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Upper Bound of Hankel Determinant for a Class of Analytic Functions

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Abstract. The aim of this study is to solve the Fekete-Szegö problem and to define upper bound for Hankel determinant $H_2(1)$ in a novel class $\mathcal{K}$ of analytical functions in the unit disc. Moreover, in a class of analytic functions on the unit disc, assuming the existence of an angular limit on the boundary point, the estimations from below of the modulus of angular derivative have been obtained.

1. Introduction

Let U be the unit disc in the complex plane $\mathbb{C}$. *Schwarz's Lemma*, which is a consequence of the Maximum Principle, says that if $g : U \to U$ is analytic with $g(z) = c_p z^p + ...$ then $\left|g(z)\right| \leq |z|^p$ for all $z \in U$ and $\left|c_p\right| \leq 1$. In addition, if the equality $\left|g(z)\right| = |z|^p$ holds for any $z \neq 0$, or $\left|c_p\right| = 1$, then g is a rotation; that is $g(z) = z^p e^{i\theta}$, θ real ([5], p.329). Schwarz lemma has several applications in the field of electrical and electronics engineering. Usage of positive real function and boundary analysis of these functions for circuit synthesis can be given as an exemplary application of the Schwarz lemma in electrical engineering. Furthermore, it is also used for analysis of transfer functions in control engineering and multi-notch filter design in signal processing [14–16].

In order to derive our main results, we will resort to the following lemma [6].

Lemma 1.1 (Jack's lemma). *Let $g(z)$ be a non-constant anaytic function in U with $g(0) = 0$. If*

$$\left|g(z_0)\right| = \max\left\{\left|g(z)\right| : |z| \leq |z_0|\right\},$$

then there exists a real number $k \geq 1$ such that

$$\frac{z_0 g'(z_0)}{g(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions $f(z) = z + c_2 z^2 + c_3 z^3 + ...$ that are analytic in U. Also, let $\mathcal{K}$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$\left|\frac{(zf(z))''}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right| < \frac{1}{3},\ z \in U. \tag{1.1}$$

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The certain anaytic functions which are in the class of $\mathcal{K}$ on the unit disc U are considered in this paper. The subject of the present paper is to discuss some properties of the function $f(z)$ which belongs to the class of $\mathcal{K}$ by applying Jack's Lemma.

In this study, we will solve the Fekete-Szegö problem where we define an upper bound for the Hankel determinant $H_2(1)$ for the class $\mathcal{K}$ of analytic function $f \in \mathcal{A}$ will satisfy the condition (1.1). In addition, the relationship between the coefficients of the Hankel determinant and the angular derivative of the function f, which provides the class $\mathcal{K}$, will be examined. In this examination, the coefficients c_2, c_3 and c_4 will be used. Let $f \in \mathcal{A}$. The q^{th} Hankel determinant of f for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas [21] as

$$H_q(n) = \begin{vmatrix} c_n & c_{n+1} & \cdots & c_{n+q-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+q-1} & c_{n+q} & \cdots & c_{n+2q-2} \end{vmatrix}, \ c_1 = 1.$$

From the Hankel determinant for $n = 1$ and $q = 2$, we have

$$H_2(1) = \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} = c_3 - c_2^2.$$

Here, the Hankel determinant $H_2(1) = c_3 - c_2^2$ is well-known as Fekete-Szegö functional [20]. In [21], the authors have obtained the upper bounds for the Hankel determinant $\left|c_2 c_4 - c_3^2\right|$. Also, in [18], the author have obtained the upper bounds for the Hankel determinant $A_n^{(k)}$. Moreover, in [19], the authors have given bounds for the Second Hankel determinant for class $\mathcal{M}_\alpha$.

Let $f \in \mathcal{K}$ and consider the following function

$$\Upsilon(z) = 2\left[\frac{z^2 f'(z)}{(f(z))^2} - 1\right]. \tag{1.2}$$

It is an analytic function in U and $\Upsilon(0) = 0$. Now, let us show that $|\Upsilon(z)| < 1$ in U. From (1.2), we have

$$\frac{z^2 f'(z)}{(f(z))^2} = 1 + \frac{1}{2}\Upsilon(z)$$

If the logarithmic differentiation of both sides is taken in the last equation, we obtain

$$\ln\left(\frac{z^2 f'(z)}{(f(z))^2}\right) = \ln\left(1 + \frac{1}{2}\Upsilon(z)\right),$$

$$2 + \frac{z f''(z)}{f'(z)} - 2\frac{z f'(z)}{f(z)} = \frac{\frac{1}{2}\Upsilon'(z)}{1 + \frac{1}{2}\Upsilon(z)}$$

and

$$\frac{(z f(z))''}{f'(z)} - 2\frac{z f'(z)}{f(z)} = \frac{z\Upsilon'(z)}{2 + \Upsilon(z)}.$$

We suppose that there exists a $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |\Upsilon(z)| = |\Upsilon(z_0)| = 1.$$

From Jack's lemma, we obtain

$$\Upsilon(z_0) = e^{i\theta} \ \text{ and } \ \frac{z_0' \Upsilon(z_0)}{\Upsilon(z_0)} = k.$$

Thus, we have that

$$\left|\frac{(z_0f(z_0))''}{f'(z_0)}-2\frac{z_0f'(z_0)}{f(z_0)}\right|=\left|\frac{z_0\Upsilon'(z_0)}{2+\Upsilon(z_0)}\right|=\left|\frac{k\Upsilon(z_0)}{2+\Upsilon(z_0)}\right|=\frac{k\left|e^{i\theta}\right|}{\left|2+e^{i\theta}\right|}$$

Since $\left|2+e^{i\theta}\right|\leq 3$ and $k\geq 1$, we take

$$\left|\frac{(z_0f(z_0))''}{f'(z_0)}-2\frac{z_0f'(z_0)}{f(z_0)}\right|\geq\frac{1}{3}.$$

This contradicts $f\in\mathcal{K}$. This means that there is no point $z_0\in U$ such that $\max_{|z|\leq|z_0|}|\Upsilon(z)|=|\Upsilon(z_0)|=1$. Hence, we take $|\Upsilon(z)|<1$ in U. From the Schwarz lemma, we obtain

$$\begin{aligned}\Upsilon(z) &= 2\left[\frac{z^2f'(z)}{(f(z))^2}-1\right]\\ &= 2\left[\left(c_3-c_2^2\right)z^2+\left(2c_4-4c_2c_3+2c_2^3\right)z^3+\ldots\right],\end{aligned}$$

$$\frac{\Upsilon(z)}{z^2}=2\left[\left(c_3-c_2^2\right)+\left(2c_4-4c_2c_3+2c_2^3\right)z+\ldots\right],$$

$$2\left|c_3-c_2^2\right|=2\left|H_2(1)\right|\leq 1$$

and

$$|H_2(1)|\leq\frac{1}{2}.$$

We thus obtain the following lemma.

Lemma 1.2. *If $f\in\mathcal{K}$, then we have the inequality*

$$|H_2(1)|\leq\frac{1}{2}. \tag{1.3}$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which are called the boundary version of Schwarz Lemma, are about estimating from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows [13]:

Lemma 1.3. *If $g:U\rightarrow U$ is analytic with $g(z)=c_pz^p+\ldots..$, g extends continuously to some boundary point c with $|c|=1$, and if $\left|g(c)\right|=1$ and $g'(c)$ exists, then we have*

$$\left|g'(c)\right|\geq p+\frac{1-\left|c_p\right|}{1+\left|c_p\right|} \tag{1.4}$$

and

$$\left|g'(c)\right|\geq p. \tag{1.5}$$

Inequality (1.5) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1–4, 7–13]. Mercer has considered some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [11]. In addition, he has obtained a new boundary Schwarz lemma , for analytic functions mapping the unit disk to itself [12].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [17]).

Lemma 1.4 (Julia-Wolff lemma). *Let g be an analytic function in U, $g(0)=0$ and $g(U)\subset U$. If, in addition, the function g has an angular limit $g(c)$ at $c\in\partial U$, $\left|g(c)\right|=1$, then the angular derivative $g'(c)$ exists and $1\leq\left|g'(c)\right|\leq\infty$.*

Corollary 1.5. *The analytic function g has a finite angular derivative $g'(c)$ if and only if g' has the finite angular limit $g'(c)$ at $c\in\partial U$.*

2. Main Results

In this section, we will discuss different versions of the boundary Schwarz lemma and the Hankel determinant for the class $\mathcal{K}$. Assuming the existence of angular limit on a boundary point, we will obtain some estimations from below for the moduli of derivatives of analytic functions from a certain class. In the inequalities obtained, the relationship between the Hankel determinant and the second angular derivative of the function $f(z)$ will be established.

Theorem 2.1. *Let $f \in \mathcal{K}$. Assume that, for some $c \in \partial U$, f has an angular limit $f(c)$ at c, $f(c) = \frac{2c}{3}$ and $f'(c) = \frac{2}{3}$. Then we have the inequality*

$$\left|f''(c)\right| \geq \frac{8}{9}\left|H_2(1)\right|. \tag{2.1}$$

Proof. Let $\Upsilon(z)$ function be the same as (1.2). In addition, since $f(c) = \frac{2c}{3}$ and $f'(c) = \frac{2}{3}$, we have

$$\Upsilon(c) = 2\left[\left(\frac{c}{f(c)}\right)^2 f'(c) - 1\right] = 2\left[\left(\frac{c}{\frac{2c}{3}}\right)^2 \frac{2}{3} - 1\right] = 1$$

and

$$|\Upsilon(c)| = 1.$$

So, from (1.5) for $p = 2$, we obtain

$$\begin{aligned}
2 \quad \leq \quad |\Upsilon'(c)| &= 2\left|\frac{\left(2cf'(c) + f''(c)c^2\right)(f(c))^2 - 2f(c)f'(c)c^2 f'(c)}{(f(c))^4}\right| \\
&= 2\left|\frac{2cf'(c)}{(f(c))^2} + \frac{f''(c)c^2}{(f(c))^2} - \frac{2c^2\left(f'(c)\right)^2}{(f(c))^3}\right| \\
&= 2\left|\frac{2c\frac{2}{3}}{\left(\frac{2c}{3}\right)^2} + \frac{f''(c)c^2}{\left(\frac{2c}{3}\right)^2} - \frac{2c^2\left(\frac{2c}{3}\right)^2}{\left(\frac{2c}{3}\right)^3}\right| \\
&= \frac{9}{2}\left|f''(c)\right|
\end{aligned}$$

and

$$\left|f''(c)\right| \geq \frac{4}{9}.$$

Moreover, from (1.3), since $|H_2(1)| \leq \frac{1}{2}$, we take

$$\left|f''(c)\right| \geq \frac{8}{9}\left|H_2(1)\right|.$$

□

The inequality (2.1) can be strengthened as below by taking into account c_2 and c_3 which are the second and third coefficients in the expansion of the function $f(z) = z + c_2z^2 + c_3z^3 + ...$

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|f''(c)\right| \geq \frac{4}{9}\left|H_2(1)\right|\left(1 + \frac{2}{1 + 2\left|H_2(1)\right|}\right). \tag{2.2}$$

Proof. Let $\Upsilon(z)$ be the same as (1.2). So, from (1.4) for $p = 2$, we obtain

$$2+\frac{1-|d_2|}{1+|d_2|}\leq|\Upsilon'(c)|=\frac{9}{2}\left|f''(c)\right|,$$

where $|d_2| = \frac{|\Upsilon''(0)|}{2!} = 2\left|c_3-c_2^2\right| = 2\,|H_2(1)|$.

Therefore, we take

$$2+\frac{1-2\,|H_2(1)|}{1+2\,|H_2(1)|}\leq\frac{9}{2}\left|f''(c)\right|$$

and

$$1+\frac{2}{1+2\,|H_2(1)|}\leq\frac{9}{2}\left|f''(c)\right|.$$

Moreover, from (1.3), since $|H_2(1)| \leq \frac{1}{2}$, we take

$$\left|f''(c)\right|\geq\frac{4}{9}\,|H_2(1)|\left(1+\frac{2}{1+2\,|H_2(1)|}\right).$$

□

In the following theorem, inequality (2.2) has been strenghened by adding the consecutive term c_4 of the function $f(z)$.

Theorem 2.3. *Let $f \in \mathcal{K}$. Assume that, for some $c \in \partial U$, f has an angular limit $f(c)$ at c, $f(c) = \frac{2c}{3}$ and $f'(c) = \frac{2}{3}$. Then we have the inequality*

$$\left|f''(c)\right|\geq\frac{8}{9}\,|H_2(1)|\left(1+\frac{(1-2\,|H_2(1)|)^2}{1-4\,|H_2(1)|^2+4\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}\right). \tag{2.3}$$

Proof. Let $\Upsilon(z)$ be the same as in the proof of Theorem 2.1 and $\vartheta(z) = z^2$. By the maximum principle, for each $z \in U$, we have the inequality $|\Upsilon(z)| \leq |\vartheta(z)|$. Therefore

$$\begin{aligned}\mu(z) &= \frac{\Upsilon(z)}{\vartheta(z)}=\frac{2\left[\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right]}{z^2}\\ &= 2\left[\left(c_3-c_2^2\right)+\left(2c_4-4c_2c_3+2c_2^3\right)z+\ldots\right]\end{aligned}$$

is analytic function in U and $\left|\mu(z)\right| \leq 1$ for $|z| < 1$. In particular, we have

$$\left|\mu(0)\right|=2\left|c_3-c_2^2\right|=2\,|H_2(1)| \tag{2.4}$$

and

$$\left|\mu'(0)\right|=2\left|2c_4-4c_2c_3+2c_2^3\right|=4\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|.$$

Furthermore, the geometric meanings of the derivative and the inequality $|\Upsilon(z)| \leq |\vartheta(z)|$ imply the inequality

$$\frac{c\Upsilon'(c)}{\Upsilon(c)}=|\Upsilon'(c)|\geq|\vartheta'(c)|=\frac{c\vartheta'(c)}{\vartheta(c)}.$$

The composite function

$$r(z)=\frac{\mu(z)-\mu(0)}{1-\overline{\mu(0)}\mu(z)}$$

is analytic in U, $r(0)=0$, $|r(z)|<1$ for $|z|<1$ and $|r(c)|=1$ for $c\in\partial U$. For $p=1$, from (1.4), we obtain

$$\begin{aligned}\frac{2}{1+|r'(0)|} &\leq |r'(c)|=\frac{1-\left|\mu(0)\right|^2}{\left|1-\overline{\mu(0)}\mu(c)\right|^2}\left|\mu'(c)\right| \\ &\leq \frac{1+\left|\mu(0)\right|}{1-\left|\mu(0)\right|}\{|\Upsilon'(c)|-|\vartheta'(c)|\} \\ &= \frac{1+2\left|H_2(1)\right|}{1-2\left|H_2(1)\right|}\left(\frac{9}{2}\left|f''(c)\right|-2\right).\end{aligned}$$

Since

$$r'(z)=\frac{1-\left|\mu(0)\right|^2}{\left(1-\overline{\mu(0)}\mu(z)\right)^2}\mu'(z)$$

and

$$|r'(0)|=\frac{\left|\mu'(0)\right|}{1-\left|\mu(0)\right|^2}=\frac{4\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{1-\left(2\left|H_2(1)\right|\right)^2}$$

we obtain

$$\frac{2}{1+\frac{4\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}{1-4\left|H_2(1)\right|^2}}\leq\frac{1+2\left|H_2(1)\right|}{1-2\left|H_2(1)\right|}\left(\frac{9}{2}\left|f''(c)\right|-2\right)$$

and

$$\left|f''(c)\right|\geq\frac{4}{9}\left(1+\frac{\left(1-2\left|H_2(1)\right|\right)^2}{1-4\left|H_2(1)\right|^2+4\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}\right).$$

Since $|H_2(1)|\leq\frac{1}{2}$, we obtain the inequality (2.3). □

If $f(z)-z$ has no zeros different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.4. *Let $f(z)\in\mathcal{K}$ and $c_3>c_2^2$ ($c_2>0, c_3>0$). Also, $f(z)-z$ has no zeros in U except $z=0$. Further assume that, for some $c\in\partial U$, f has an angular limit $f(c)$ at c, $f(c)=\frac{2c}{3}$ and $f'(c)=\frac{2}{3}$. Then we have the inequality*

$$\left|f''(c)\right|\geq\frac{4}{9}\left(1-\frac{1}{2}\frac{H_2(1)\ln\left(2H_2(1)\right)}{H_2(1)\ln\left(2H_2(1)\right)-\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|}\right) \tag{2.5}$$

and

$$\left|c_4-c_2\left(c_2^2+2H_2(1)\right)\right|\leq\left|H_2(1)\ln\left(2H_2(1)\right)\right|. \tag{2.6}$$

Proof. Let $c_3>c_2^2$ and $\Upsilon(z)$, $\mu(z)$ be as in the proof of Theorem 2.3. Having in mind inequality (2.4), we denote by $\ln\mu(z)$ the analytic branch of the logarithm normed by the condition

$$\ln\mu(0)=\ln\left(2\left(c_3-c_2^2\right)\right)=\ln 2H_2(1)<0.$$

The function

$$m(z) = \frac{\ln \mu(z) - \ln \mu(0)}{\ln \mu(z) + \ln \mu(0)}$$

is analytic in the unit disc U, $|m(z)| < 1$ for $z \in U$, $m(0) = 0$ and $|m(c)| = 1$ for $c \in \partial U$. From (1.4) for $p = 1$, we obtain

$$\begin{aligned} \frac{2}{1 + |m'(0)|} &\leq |m'(c)| = \frac{\left|2 \ln \mu(0)\right|}{\left|\ln \mu(c) + \ln \mu(0)\right|^2} \left|\frac{\mu'(c)}{\mu(c)}\right| \\ &= \frac{-2 \ln \mu(0)}{\ln^2 \mu(0) + \arg^2 \mu(c)} \left\{|\Upsilon'(c)| - |\vartheta'(c)|\right\}. \end{aligned}$$

Since

$$\begin{aligned} |m'(0)| &= \frac{1}{\left|2 \ln \mu(0)\right|} \left|\frac{\mu'(0)}{\mu(0)}\right| = \frac{-1}{2 \ln (2H_2(1))} \frac{4\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{2\,|H_2(1)|} \\ &= \frac{-1}{2 \ln (2H_2(1))} \frac{4\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{2H_2(1)} \\ &= \frac{-1}{\ln (2H_2(1))} \frac{\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{H_2(1)}, \end{aligned}$$

we take

$$\frac{1}{1 - \frac{\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{H_2(1) \ln (2H_2(1))}} \leq \frac{-\ln (2\,|H_2(1)|)}{\ln^2 (2\,|H_2(1)|) + \arg^2 \mu(c)} \left(\frac{9}{2}\left|f''(c)\right| - 2\right).$$

Replacing $\arg^2 \mu(c)$ by zero, we take

$$\frac{1}{1 - \frac{\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{H_2(1) \ln (2H_2(1))}} \leq \frac{-1}{\ln (2\,|H_2(1)|)} \left(\frac{9}{2}\left|f''(c)\right| - 2\right),$$

$$2 - \frac{H_2(1) \ln (2H_2(1))}{H_2(1) \ln (2H_2(1)) - \left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|} \leq \frac{9}{2}\left|f''(c)\right|$$

and

$$\left|f''(c)\right| \geq \frac{4}{9}\left(1 - \frac{1}{2} \frac{H_2(1) \ln (2H_2(1))}{H_2(1) \ln (2H_2(1)) - \left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}\right).$$

Similarly, the function $m(z)$ satisfies the assumptions of the Schwarz lemma, we obtain

$$\begin{aligned} 1 &\geq |m'(0)| = \frac{\left|2 \ln \mu(0)\right|}{\left|\ln \mu(0) + \ln \mu(0)\right|^2} \left|\frac{\mu'(0)}{\mu(0)}\right| = \frac{-1}{2 \ln \mu(0)} \left|\frac{\mu'(0)}{\mu(0)}\right| \\ &= \frac{-1}{2 \ln (2H_2(1))} \frac{4\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{2\,|H_2(1)|} \\ &= \frac{-1}{\ln (2H_2(1))} \frac{\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right|}{|H_2(1)|} \end{aligned}$$

and

$$\left|c_4 - c_2\left(c_2^2 + 2H_2(1)\right)\right| \leq |H_2(1)\ln(2H_2(1))|.$$

□

Theorem 2.5. *Under hypotheses of Theorem 2.4, we have*

$$\left|f''(c)\right| \geq \frac{4}{9}\left(2 - \frac{1}{4}\ln(2H_2(1))\right). \tag{2.7}$$

Proof. From the proof of Theorem 2.4, using the inequality (1.5) for the function $g(z)$, for $p = 1$ we obtain

$$1 \leq |m'(c)| = \frac{\left|2\ln\mu(0)\right|}{\left|\ln\mu(c) + \ln\mu(0)\right|^2}\left|\frac{\mu'(c)}{\mu(c)}\right| = \frac{-2}{\ln(2H_2(1))}\left(\frac{9}{2}\left|f''(c)\right| - 2\right)$$

and

$$\left|f''(c)\right| \geq \frac{4}{9}\left(2 - \frac{1}{4}\ln(2H_2(1))\right).$$

□

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