{X} = C_1, \\ \mathcal{X} * \mathcal{B}_1 = \mathcal{D}_1, \end{cases} \tag{20}$$

and

$$\mathcal{A}_2 * \mathcal{X} * \mathcal{B}_2 = C_2. \tag{21}$$

It follows from Theorem 3.1 that the system (20) is consistent if and only if

$$\mathcal{R}_{\mathcal{A}_1} * C_1 = 0,\ \mathcal{D}_1 * \mathcal{L}_{\mathcal{B}_1} = 0,\ \mathcal{A}_1 * \mathcal{D}_1 = C_1 * \mathcal{B}_1. \tag{22}$$

In this case, the general solution $\mathcal{X}$ has the form of (11). Substituting (11) into

$$\mathcal{A}_2 * \mathcal{X} * \mathcal{B}_2 = C_2 \tag{23}$$

yields

$$\mathcal{A}_2 * \mathcal{A}_1^{\dagger} * C_1 * \mathcal{B}_2 + \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^{\dagger} * \mathcal{B}_2 + \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1} * \mathcal{Y} * \mathcal{R}_{\mathcal{B}_1} * \mathcal{B}_2 = C_2, \tag{24}$$

i.e.,

$$\mathcal{A}_3 * \mathcal{Y} * \mathcal{B}_3 = C_3, \tag{25}$$

where $\mathcal{A}_3, \mathcal{B}_3$, and C_3 are given by (17). Thus by $\mathcal{R}_{\mathcal{A}_3} * \mathcal{A}_3 = 0$ and $\mathcal{B}_3 * \mathcal{L}_{\mathcal{B}_3} = 0$,

$$\mathcal{R}_{\mathcal{A}_3} * C_3 = \mathcal{R}_{\mathcal{A}_3} * \mathcal{A}_3 * \mathcal{Y} * \mathcal{B}_3 = 0, \tag{26}$$

$$C_3 * \mathcal{L}_{\mathcal{B}_3} = \mathcal{A}_3 * \mathcal{Y} * \mathcal{B}_3 * \mathcal{L}_{\mathcal{B}_3} = 0. \tag{27}$$

"if"-part. Suppose that the equations in (18) hold. By virtue of $\mathcal{R}_{\mathcal{A}_1} * C_1 = 0$, $\mathcal{D}_1 * \mathcal{L}_{\mathcal{B}_1} = 0$, $\mathcal{A}_1 * \mathcal{D}_1 = C_1 * \mathcal{B}_1$, we have

$$\begin{aligned} &\mathcal{A}_1 * [\mathcal{A}_1^{\dagger} * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^{\dagger} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{A}_3^{\dagger} * C_3 * \mathcal{B}_3^{\dagger} * \mathcal{R}_{\mathcal{B}_1} \\ &\quad + \mathcal{L}_{\mathcal{A}_1} * \mathcal{L}_{\mathcal{A}_3} * \mathcal{Z}_1 * \mathcal{R}_{\mathcal{B}_1} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{Z}_2 * \mathcal{R}_{\mathcal{B}_3} * \mathcal{R}_{\mathcal{B}_1}] \\ =&\mathcal{A}_1 * \mathcal{A}_1^{\dagger} * C_1 \\ =&C_1, \end{aligned}$$

$$\begin{aligned} &[\mathcal{A}_1^{\dagger} * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^{\dagger} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{A}_3^{\dagger} * C_3 * \mathcal{B}_3^{\dagger} * \mathcal{R}_{\mathcal{B}_1} \\ &\quad + \mathcal{L}_{\mathcal{A}_1} * \mathcal{L}_{\mathcal{A}_3} * \mathcal{Z}_1 * \mathcal{R}_{\mathcal{B}_1} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{Z}_2 * \mathcal{R}_{\mathcal{B}_3} * \mathcal{R}_{\mathcal{B}_1}] * \mathcal{B}_1 \\ =&\mathcal{A}_1^{\dagger} * C_1 * \mathcal{B}_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^{\dagger} * \mathcal{B}_1 \\ =&\mathcal{A}_1^{\dagger} * C_1 * \mathcal{B}_1 + \mathcal{D}_1 * \mathcal{B}_1^{\dagger} * \mathcal{B}_1 - \mathcal{A}_1^{\dagger} * \mathcal{A}_1 * \mathcal{D}_1 * \mathcal{B}_1^{\dagger} * \mathcal{B}_1 \\ =&\mathcal{A}_1^{\dagger} * C_1 * \mathcal{B}_1 + \mathcal{D}_1 - \mathcal{A}_1^{\dagger} * C_1 * \mathcal{B}_1 \\ =&\mathcal{D}_1. \end{aligned}$$

Now we prove that the $\mathcal{X}$ that has the form of (19) is also a solution of $\mathcal{A}_2 * \mathcal{X} * \mathcal{B}_2 = C_2$. It follows from

$\mathcal{R}_{\mathcal{A}_3} * C_3 = 0$ and $C_3 * \mathcal{L}_{\mathcal{B}_3} = 0$ that

$$\begin{aligned}
&\mathcal{A}_2 * [\mathcal{A}_1^\dagger * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{L}_{\mathcal{A}_1} * \mathcal{A}_3^\dagger * C_3 * \mathcal{B}_3^\dagger * \mathcal{R}_{\mathcal{B}_1} \\
&+ \mathcal{L}_{\mathcal{A}_1} * \mathcal{L}_{\mathcal{A}_3} * \mathcal{Z}_1 * \mathcal{R}_{\mathcal{B}_1} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{Z}_2 * \mathcal{R}_{\mathcal{B}_3} * \mathcal{R}_{\mathcal{B}_1}] * \mathcal{B}_2 \\
=&\mathcal{A}_2 * \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_2 + \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_2 + \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1} * \mathcal{A}_3^\dagger * C_3 * \mathcal{B}_3^\dagger * \mathcal{R}_{\mathcal{B}_1} * \mathcal{B}_2 \\
&+ \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1} * \mathcal{L}_{\mathcal{A}_3} * \mathcal{Z}_1 * \mathcal{R}_{\mathcal{B}_1} * \mathcal{B}_2 + \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1} * \mathcal{Z}_2 * \mathcal{R}_{\mathcal{B}_3} * \mathcal{R}_{\mathcal{B}_1} * \mathcal{B}_2 \\
=&\mathcal{A}_2 * \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_2 + \mathcal{A}_3 * \mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_2 + \mathcal{A}_3 * \mathcal{A}_3^\dagger * C_3 * \mathcal{B}_3^\dagger * \mathcal{B}_3 \\
&+ \mathcal{A}_3 * \mathcal{L}_{\mathcal{A}_3} * \mathcal{Z}_1 * \mathcal{R}_{\mathcal{B}_1} * \mathcal{B}_2 + \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1} * \mathcal{Z}_2 * \mathcal{R}_{\mathcal{B}_3} * \mathcal{B}_3 \\
=&\mathcal{A}_2 * \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_2 + \mathcal{A}_3 * \mathcal{D}_1 * \mathcal{B}_1^\dagger * \mathcal{B}_2 + C_3 \\
=&C_2.
\end{aligned}$$

Hence, the tensor $\mathcal{X}$ has the form of (19) is a solution to the system (16).

Now we prove that for an arbitrary solution $\mathcal{X}_0$ of the system (16) can be expressed as the form of (19). Let

$$\mathcal{Z}_1 = \mathcal{X}_0 * \mathcal{B}_3 * \mathcal{B}_3^\dagger, \quad \mathcal{Z}_2 = \mathcal{X}_0.$$

Using the fact that $\mathcal{A}_3 * \mathcal{X}_0 * \mathcal{B}_3 = C_3$ and (18), we obtain

$$\begin{aligned}
\mathcal{X} =&\mathcal{A}_1^\dagger * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{L}_{\mathcal{A}_1} * \mathcal{A}_3^\dagger * C_3 * \mathcal{B}_3^\dagger * \mathcal{R}_{\mathcal{B}_1} \\
&+ \mathcal{L}_{\mathcal{A}_1} * \mathcal{L}_{\mathcal{A}_3} * \mathcal{X}_0 * \mathcal{B}_3 * \mathcal{B}_3^\dagger * \mathcal{R}_{\mathcal{B}_1} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{X}_0 * \mathcal{R}_{\mathcal{B}_3} * \mathcal{R}_{\mathcal{B}_1} \\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{L}_{\mathcal{A}_1} * [\mathcal{A}_3^\dagger * C_3 * \mathcal{B}_3^\dagger + \mathcal{L}_{\mathcal{A}_3} * \mathcal{X}_0 * \mathcal{B}_3 * \mathcal{B}_3^\dagger + \mathcal{X}_0 * \mathcal{R}_{\mathcal{B}_3}] * \mathcal{R}_{\mathcal{B}_1} \\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{L}_{\mathcal{A}_1} * [\mathcal{A}_3^\dagger * C_3 * \mathcal{B}_3^\dagger + \mathcal{X}_0 * \mathcal{B}_3 * \mathcal{B}_3^\dagger \\
&- \mathcal{A}_3^\dagger * \mathcal{A}_3 * \mathcal{X}_0 * \mathcal{B}_3 * \mathcal{B}_3^\dagger + \mathcal{X}_0 - \mathcal{X}_0 * \mathcal{B}_3 * \mathcal{B}_3^\dagger] * \mathcal{R}_{\mathcal{B}_1} \\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{L}_{\mathcal{A}_1} * [\mathcal{A}_3^\dagger * C_3 * \mathcal{B}_3^\dagger - \mathcal{A}_3^\dagger * C_3 * \mathcal{B}_3^\dagger + \mathcal{X}_0] * \mathcal{R}_{\mathcal{B}_1} \\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{L}_{\mathcal{A}_1} * \mathcal{X}_0 * \mathcal{R}_{\mathcal{B}_1} \\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{D}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * \mathcal{A}_1 * \mathcal{D}_1 * \mathcal{B}_1^\dagger + \mathcal{X}_0 * \mathcal{R}_{\mathcal{B}_1} - \mathcal{A}_1^\dagger * \mathcal{A}_1 * \mathcal{X}_0 * \mathcal{R}_{\mathcal{B}_1} \\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{D}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 * \mathcal{B}_1^\dagger + \mathcal{X}_0 - \mathcal{X}_0 * \mathcal{B}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * C_1 * \mathcal{R}_{\mathcal{B}_1} \\
=&\mathcal{A}_1^\dagger * C_1 + \mathcal{D}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 * \mathcal{B}_1^\dagger + \mathcal{X}_0 - \mathcal{D}_1 * \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger * C_1 + \mathcal{A}_1^\dagger * C_1 * \mathcal{B}_1 * \mathcal{B}_1^\dagger \\
=&\mathcal{X}_0.
\end{aligned}$$

□

Now we give the algorithm for finding the general solution to (16).

Algorithm 4.2. *The solution to the system (16).*

Input: $\mathcal{A}_1, \mathcal{B}_1, C_1, \mathcal{D}_1, \mathcal{A}_2, \mathcal{B}_2$ *and* C_2.

(1) : *Compute* $\mathcal{A}_3, \mathcal{B}_3, C_3$ *by (17).*

(2) : *If any of equations in (18) fails, return "No solution".*

(3) : *Else compute the Moore-Penrose inverses* $\mathcal{A}_1^\dagger, \mathcal{B}_1^\dagger, \mathcal{A}_3^\dagger$, *and* $\mathcal{B}_3^\dagger$.

(4) : *Compute* $\mathcal{L}_{\mathcal{A}_1}, \mathcal{L}_{\mathcal{A}_3}, \mathcal{R}_{\mathcal{B}_1}$ *and* $\mathcal{R}_{\mathcal{B}_3}$ *by (7).*

Output: $\mathcal{X}$ *in the form (19).*

We give two examples to illustrate Theorem 4.1.

Example 4.3. *For the system (16), let* $\mathcal{A}_1, \mathcal{B}_1, C_1, \mathcal{D}_1, \mathcal{A}_2, \mathcal{B}_2$ *and* C_2 *be given tensors, where* $\mathcal{A}_1 \in \mathbb{R}^{4\times5\times4}$, $\mathcal{B}_1 \in \mathbb{R}^{4\times3\times4}$, $C_1 \in \mathbb{R}^{4\times4\times4}$, $\mathcal{D}_1 \in \mathbb{R}^{5\times3\times4}$, $\mathcal{A}_2 \in \mathbb{R}^{3\times5\times4}$, $\mathcal{B}_2 \in \mathbb{R}^{4\times2\times4}$, $C_2 \in \mathbb{R}^{3\times2\times4}$, *and*

$$\mathcal{A}_1(:,:,1) = \begin{bmatrix} 8 & 2 & 7 & 5 & 2 \\ 7 & 8 & 8 & 2 & 7 \\ 8 & 5 & 8 & 5 & 8 \\ 2 & 2 & 8 & 8 & 5 \end{bmatrix}, \ \mathcal{A}_1(:,:,2) = \begin{bmatrix} 5 & 4 & 2 & 9 & 5 \\ 9 & 3 & 5 & 6 & 6 \\ 6 & 4 & 2 & 4 & 8 \\ 8 & 7 & 2 & 7 & 6 \end{bmatrix},$$

$$\mathcal{A}_1(:,:,3)=\begin{bmatrix}5&4&9&7&9\\5&4&3&7&2\\1&1&7&5&1\\9&5&2&2&6\end{bmatrix},\ \mathcal{A}_1(:,:,4)=\begin{bmatrix}6&4&4&3&1\\5&6&5&7&1\\6&4&2&5&4\\7&2&1&5&4\end{bmatrix},$$

$$\mathcal{B}_1(:,:,1)=\begin{bmatrix}13&18&9\\19&10&11\\17&5&12\\12&9&15\end{bmatrix},\ \mathcal{B}_1(:,:,2)=\begin{bmatrix}10&18&13\\19&13&5\\9&18&7\\16&5&11\end{bmatrix},$$

$$\mathcal{B}_1(:,:,3)=\begin{bmatrix}8&13&7\\13&8&12\\9&5&14\\15&5&6\end{bmatrix},\ \mathcal{B}_1(:,:,4)=\begin{bmatrix}6&9&7\\9&14&6\\6&8&18\\11&14&13\end{bmatrix},$$

$$C_1(:,:,1)=\begin{bmatrix}588&662&697&822\\664&718&703&871\\574&600&632&749\\640&676&646&847\end{bmatrix},\ C_1(:,:,2)=\begin{bmatrix}697&741&684&745\\680&744&686&760\\631&696&601&673\\659&643&655&767\end{bmatrix},$$

$$C_1(:,:,3)=\begin{bmatrix}682&681&602&857\\708&721&679&791\\580&634&633&785\\560&660&642&762\end{bmatrix},\ C_1(:,:,4)=\begin{bmatrix}666&711&715&730\\674&707&714&835\\619&620&656&738\\583&747&649&715\end{bmatrix},$$

$$\mathcal{D}_1(:,:,1)=\begin{bmatrix}1324&1139&1105\\1054&989&907\\1300&1232&1177\\1552&1358&1278\\1328&1139&1114\end{bmatrix},\ \mathcal{D}_1(:,:,2)=\begin{bmatrix}1374&1160&1092\\1030&994&953\\1383&1243&1193\\1494&1428&1382\\1375&1074&1225\end{bmatrix},$$

$$\mathcal{D}_1(:,:,3)=\begin{bmatrix}1350&1236&1180\\1114&935&936\\1282&1245&1120\\1553&1309&1300\\1302&1168&1173\end{bmatrix},\ \mathcal{D}_1(:,:,4)=\begin{bmatrix}1451&1058&1177\\1091&965&940\\1324&1200&1225\\1528&1322&1220\\1326&1140&1176\end{bmatrix},$$

$$\mathcal{A}_2(:,:,1)=\begin{bmatrix}8&8&8&6&4\\7&6&4&4&4\\6&5&3&3&3\end{bmatrix},\ \mathcal{A}_2(:,:,2)=\begin{bmatrix}4&4&5&6&3\\7&6&4&3&4\\8&8&6&6&6\end{bmatrix},$$

$$\mathcal{A}_2(:,:,3)=\begin{bmatrix}7&8&4&8&3\\7&8&8&4&6\\6&5&5&3&5\end{bmatrix},\ \mathcal{A}_2(:,:,4)=\begin{bmatrix}5&7&3&6&5\\7&8&8&3&5\\5&7&8&8&6\end{bmatrix},$$

$$\mathcal{B}_2(:,:,1)=\begin{bmatrix}3&2\\1&1\\4&2\\1&3\end{bmatrix},\ \mathcal{B}_2(:,:,2)=\begin{bmatrix}6&3\\6&3\\5&3\\1&6\end{bmatrix},\ \mathcal{B}_2(:,:,3)=\begin{bmatrix}2&6\\1&5\\5&6\\1&5\end{bmatrix},\ \mathcal{B}_2(:,:,4)=\begin{bmatrix}4&2\\6&2\\4&3\\3&5\end{bmatrix},$$

$$C_2(:,:,1) = 10^4 * \begin{bmatrix} 3.9798 & 4.4064 \\ 3.9508 & 4.3906 \\ 3.9236 & 4.4248 \end{bmatrix}, \; C_2(:,:,2) = 10^4 * \begin{bmatrix} 3.9418 & 4.4213 \\ 3.9445 & 4.3952 \\ 3.9778 & 4.4052 \end{bmatrix},$$

$$C_2(:,:,3) = 10^4 * \begin{bmatrix} 3.9845 & 4.4132 \\ 3.9483 & 4.3851 \\ 3.9210 & 4.4102 \end{bmatrix}, \; C_2(:,:,4) = 10^4 * \begin{bmatrix} 3.9469 & 4.4419 \\ 3.9435 & 4.3791 \\ 3.9829 & 4.3823 \end{bmatrix},$$

where $\mathcal{A}(:,:,k)$ means the kth frontal slice of the tensor $\mathcal{A}$. We consider the system (16). By using the Algorithm 4.2, we have

$$\mathcal{R}_{\mathcal{A}_1} * C_1 = 0, \; \mathcal{D}_1 * \mathcal{L}_{\mathcal{B}_1} = 0, \; \mathcal{A}_1 * \mathcal{D}_1 = C_1 * \mathcal{B}_1, \; \mathcal{R}_{\mathcal{A}_3} * C_3 = 0, \; C_3 * \mathcal{L}_{\mathcal{B}_3} = 0.$$

Hence, the system (16) has a solution. The general solution to the system (16) is given as follows

$$\mathcal{X} = \mathcal{X}_0 + \mathcal{L}_{\mathcal{A}_1} * \mathcal{L}_{\mathcal{A}_3} * \mathcal{Z}_1 * \mathcal{R}_{\mathcal{B}_1} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{Z}_2 * \mathcal{R}_{\mathcal{B}_3} * \mathcal{R}_{\mathcal{B}_1}, \tag{28}$$

where

$$\mathcal{L}_{\mathcal{A}_1}(:,:,1) = \begin{bmatrix} 0.2088 & -0.1410 & 0.0615 & -0.1326 & -0.0905 \\ -0.1410 & 0.2956 & -0.0514 & 0.1678 & 0.0337 \\ 0.0615 & -0.0514 & 0.0360 & 0.0246 & -0.0356 \\ -0.1326 & 0.1678 & 0.0246 & 0.3796 & -0.0035 \\ -0.0905 & 0.0337 & -0.0356 & -0.0035 & 0.0799 \end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_1}(:,:,2) = \begin{bmatrix} 0.0656 & -0.0929 & 0.0270 & -0.0651 & 0.0487 \\ 0.0559 & -0.0020 & 0.0606 & 0.1795 & -0.1250 \\ 0.0272 & -0.0868 & 0.0071 & -0.0659 & 0.0159 \\ 0.0259 & -0.2031 & 0.0236 & -0.0357 & -0.0689 \\ 0.0135 & 0.0696 & -0.0187 & -0.0405 & -0.0349 \end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_1}(:,:,3) = \begin{bmatrix} 0.1502 & -0.0713 & 0.0035 & -0.2407 & -0.0303 \\ -0.0713 & -0.1648 & 0.0169 & 0.0656 & 0.0515 \\ 0.0035 & 0.0169 & -0.0213 & -0.0827 & 0.0241 \\ -0.2407 & 0.0656 & -0.0827 & 0.0390 & 0.1622 \\ -0.0303 & 0.0515 & 0.0241 & 0.1622 & -0.0030 \end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_1}(:,:,4) = \begin{bmatrix} 0.0656 & 0.0559 & 0.0272 & 0.0259 & 0.0135 \\ -0.0929 & -0.0020 & -0.0868 & -0.2031 & 0.0696 \\ 0.0270 & 0.0606 & 0.0071 & 0.0236 & -0.0187 \\ -0.0651 & 0.1795 & -0.0659 & -0.0357 & -0.0405 \\ 0.0487 & -0.1250 & 0.0159 & -0.0689 & -0.0349 \end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_3}(:,:,1) = \begin{bmatrix} 0.7912 & 0.1410 & -0.0615 & 0.1326 & 0.0905 \\ 0.1410 & 0.7044 & 0.0514 & -0.1678 & -0.0337 \\ -0.0615 & 0.0514 & 0.9640 & -0.0246 & 0.0356 \\ 0.1326 & -0.1678 & -0.0246 & 0.6204 & 0.0035 \\ 0.0905 & -0.0337 & 0.0356 & 0.0035 & 0.9201 \end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_3}(:,:,2) = \begin{bmatrix} -0.0656 & 0.0929 & -0.0270 & 0.0651 & -0.0487 \\ -0.0559 & 0.0020 & -0.0606 & -0.1795 & 0.1250 \\ -0.0272 & 0.0868 & -0.0071 & 0.0659 & -0.0159 \\ -0.0259 & 0.2031 & -0.0236 & 0.0357 & 0.0689 \\ -0.0135 & -0.0696 & 0.0187 & 0.0405 & 0.0349 \end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_3}(:,:,3) = \begin{bmatrix} -0.1502 & 0.0713 & -0.0035 & 0.2407 & 0.0303 \\ 0.0713 & 0.1648 & -0.0169 & -0.0656 & -0.0515 \\ -0.0035 & -0.0169 & 0.0213 & 0.0827 & -0.0241 \\ 0.2407 & -0.0656 & 0.0827 & -0.0390 & -0.1622 \\ 0.0303 & -0.0515 & -0.0241 & -0.1622 & 0.0030 \end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_3}(:,:,4) = \begin{bmatrix} -0.0656 & -0.0559 & -0.0272 & -0.0259 & -0.0135 \\ 0.0929 & 0.0020 & 0.0868 & 0.2031 & -0.0696 \\ -0.0270 & -0.0606 & -0.0071 & -0.0236 & 0.0187 \\ 0.0651 & -0.1795 & 0.0659 & 0.0357 & 0.0405 \\ -0.0487 & 0.1250 & -0.0159 & 0.0689 & 0.0349 \end{bmatrix},$$

$$\mathcal{R}_{\mathcal{B}_1}(:,:,1) = \begin{bmatrix} 0.2192 & -0.0988 & -0.1146 & 0.1827 \\ -0.0988 & 0.1843 & 0.0316 & -0.1150 \\ -0.1146 & 0.0316 & 0.1326 & -0.0937 \\ 0.1827 & -0.1150 & -0.0937 & 0.4639 \end{bmatrix},$$

$$\mathcal{R}_{\mathcal{B}_1}(:,:,2) = \begin{bmatrix} -0.0309 & -0.0064 & -0.0633 & -0.2045 \\ -0.1302 & 0.0165 & 0.1512 & -0.0491 \\ 0.0703 & 0.0198 & 0.0356 & 0.0486 \\ 0.1300 & -0.2418 & -0.0985 & -0.0213 \end{bmatrix},$$

$$\mathcal{R}_{\mathcal{B}_1}(:,:,3) = \begin{bmatrix} -0.0901 & 0.1202 & -0.0226 & 0.0724 \\ 0.1202 & -0.0204 & 0.0199 & 0.0970 \\ -0.0226 & 0.0199 & 0.0476 & -0.2053 \\ 0.0724 & 0.0970 & -0.2053 & 0.0630 \end{bmatrix},$$

$$\mathcal{R}_{\mathcal{B}_1}(:,:,4) = \begin{bmatrix} -0.0309 & -0.1302 & 0.0703 & 0.1300 \\ -0.0064 & 0.0165 & 0.0198 & -0.2418 \\ -0.0633 & 0.1512 & 0.0356 & -0.0985 \\ -0.2045 & -0.0491 & 0.0486 & -0.0213 \end{bmatrix},$$

$$\mathcal{R}_{\mathcal{B}_3}(:,:,1) = \begin{bmatrix} 0.7808 & 0.0988 & 0.1146 & -0.1827 \\ 0.0988 & 0.8157 & -0.0316 & 0.1150 \\ 0.1146 & -0.0316 & 0.8674 & 0.0937 \\ -0.1827 & 0.1150 & 0.0937 & 0.5361 \end{bmatrix},$$

$$\mathcal{R}_{\mathcal{B}_3}(:,:,2) = \begin{bmatrix} 0.0309 & 0.0064 & 0.0633 & 0.2045 \\ 0.1302 & -0.0165 & -0.1512 & 0.0491 \\ -0.0703 & -0.0198 & -0.0356 & -0.0486 \\ -0.1300 & 0.2418 & 0.0985 & 0.0213 \end{bmatrix},$$

$$\mathcal{R}_{\mathcal{B}_3}(:,:,3) = \begin{bmatrix} 0.0901 & -0.1202 & 0.0226 & -0.0724 \\ -0.1202 & 0.0204 & -0.0199 & -0.0970 \\ 0.0226 & -0.0199 & -0.0476 & 0.2053 \\ -0.0724 & -0.0970 & 0.2053 & -0.0630 \end{bmatrix},$$

$$\mathcal{R}_{\mathcal{B}_3}(:,:,4) = \begin{bmatrix} 0.0309 & 0.1302 & -0.0703 & -0.1300 \\ 0.0064 & -0.0165 & -0.0198 & 0.2418 \\ 0.0633 & -0.1512 & -0.0356 & 0.0985 \\ 0.2045 & 0.0491 & -0.0486 & 0.0213 \end{bmatrix},$$

$$\mathcal{X}_0(:,:,1) = \begin{bmatrix} 2 & 5 & 6 & 2 \\ 5 & 3 & 7 & 6 \\ 7 & 3 & 9 & 13 \\ 13 & 5 & 7 & 8 \\ 8 & 7 & 2 & 12 \end{bmatrix}, \mathcal{X}_0(:,:,2) = \begin{bmatrix} 5 & 12 & 10 & 5 \\ 5 & 7 & 3 & 9 \\ 12 & 7 & 10 & 4 \\ 11 & 7 & 3 & 13 \\ 6 & 7 & 7 & 2 \end{bmatrix},$$

$$\mathcal{X}_0(:,:,3) = \begin{bmatrix} 4 & 13 & 2 & 12 \\ 9 & 5 & 5 & 8 \\ 5 & 4 & 4 & 8 \\ 3 & 9 & 10 & 8 \\ 4 & 2 & 10 & 12 \end{bmatrix}, \mathcal{X}_0(:,:,4) = \begin{bmatrix} 7 & 6 & 9 & 10 \\ 6 & 4 & 5 & 3 \\ 9 & 9 & 7 & 2 \\ 5 & 11 & 5 & 8 \\ 3 & 7 & 11 & 10 \end{bmatrix}.$$

Example 4.4. *For the system (16), let $\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1, \mathcal{A}_2, \mathcal{B}_2$ and $\mathcal{C}_2$ be given tensors, where $\mathcal{A}_1 \in \mathbb{R}^{2\times3\times3}$, $\mathcal{B}_1 \in \mathbb{R}^{3\times2\times3}$, $\mathcal{C}_1 \in \mathbb{R}^{2\times3\times3}$, $\mathcal{D}_1 \in \mathbb{R}^{3\times2\times3}$, $\mathcal{A}_2 \in \mathbb{R}^{3\times3\times3}$, $\mathcal{B}_2 \in \mathbb{R}^{3\times2\times3}$, $\mathcal{C}_2 \in \mathbb{R}^{3\times2\times3}$, and*

$$\mathcal{A}_1(:,:,1) = \begin{bmatrix} 8 & 8 & 9 \\ 8 & 5 & 9 \end{bmatrix}, \mathcal{A}_1(:,:,2) = \begin{bmatrix} 8 & 6 & 7 \\ 6 & 5 & 8 \end{bmatrix}, \mathcal{A}_1(:,:,3) = \begin{bmatrix} 8 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix},$$

$$\mathcal{B}_1(:,:,1) = \begin{bmatrix} 4 & 5 \\ 6 & 4 \\ 9 & 9 \end{bmatrix}, \mathcal{B}_1(:,:,2) = \begin{bmatrix} 7 & 8 \\ 5 & 7 \\ 6 & 6 \end{bmatrix}, \mathcal{B}_1(:,:,3) = \begin{bmatrix} 7 & 8 \\ 5 & 6 \\ 8 & 8 \end{bmatrix},$$

$$\mathcal{C}_1(:,:,1) = \begin{bmatrix} 77 & 94 & 73 \\ 66 & 77 & 90 \end{bmatrix}, \mathcal{C}_1(:,:,2) = \begin{bmatrix} 68 & 77 & 84 \\ 98 & 68 & 60 \end{bmatrix}, \mathcal{C}_1(:,:,3) = \begin{bmatrix} 91 & 52 & 63 \\ 93 & 80 & 87 \end{bmatrix},$$

$$\mathcal{D}_1(:,:,1) = \begin{bmatrix} 74 & 79 \\ 89 & 61 \\ 76 & 68 \end{bmatrix}, \mathcal{D}_1(:,:,2) = \begin{bmatrix} 64 & 81 \\ 92 & 94 \\ 98 & 99 \end{bmatrix}, \mathcal{D}_1(:,:,3) = \begin{bmatrix} 82 & 74 \\ 89 & 53 \\ 70 & 90 \end{bmatrix},$$

$$\mathcal{A}_2(:,:,1) = \begin{bmatrix} 8 & 3 & 7 \\ 8 & 6 & 4 \\ 4 & 6 & 5 \end{bmatrix}, \mathcal{A}_2(:,:,2) = \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 6 \\ 3 & 8 & 7 \end{bmatrix}, \mathcal{A}_2(:,:,3) = \begin{bmatrix} 8 & 8 & 5 \\ 5 & 8 & 4 \\ 8 & 3 & 6 \end{bmatrix},$$

$$\mathcal{B}_2(:,:,1) = \begin{bmatrix} 3 & 4 \\ 5 & 5 \\ 1 & 1 \end{bmatrix}, \mathcal{B}_2(:,:,2) = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ 3 & 6 \end{bmatrix}, \mathcal{B}_2(:,:,3) = \begin{bmatrix} 2 & 6 \\ 3 & 2 \\ 3 & 6 \end{bmatrix},$$

$$\mathcal{C}_2(:,:,1) = \begin{bmatrix} 170 & 197 \\ 270 & 171 \\ 299 & 110 \end{bmatrix}, \mathcal{C}_2(:,:,2) = \begin{bmatrix} 283 & 294 \\ 241 & 245 \\ 188 & 232 \end{bmatrix}, \mathcal{C}_2(:,:,3) = \begin{bmatrix} 203 & 194 \\ 276 & 162 \\ 112 & 139 \end{bmatrix}.$$

Note that $\mathcal{A}_1 * \mathcal{D}_1 \neq \mathcal{C}_1 * \mathcal{B}_1$. Hence, the system (16) has no solution.

5. Symmetric solution to the system (4)

Based on Theorem 4.1, we in this section consider the solvability conditions and general symmetric solution to the system

$$\begin{cases} \mathcal{A}_1 * \mathcal{X} = C_1, \\ \mathcal{A}_2 * \mathcal{X} * \mathcal{A}_2^T = C_2, \end{cases} \quad \mathcal{X} = \mathcal{X}^T, \tag{29}$$

where

$$\mathcal{A}_1 \in \mathbb{R}^{t_1\times l\times n_3\times\cdots\times n_p},\ C_1 \in \mathbb{R}^{t_1\times l\times n_3\times\cdots\times n_p},\ \mathcal{A}_2 \in \mathbb{R}^{t_2\times l\times n_3\times\cdots\times n_p},\ C_2 \in \mathbb{R}^{t_2\times t_2\times n_3\times\cdots\times n_p},$$

and $\mathcal{X} \in \mathbb{R}^{l\times l\times n_3\times\cdots\times n_p}$ is unknown.

Theorem 5.1. *Let $\mathcal{A}_1, C_1, \mathcal{A}_2$ and $C_2 = C_2^T$ be given tensors. Denote*

$$\mathcal{A}_3 = \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1},\ C_3 = C_2 - \mathcal{A}_2 * \mathcal{A}_1^\dagger * C_1 * (\mathcal{A}_2)^T - \mathcal{A}_2 * \mathcal{L}_{\mathcal{A}_1} * C_1^T * (\mathcal{A}_1^\dagger)^T * \mathcal{A}_2^T. \tag{30}$$

The system of tensor equations (29) has a symmetric solution if and only if

$$\mathcal{R}_{\mathcal{A}_1} * C_1 = 0,\ \mathcal{A}_1 * C_1^T = C_1 * \mathcal{A}_1^T,\ \mathcal{R}_{\mathcal{A}_3} * C_3 = 0. \tag{31}$$

In this case, the general symmetric solution to (29) can be expressed as

$$\begin{aligned} \mathcal{X} =& \frac{1}{2}[\mathcal{A}_1^\dagger * C_1 + C_1^T * (\mathcal{A}_1^\dagger)^T + \mathcal{L}_{\mathcal{A}_1} * C_1^T * (\mathcal{A}_1^\dagger)^T + \mathcal{A}_1^\dagger * C_1 * \mathcal{L}_{\mathcal{A}_1} + \\ & \mathcal{L}_{\mathcal{A}_1} * \mathcal{A}_3^\dagger * (C_3 + C_3^T) * (\mathcal{A}_3^\dagger)^T * \mathcal{L}_{\mathcal{A}_1}] + \mathcal{L}_{\mathcal{A}_1} * \mathcal{W} * \mathcal{L}_{\mathcal{A}_3} * \mathcal{L}_{\mathcal{A}_1} + \mathcal{L}_{\mathcal{A}_1} * \mathcal{L}_{\mathcal{A}_3} * \mathcal{W}^T * \mathcal{L}_{\mathcal{A}_1}, \end{aligned} \tag{32}$$

where $\mathcal{W}$ is an arbitrary tensor with suitable order.

Proof. The system (29) has a symmetric solution if and only if the system of tensor equations has a solution

$$\begin{cases} \mathcal{A}_1 * \mathcal{Y} = C_1, \\ \mathcal{Y} * \mathcal{A}_1^T = C_1^T, \\ \mathcal{A}_2 * \mathcal{Y} * \mathcal{A}_2^T = C_2. \end{cases} \tag{33}$$

The general symmetric solution to the system (29) can be expressed as $\frac{\mathcal{Y}+\mathcal{Y}^T}{2}$. We obtain the solvability conditions and the expression of general symmetric solution by Theorem 4.1. □

We present the algorithm for finding the general symmetric solution to the system (29).

Algorithm 5.2. *The general symmetric solution to the system (29).*
Input: *$\mathcal{A}_1, C_1, \mathcal{A}_2$ and C_2.*
(1) : *Compute $\mathcal{A}_3$ and C_3 by (30).*
(2) : *If any of equations in (31) fails, return "No solution".*
(3) : *Else compute the Moore-Penrose inverses $\mathcal{A}_1^\dagger$ and $\mathcal{A}_3^\dagger$.*
(4) : *Compute $\mathcal{L}_{\mathcal{A}_1}$ and $\mathcal{L}_{\mathcal{A}_3}$ by (7).*
Output: *$\mathcal{X}$ in the form (32).*

We provide two examples to illustrate Theorem 5.1.

Example 5.3. *For the system (29), let $\mathcal{A}_1, C_1, \mathcal{A}_2$ and C_2 be given tensors, where $\mathcal{A}_1 \in \mathbb{R}^{3\times4\times3}$, $C_1 \in \mathbb{R}^{3\times4\times3}$, $\mathcal{A}_2 \in \mathbb{R}^{2\times4\times3}$, $C_2 \in \mathbb{R}^{2\times2\times3}$,*
and

$$\mathcal{A}_1(:,:,1) = \begin{bmatrix} 2 & 3 & 7 & 2 \\ 2 & 7 & 1 & 9 \\ 8 & 2 & 7 & 6 \end{bmatrix},\ \mathcal{A}_1(:,:,2) = \begin{bmatrix} 3 & 5 & 5 & 5 \\ 4 & 1 & 2 & 1 \\ 9 & 6 & 7 & 9 \end{bmatrix},\ \mathcal{A}_1(:,:,3) = \begin{bmatrix} 8 & 9 & 5 & 8 \\ 1 & 4 & 5 & 9 \\ 6 & 7 & 1 & 5 \end{bmatrix},$$

$$C_1(:,:,1)=\begin{bmatrix}419 & 538 & 495 & 384\\ 302 & 325 & 359 & 234\\ 485 & 598 & 511 & 452\end{bmatrix},\ C_1(:,:,2)=\begin{bmatrix}384 & 471 & 441 & 417\\ 320 & 401 & 396 & 331\\ 476 & 537 & 499 & 387\end{bmatrix},$$

$$C_1(:,:,3)=\begin{bmatrix}428 & 451 & 480 & 327\\ 270 & 311 & 281 & 239\\ 464 & 568 & 591 & 449\end{bmatrix},\ \mathcal{A}_2(:,:,1)=\begin{bmatrix}6 & 4 & 8 & 2\\ 9 & 2 & 7 & 1\end{bmatrix},$$

$$\mathcal{A}_2(:,:,2)=\begin{bmatrix}2 & 9 & 2 & 6\\ 2 & 7 & 4 & 3\end{bmatrix},\ \mathcal{A}_2(:,:,3)=\begin{bmatrix}6 & 2 & 7 & 3\\ 2 & 2 & 3 & 5\end{bmatrix},$$

$$C_2(:,:,1)=\begin{bmatrix}22923 & 18820\\ 18820 & 15721\end{bmatrix},\ C_2(:,:,2)=\begin{bmatrix}23465 & 19221\\ 19258 & 15638\end{bmatrix},\ C_2(:,:,3)=\begin{bmatrix}23465 & 19258\\ 19221 & 15638\end{bmatrix}.$$

We get $\mathcal{L}_{\mathcal{A}_1}$, $\mathcal{L}_{\mathcal{A}_3}$ and the special solution

$$\begin{aligned}X_0=&\frac{1}{2}[\mathcal{A}_1^{\dagger}*C_1+C_1^T*(\mathcal{A}_1^{\dagger})^T+\mathcal{L}_{\mathcal{A}_1}*C_1^T*(\mathcal{A}_1^{\dagger})^T+\mathcal{A}_1^{\dagger}*C_1*\mathcal{L}_{\mathcal{A}_1}+\\ &\mathcal{L}_{\mathcal{A}_1}*\mathcal{A}_3^{\dagger}*(C_3+C_3^T)*(\mathcal{A}_3^{\dagger})^T*\mathcal{L}_{\mathcal{A}_1}],\end{aligned}\tag{34}$$

by Algorithm 5.2. The results are as follows,

$$\mathcal{L}_{\mathcal{A}_1}(:,:,1)=\begin{bmatrix}0.0404 & -0.0136 & -0.0610 & 0.0249\\ -0.0136 & 0.4721 & 0.0874 & -0.2675\\ -0.0610 & 0.0874 & 0.3224 & -0.1116\\ 0.0249 & -0.2675 & -0.1116 & 0.1651\end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_1}(:,:,2)=\begin{bmatrix}-0.0090 & -0.0123 & -0.0801 & 0.0183\\ 0.1349 & 0.0293 & -0.2387 & 0.0493\\ 0.0486 & -0.2989 & 0.0297 & 0.1624\\ -0.0813 & 0.0323 & 0.1068 & -0.0500\end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_1}(:,:,3)=\begin{bmatrix}-0.0090 & 0.1349 & 0.0486 & -0.0813\\ -0.0123 & 0.0293 & -0.2989 & 0.0323\\ -0.0801 & -0.2387 & 0.0297 & 0.1068\\ 0.0183 & 0.0493 & 0.1624 & -0.0500\end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_3}(:,:,1)=\begin{bmatrix}0.9596 & 0.0136 & 0.0610 & -0.0249\\ 0.0136 & 0.5279 & -0.0874 & 0.2675\\ 0.0610 & -0.0874 & 0.6776 & 0.1116\\ -0.0249 & 0.2675 & 0.1116 & 0.8349\end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_3}(:,:,2)=\begin{bmatrix}0.0090 & 0.0123 & 0.0801 & -0.0183\\ -0.1349 & -0.0293 & 0.2387 & -0.0493\\ -0.0486 & 0.2989 & -0.0297 & -0.1624\\ 0.0813 & -0.0323 & -0.1068 & 0.0500\end{bmatrix},$$

$$\mathcal{L}_{\mathcal{A}_3}(:,:,3) = \begin{bmatrix} 0.0090 & -0.1349 & -0.0486 & 0.0813 \\ 0.0123 & -0.0293 & 0.2989 & -0.0323 \\ 0.0801 & 0.2387 & -0.0297 & -0.1068 \\ -0.0183 & -0.0493 & -0.1624 & 0.0500 \end{bmatrix},$$

$$\mathcal{X}_0(:,:,1) = \begin{bmatrix} 10 & 7 & 7 & 5 \\ 7 & 7 & 9 & 1 \\ 7 & 9 & 6 & 2 \\ 5 & 1 & 2 & 0 \end{bmatrix}, \mathcal{X}_0(:,:,2) = \begin{bmatrix} 5 & 9 & 6 & 3 \\ 10 & 12 & 8 & 5 \\ 4 & 3 & 12 & 9 \\ 8 & 11 & 12 & 8 \end{bmatrix}, \mathcal{X}_0(:,:,3) = \begin{bmatrix} 5 & 10 & 4 & 8 \\ 9 & 12 & 3 & 11 \\ 6 & 8 & 12 & 12 \\ 3 & 5 & 9 & 8 \end{bmatrix}.$$

Example 5.4. *For the system (29), let $\mathcal{A}_1, C_1, \mathcal{A}_2$ and C_2 be given tensors, where $\mathcal{A}_1 \in \mathbb{R}^{3\times4\times3}$, $C_1 \in \mathbb{R}^{3\times4\times3}$, $\mathcal{A}_2 \in \mathbb{R}^{2\times4\times3}$, $C_2 \in \mathbb{R}^{2\times2\times3}$, and*

$$\mathcal{A}_1(:,:,1) = \begin{bmatrix} 8 & 5 & 5 & 1 \\ 2 & 4 & 3 & 6 \\ 3 & 1 & 6 & 4 \end{bmatrix}, \mathcal{A}_1(:,:,2) = \begin{bmatrix} 6 & 4 & 2 & 1 \\ 3 & 5 & 3 & 2 \\ 3 & 2 & 1 & 7 \end{bmatrix}, \mathcal{A}_1(:,:,3) = \begin{bmatrix} 5 & 6 & 3 & 7 \\ 5 & 7 & 9 & 1 \\ 9 & 4 & 8 & 9 \end{bmatrix},$$

$$C_1(:,:,1) = \begin{bmatrix} 5 & 3 & 9 & 9 \\ 2 & 4 & 8 & 7 \\ 3 & 5 & 3 & 8 \end{bmatrix}, C_1(:,:,2) = \begin{bmatrix} 5 & 7 & 3 & 1 \\ 1 & 1 & 9 & 8 \\ 9 & 9 & 2 & 8 \end{bmatrix}, C_1(:,:,3) = \begin{bmatrix} 4 & 4 & 6 & 3 \\ 1 & 9 & 9 & 1 \\ 6 & 8 & 2 & 1 \end{bmatrix},$$

$$\mathcal{A}_2(:,:,1) = \begin{bmatrix} 6 & 5 & 1 & 4 \\ 9 & 3 & 4 & 8 \end{bmatrix}, \mathcal{A}_2(:,:,2) = \begin{bmatrix} 9 & 4 & 4 & 1 \\ 1 & 9 & 4 & 6 \end{bmatrix}, \mathcal{A}_2(:,:,3) = \begin{bmatrix} 2 & 9 & 3 & 3 \\ 4 & 7 & 7 & 4 \end{bmatrix},$$

$$C_2(:,:,1) = \begin{bmatrix} 6 & 7 \\ 3 & 6 \end{bmatrix}, C_2(:,:,2) = \begin{bmatrix} 2 & 2 \\ 3 & 6 \end{bmatrix}, C_2(:,:,3) = \begin{bmatrix} 2 & 1 \\ 8 & 8 \end{bmatrix}.$$

Since $\mathcal{A}_1 * C_1^T \neq C_1 * \mathcal{A}_1^T$, the system has no solution.

6. Conclusion

We have established some practical necessary and sufficient conditions for the existence of a solution to the system of tensor equations (2) under t-product. We have presented the general solution to the system (2) in terms of Moore-Penrose inverses. As an application of the system (2), we have given some necessary and sufficient conditions for the existence of a symmetric solution to the system of tensor equations (4). We have also provided the general symmetric solution to the system (4) when its solvability conditions are satisfied. Moreover, we have given some algorithms and numerical examples.

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