Filomat 35:11 (2021), 3745–3758
https://doi.org/10.2298/FIL2111745L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Selective Separability in (a)Topological Spaces

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Abstract. In this paper, we study selective versions of separability in (a)topological spaces with the help of some strong and weak forms of open sets. For this we use the notions of semi-closure, pre-closure, α-closure, β-closure and δ-closure and their respective density in (a)topological spaces. The interrelationships between different types of selective versions of separability in (a)topological spaces have been given by suitable counterexamples. Sufficient conditions are given for (a)topological spaces to be $(a)R^t$-separable and $(a)M^t$-separable for each $t \in \{s, p, \alpha, \beta, \delta\}$. It is shown that under some condition $(a)M^t$-separability and $(a)R^t$-separability are equivalent.

1. Introduction

At first we recall the two classical selection principles in topological spaces which have been systematically investigated in the last two decades, though their roots go back to 1920s and 1930s. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:
$S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

For each sequence $< A_n : n \in \mathbb{N} >$ of elements of $\mathcal{A}$, there is a sequence $< B_n : n \in \mathbb{N} >$ of finite sets such that for each n, $B_n \subseteq A_n$ and $\cup_{n\in\mathbb{N}} B_n \in \mathcal{B}$ (see [24, 26]).
$S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

For each sequence $< A_n : n \in \mathbb{N} >$ of elements of $\mathcal{A}$, there is a sequence $< a_n : n \in \mathbb{N} >$ such that for each n, $a_n \in A_n$ and $\{a_n : n \in \mathbb{N}\} \in \mathcal{B}$ (see [24, 26]).

In recent years, selection principles in topological spaces have been studied much in the literature. Several papers on selective version of separability have been published in the last few years. Further, several weak variant of selection principles in topological spaces have appeared in the literature and studied in detail by a number of authors. Also, there are some recent papers which deals with selection principles in bitopological spaces [25, 26, 28, 31, 33, 35, 36]. In selection principles theory, authors study mainly in two directions: (1). the closure operator is used in the definition of selection principles [2, 3, 7, 14–16, 21–23, 34, 40] and (2). sequences of open covers are replaced by sequences of covers by some weak form of open sets [27, 30, 38, 39]. In this paper, we study selective separability in first direction by using semi-closure, pre-closure, α-closure, β-closure and δ-closure in (a)topological spaces which is more general than bitopological spaces [20], (ω)topological spaces [9–11] and $(\aleph_0)$topological spaces [8]. Let D denote the

2020 *Mathematics Subject Classification*. Primary 54D65; Secondary 54A10, 54A05, 54E55
Keywords. Selection principles, (a)topological spaces, selective separability, groupable
Received: 15 August 2020; Revised: 25 October 2020; Accepted: 30 October 2020
Communicated by Ljubiša D.R. Kočinac
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family of all dense subsets of a topological space X. Selection principles in the context of separability, that is, $S_{fin}(D,D)$ and $S_1(D,D)$ in topological spaces have been introduced and studied in [42]. These principles appeared in a natural way in a study of hyperspace topologies in [12, 17]. The selection properties $S_{fin}(D,D^{gp})$ and $S_1(D,D^{gp})$ were introduced in [17], where D^{gp} is the family of all groupable dense subsets of a topological space X. Later on, these four selection properties have been studied by a several authors [1, 4–6, 19, 37] in a systematic way. The selection properties $S_{fin}(D,D)$, $S_1(D,D)$, $S_{fin}(D,D^{gp})$ and $S_1(D,D^{gp})$ are called M-separability, R-separability, GN-separability and H-separability (in a little bit modified form), respectively (see [5]). Study of selection principles in bitopological spaces began with [28] and continued in [29]. In the papers [28, 29, 32], variations of selective separability and tightness in bitopological function space $(C(X),\tau_k,\tau_p)$ were investigated and in [31], Lyakhovets et al. continue to study the selective properties and the corresponding topological games in $(C(X),\tau_k,\tau_p)$. In [33], S. Özçağ studied selective versions of separability in bitopological spaces by using the notions of θ-closure and θ-density. In this paper, we introduce and study the notion of selective version of separability in (a)topological spaces using the notion of semi-closure, pre-closure, α-closure, β-closure and δ-closure and their respective density.

This paper is organized as follows: In Section 2, we introduce and study various weak forms of open set in (a)topological spaces. We give counterexamples that show the interrelations between them. In Section 3, we study δ-open sets in (a)topological spaces. In Section 4, we discuss various selection properties by using notion of semi-closure, pre-closure, α-closure, β-closure and δ-closure and their respective density in (a)topological spaces and provide interrelationships between them. In Section 5 and 6, we study R^t-separability, M^t-separability, H^t-separability and GN^t-separability in (a)topological spaces for each $t \in \{s,p,\alpha,\beta,\delta\}$. Sufficient conditions are given for (a)topological spaces to be (a)R^t-separable and (a)M^t-separable for each $t \in \{s,p,\alpha,\beta,\delta\}$. It is shown that under some condition (a)M^t-separability and (a)R^t-separability are equivalent.

Throughout the paper, $(X,\{\tau_n\})$ denote an (a)topological space on which no separation axioms are assumed unless explicitly stated. If there is no scope of confusion, we denote (a)topological space $(X,\{\tau_n\})$ by X. For a subset A of an (a)topological space X, (τ_n)interior (resp. (τ_n)closure) of A denoted by τ_n-int(A) (resp. τ_n-cl(A)). $\mathbb{N}$ denotes the set of natural numbers and $k,l,m,n,m_0,n_0 \in \mathbb{N}$. For general notion of topology, we follow [18]. For other basic notions regarding selection principles, one can see [24, 26, 41, 43].

2. Weak Forms of Open Sets in (a)Topological Spaces

Definition 2.1. ([13]) If $< \tau_n : n \in \mathbb{N} >$ is a sequence of topologies on a set X, then the pair $(X,\{\tau_n\})$ is called an (a)topological space (in short, (a)space).

Definition 2.2. A subset A of $(X,\{\tau_n\})$ is said to be:

1. ([44]) (m,n)-semi-open ((m,n)-s-open) if $A \subseteq \tau_m$-cl$(\tau_n$-int$(A))$, or equivalently, if there exists a τ_n-open set U such that $U \subseteq A \subseteq \tau_m$-cl$(U)$.

2. (m,n)-pre-open ((m,n)-p-open) if $A \subseteq \tau_n$-int$(\tau_m$-cl$(A))$, or equivalently, if there exists a τ_n-open set U such that $A \subseteq U \subseteq \tau_m$-cl$(A)$.

3. (m,n)-α-open if $A \subseteq \tau_n$-int$(\tau_m$-cl$(\tau_n$-int$(A)))$, or equivalently, if there exists a τ_n-open set U such that $U \subseteq A \subseteq \tau_n$-int$(\tau_m$-cl$(U))$.

4. (m,n)-β-open if $A \subseteq \tau_m$-cl$(\tau_n$-int$(\tau_m$-cl$(A)))$.

It is clear that every τ_n-open set of X is (m,n)-α-open and hence, (m,n)-pre-open, (m,n)-semi-open and (m,n)-β-open for all $m \in \mathbb{N}$.

Evidently, we have

$$\begin{array}{ccccc} (m,n)\text{-}\alpha\text{-open} & \Rightarrow & (m,n)\text{-pre-open} & \Rightarrow & (m,n)\text{-}\beta\text{-open} \\ \Uparrow & & & & \Uparrow \\ \tau_n\text{-open} & \Rightarrow & (m,n)\text{-}\alpha\text{-open} & \Rightarrow & (m,n)\text{-semi-open} \end{array}$$

Diagram 1

The collection of all (m, n)-semi-open (resp. (m, n)-pre-open, (m, n)-α-open and (m, n)-β-open) sets is closed under arbitrary union. However, finite intersection of (m, n)-semi-open (resp. (m, n)-pre-open, (m, n)-α-open and (m, n)-β-open) sets need not be (m, n)-semi-open (resp. (m, n)-pre-open, (m, n)-α-open and (m, n)-β-open) (see Example 2.3). The complement of (m, n)-semi-open set, (m, n)-pre-open set, (m, n)-α-open set and (m, n)-β-open set is (m, n)-semi-closed, (m, n)-pre-closed, (m, n)-α-closed and (m, n)-β-closed, respectively. For each $t \in \{s, p, \alpha, \beta\}$, the (m, n)-t-interior of A denoted by $\tau_{(m,n)}$-int$_t(A)$ is the union of all (m, n)-t-open sets contained in A and the (m, n)-t-closure of A denoted by $\tau_{(m,n)}$-cl$_t(A)$ is the intersection of all (m, n)-t-closed sets containing A. By $S_{(m,n)}X$, $P_{(m,n)}X$, $\alpha_{(m,n)}X$ and $\beta_{(m,n)}X$, we denote the family of all (m, n)-semi-open sets, (m, n)-pre-open sets, (m, n)-α-open sets and (m, n)-β-open sets, respectively.

Example 2.3. Consider the (a)space $(\mathbb{Z}, \{\tau_n\})$ on $\mathbb{Z}$, where $\tau_n = \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, \mathbb{Z} - 2\mathbb{Z}\}$ if n is odd and τ_n is the cofinite topology if n is even. Consider the set $A = 2\mathbb{Z} \cup \{1\}$ and $B = \mathbb{Z} - 2\mathbb{Z}$. Both A and B are $(2, 1)$-α-open as τ_1-int(τ_2-cl(τ_1-int(A))) $= \mathbb{Z}$ and τ_1-int(τ_2-cl(τ_1-int(B))) $= \mathbb{Z}$. But, τ_2-cl(τ_1-int(τ_2-cl($A \cap B$))) $= \emptyset$ witness that $A \cap B$ is not $(2, 1)$-β-open. Thus, $S_{(m,n)}X$, $P_{(m,n)}X$, $\alpha_{(m,n)}X$ and $\beta_{(m,n)}X$ are not closed under finite intersection. □

Proposition 2.4. *Let X be an (a)space and A be a subset of X. Then A is:*

1. *(m, n)-semi-closed if and only if $A \supseteq \tau_m$-int(τ_n-cl(A)).*
2. *(m, n)-pre-closed if and only if $A \supseteq \tau_n$-cl(τ_m-int(A)).*
3. *(m, n)-α-closed if and only if $A \supseteq \tau_n$-cl(τ_m-int(τ_n-cl(A))).*
4. *(m, n)-β-closed if and only if $A \supseteq \tau_m$-int(τ_n-cl(τ_m-int(A))).*

Proof. (1) Let A be a (m, n)-semi-closed set in X. Then $X - A$ is (m, n)-semi-open, so $X - A \subseteq \tau_m$-cl(τ_n-int($X - A$)). It follows that $A \supseteq X - (\tau_m$-cl(τ_n-int($X - A$))) $= \tau_m$-int($X - \tau_n$-int($X - A$)) $= \tau_m$-int(τ_n-cl(A)). Conversely, we have $A \supseteq \tau_m$-int(τ_n-cl(A)) which implies that $X - A \subseteq X - (\tau_m$-int($\tau_n$-cl($A$))) $= \tau_m$-cl($X - \tau_n$-cl(A)) $= \tau_m$-cl(τ_n-int($X - A$)). So $X - A$ is (m, n)-semi-open and therefore, A is (m, n)-semi-closed.

Proof of parts (2), (3) and (4) can be proved in a similar manner. □

Proposition 2.5. *Let X be an (a)space and A be a subset of X. Then:*

1. *$\tau_{(m,n)}$-int$_s(A) = A \cap \tau_m$-cl(τ_n-int(A)).*
2. *$\tau_{(m,n)}$-int$_\alpha(A) = A \cap \tau_n$-int($\tau_m$-cl($\tau_n$-int($A$))).*
3. *$\tau_{(m,n)}$-int$_p(A) \subseteq A \cap \tau_n$-int($\tau_m$-cl($A$)).*
4. *$\tau_{(m,n)}$-int$_\beta(A) \subseteq A \cap \tau_m$-cl($\tau_n$-int($\tau_m$-cl($A$))).*
5. *$\tau_{(m,n)}$-cl$_s(A) = A \cup \tau_m$-int(τ_n-cl(A)).*
6. *$\tau_{(m,n)}$-cl$_\alpha(A) = A \cup \tau_n$-cl($\tau_m$-int($\tau_n$-cl($A$))).*
7. *$\tau_{(m,n)}$-cl$_p(A) \supseteq A \cup \tau_n$-cl($\tau_m$-int($A$)).*
8. *$\tau_{(m,n)}$-cl$_\beta(A) \supseteq A \cup \tau_m$-int($\tau_n$-cl($\tau_m$-int($A$))).*

Proof. We will prove relations (1), (2), (3) and (4) only as (5), (6), (7) and (8) are easy consequences of (1), (2), (3) and (4), respectively.

(1) Let $x \in \tau_{(m,n)}$-int$_s(A)$. Then there exists a (m, n)-semi-open set U such that $x \in U \subseteq A$. So $U \subseteq \tau_m$-cl(τ_n-int(U)) $\subseteq \tau_m$-cl(τ_n-int(A)). Therefore, $x \in A \cap \tau_m$-cl(τ_n-int(A)). Conversely, we have τ_n-int(A) $\subseteq A \cap \tau_m$-cl(τ_n-int(A)) $\subseteq \tau_m$-cl(τ_n-int(A)). So, $U = \tau_n$-int(A) is a τ_n-open set such that $U \subseteq A \cap \tau_m$-cl($\tau_n$-int($A$)) $\subseteq \tau_m$-cl(U).

Therefore, $A \cap \tau_m$-cl(τ_n-int(A)) is (m,n)-semi-open set contained in A and thus, $A \cap \tau_m$-cl(τ_n-int(A)) $\subseteq \tau_{(m,n)}$-int$_s(A)$.

(2) Let $x \in \tau_{(m,n)}$-int$_\alpha(A)$. Then there exists a (m,n)-α-open set U such that $x \in U \subseteq A$. So $U \subseteq \tau_n$-int(τ_m-cl(τ_n-int(U))) $\subseteq \tau_n$-int(τ_m-cl(τ_n-int(A))). Therefore, $x \in A \cap \tau_n$-int(τ_m-cl(τ_n-int(A))). Conversely, τ_n-int(A) $\subseteq A \cap \tau_n$-int(τ_m-cl(τ_n-int(A))) $\subseteq \tau_n$-int(τ_m-cl(τ_n-int(A))). This implies that $U = \tau_n$-int(A) is a τ_n-open set such that $U \subseteq A \cap \tau_n$-int($\tau_m$-cl($\tau_n$-int($A$))) $\subseteq \tau_n$-int(τ_m-cl(U)). So $A \cap \tau_n$-int(τ_m-cl(τ_n-int(A))) is (m,n)-α-open set contained in A. Thus, $A \cap \tau_n$-int(τ_m-cl(τ_n-int(A))) $\subseteq \tau_{(m,n)}$-int$_\alpha(A)$.

(3) Let $x \in \tau_{(m,n)}$-int$_p(A)$. Then there exists a (m,n)-pre-open set U such that $x \in U \subseteq A$. So $U \subseteq \tau_n$-int(τ_m-cl(U)) $\subseteq \tau_n$-int(τ_m-cl(A)). Therefore, $x \in A \cap \tau_n$-int(τ_m-cl(A)).

(4) Let $x \in \tau_{(m,n)}$-int$_\beta(A)$. Then there exists a (m,n)-β-open set U such that $x \in U \subseteq A$. So $U \subseteq \tau_m$-cl(τ_n-int(τ_m-cl(U))) $\subseteq \tau_m$-cl(τ_n-int(τ_m-cl(A))). Therefore, $x \in A \cap \tau_m$-cl(τ_n-int(τ_m-cl(A))). □

Definition 2.6. A subset A of $(X, \{\tau_n\})$ is said to be:

1. ([44]) (a)-semi-open if A is (m,n)-semi-open for all $m \neq n$.
2. (a)-pre-open if A is (m,n)-pre-open for all $m \neq n$.
3. (a)-α-open if A is (m,n)-α-open for all $m \neq n$.
4. (a)-β-open if A is (m,n)-β-open for all $m \neq n$.
5. (a)-open if A is τ_n-open for all $n \in \mathbb{N}$.

The collection of all (a)-open (resp. (a)-semi-open, (a)-pre-open, (a)-α-open and (a)-β-open) sets is closed under arbitrary union. Indeed, the collection of (a)-open sets form a topology on X. The complement of (a)-semi-open set, (a)-pre-open set, (a)-α-open set, (a)-β-open set and (a)-open set is (a)-semi-closed, (a)-pre-closed, (a)-α-closed, (a)-β-closed and (a)-closed, respectively. By $S(X)$, $P(X)$, $\alpha(X)$, $\beta(X)$ and $O(X)$, we denote the family of all (a)-semi-open sets, (a)-pre-open sets, (a)-α-open sets, (a)-β-open sets and (a)-open sets, respectively.

It is clear that every (a)-open set is (a)-α-open and hence, (a)-semi-open, (a)-pre-open and (a)-β-open.

Note that

$$\begin{array}{ccccc} (a)\text{-}\alpha\text{-open} & \Rightarrow & (a)\text{-pre-open} & \Rightarrow & (a)\text{-}\beta\text{-open} \\ \Uparrow & & & & \Uparrow \\ (a)\text{-open} & \Rightarrow & (a)\text{-}\alpha\text{-open} & \Rightarrow & (a)\text{-semi-open} \end{array}$$

Diagram 2

Following examples show that no implication in the above diagram is reversible.

Example 2.7. Let F be a finite subset of $\mathbb{Z}$ having at least two elements. Consider the (a)space $(\mathbb{Z}, \{\tau_n\})$ on $\mathbb{Z}$, where $\tau_n = \{\emptyset, \mathbb{Z}, F\}$ if n is odd and τ_n is the cofinite topology if n is even. Let $w \in F$. Then $A = \mathbb{Z} - \{w\}$ is (a)-pre-open and (a)-β-open but not (a)-semi-open and hence, not (a)-α-open.

Example 2.8. Consider the (a)space $(\mathbb{R}, \{\tau_n\})$ on $\mathbb{R}$, where $\tau_n = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$ is the right ray topology if n is odd and τ_n is the cocountable topology if n is even. For each $a \in \mathbb{R}$, τ_1-cl($\{a\}$) $= (-\infty, a]$. It is observe that τ_1-cl($(10,\infty)$) $= \mathbb{R}$ and τ_2-cl($(10,\infty)$) $= \mathbb{R}$. So $(10,\infty)$ is (m,n)-pre-open for all $m \neq n$ and therefore, (a)-pre-open. But τ_2-int($(10,\infty)$) $= \emptyset$. So $(10,\infty) \nsubseteq \tau_1$-cl($\tau_2$-int$(10,\infty)$). Therefore, $(10,\infty)$ is not $(1,2)$-semi-open and hence, not (a)-semi-open. Thus, $(10,\infty)$ is (a)-pre-open but not (a)-α-open and (a)-β-open but not (a)-semi-open. □

Every (a)-pre-open set is (a)-β-open and every (a)-α-open set is (a)-semi-open but not vice-versa. To show that converse need not be true, we recall the digital topology on $\mathbb{Z}$. Digital topology τ on $\mathbb{Z}$ is generated by $\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$.

$$\tau\text{-cl}(\{m\}) = \begin{cases} \{m\}, & \text{if } m \text{ is even;} \\ \{m-1, m, m+1\}, & \text{if } m \text{ is odd.} \end{cases}$$

and

$$\tau\text{-cl}(\{2n-1, 2n, 2n+1\}) = \{2n-2, 2n-1, 2n, 2n+1, 2n+2\}.$$

We consider the following example to show that not every (a)-β-open set is (a)-pre-open and not every (a)-semi-open set is (a)-α-open.

Example 2.9. Consider the (a)space $(\mathbb{Z}, \{\tau_n\})$, where
τ_1 is the digital topology on $\mathbb{Z}$ generated by $\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$,
τ_2 is the topology on $\mathbb{Z}$ generated by $\{\{3n+1, 3n+2, 3n+3, 3n+4\} : n \in \mathbb{Z}\}$,
τ_3 is the topology on $\mathbb{Z}$ generated by $\{\{4n+1, 4n+2, 4n+3, 4n+4, 4n+5\} : n \in \mathbb{Z}\}$,
τ_4 is the topology on $\mathbb{Z}$ generated by $\{\{5n+1, 5n+2, 5n+3, 5n+4, 5n+5, 5n+6\} : n \in \mathbb{Z}\}$,
.
.
.
τ_k is the topology on $\mathbb{Z}$ generated by $\{\{(k+1)n+1, (k+1)n+2, ..., (k+1)n+(k+2)\} : n \in \mathbb{Z}\}$ for each $k \in \mathbb{N}$.

Consider the set $G = \{1, 2\}$. Then τ_n-int$(G) = \{1\}$ for all $n \in \mathbb{N}$ and τ_m-cl$(\{1\}) \supseteq \{0, 1, 2\}$ for all $m \neq n$.
So, $G = \{1, 2\} \subseteq \tau_m$-cl$(\tau_n$-int$(G))$ for all $m \neq n$. Thus, G is (a)-semi-open and hence, (a)-β-open.
Observe that τ_3-int$(\tau_1$-cl$(G)) = \tau_3$-int$(\{0, 1, 2\}) = \{1\}$. Thus, G is not a $(1, 3)$-pre-open set. Therefore, G is not (a)-pre-open and hence, not (a)-α-open.
Conclusively, G is (a)-β-open but not (a)-pre-open and G is (a)-semi-open but not (a)-α-open in $(\mathbb{Z}, \{\tau_n\})$. □

It is clear from Example 2.8 and Example 2.9 that there is no relation between (a)semi-open and (a)-pre-open. Following example shows that there exists a set in some (a)space which is (a)-semi-open, (a)-pre-open, (a)-α-open and (a)-β-open but not (a)-open.

Example 2.10. Consider the (a)space $(\mathbb{R}, \{\tau_n\})$ on $\mathbb{R}$, where τ_n is the right ray topology if n is odd and τ_n is the cocountable topology if n is even. Consider the set $A = (-\infty, 9) \cup (10, \infty) \cup [(\mathbb{R} - \mathbb{Q}) \cap (9, 10)] = \mathbb{R} - (\mathbb{Q} \cap (9, 10))$. A is not (a)-open as A is not open in $(\mathbb{R}, \tau_1)$. We have τ_n-int$(\tau_m$-cl$(A)) = \mathbb{R}$ for all $m \neq n$. Thus, A is (a)-pre-open as well as (a)-β-open. Also, τ_m-cl$(\tau_n$-int$(A)) = \mathbb{R}$ for all $m \neq n$. Thus, A is (a)-semi-open as well as (a)-α-open. Hence, A is (a)-semi-open, (a)-pre-open, (a)-β-open and (a)-α-open but not (a)-open. □

3. Strong Form of Open Sets in (a)Topological Spaces

Recall that in a topological space (X, τ), a point $x \in X$ is said to be δ-cluster point [45] of a subset $S \subseteq X$ if for every τ-open set G containing x, τ-int$(\tau$-cl$(G)) \cap S \neq \emptyset$. Following Tyagi et. al. [44], we define δ-open sets in (a)spaces as follows:

Definition 3.1. Let X be an (a)space and S be a subset of X. A point $x \in X$ is said to be (m, n)-δ-cluster point of S if for every τ_n-open set G containing x, τ_n-int$(\tau_m$-cl$(G)) \cap S \neq \emptyset$. The set of all (m, n)-δ-cluster points of S is called the (m, n)-δ-closure of S and denoted by $\tau_{(m,n)}$-cl$_\delta(S)$. S is said to be (m, n)-δ-closed if $\tau_{(m,n)}$-cl$_\delta(S) = S$. The complement of (m, n)-δ-closed set is (m, n)-δ-open. In case, S is (m, n)-δ-open for all $m \neq n$, S is (a)-δ-open.

Remark 3.2. The collection of all (m, n)-δ-open sets forms a topology.

Theorem 3.3. *Let A be a subset of an (a)space X. Then A is (m,n)-δ-open if and only if for each $x \in A$ there exists a τ_n-open set U such that $x \in U \subseteq \tau_n$-int$(\tau_m$-cl$(U)) \subseteq A$.*

Proof. Let $A \subseteq X$ be (m,n)-δ-open. Then $\tau_{(m,n)}$-cl$_\delta(X-A) = X-A$. So for each $x \in A$ there is a τ_n-open set U containing x such that τ_n-int$(\tau_m$-cl$(U)) \cap (X-A) = \emptyset$. Therefore, for each $x \in A$ there exists a τ_n-open set U such that $x \in U \subseteq \tau_n$-int$(\tau_m$-cl$(U)) \subseteq A$. Conversely, let for each $x \in A$ there exists a τ_n-open set U such that $x \in U \subseteq \tau_n$-int$(\tau_m$-cl$(U)) \subseteq A$. Therefore, τ_n-int$(\tau_m$-cl$(U)) \cap (X-A) = \emptyset$. So $\tau_{(m,n)}$-cl$_\delta(X-A) = X-A$. Thus, $A \subseteq X$ is (m,n)-*delta*-open. □

Theorem 3.4. *Let A be a subset of an (a)space X. A point $x \in \tau_{(m,n)}$-cl$_\delta(A)$ if and only if every (m,n)-δ-open set containing x intersects A.*

Proof. Let $x \in \tau_{(m,n)}$-cl$_\delta(A)$ and U be a (m,n)-δ-open set containing x. There exist a τ_n-open set V such that $x \in V \subseteq \tau_n$-int$(\tau_m$-cl$(V)) \subseteq U$. Since $x \in \tau_{(m,n)}$-cl$_\delta(A)$, τ_n-int$(\tau_m$-cl$(V)) \cap A \neq \emptyset$. It follows that $U \cap A \neq \emptyset$. Conversely, let U be a τ_n-open set containing x. The set τ_n-int$(\tau_m$-cl$(U))$ is a (m,n)-δ-open set. Indeed, for each $x \in \tau_n$-int$(\tau_m$-cl$(U))$ there exists a τ_n-open set G such that $x \in G \subseteq \tau_n$-int$(\tau_m$-cl$(U))$. Since τ_n-int$(\tau_m$-cl$(G)) \subseteq \tau_n$-int$(\tau_m$-cl$(U))$, it follows that τ_n-int$(\tau_m$-cl$(U))$ is (m,n)-δ-open. So τ_n-int$(\tau_m$-cl$(U)) \cap A \neq \emptyset$ and therefore, $x \in \tau_{(m,n)}$-cl$_\delta(A)$. □

Remark 3.5. Theorem 3.4 emphasize that the (m,n)-δ-closure of a set A is the intersection of all (m,n)-δ-closed subsets of X containing A.

Proposition 3.6. *Every (a)-δ-open set is (a)-open.*

Proof. Let G be an (a)-δ-open set in X. By definition, G is (m,n)-δ-open for all $m \neq n$. Then $\tau_{(m,n)}$-cl$_\delta(X-G) = X-G$ for all $m \neq n$. But τ_n-cl$(A) \subseteq \tau_{(m,n)}$-cl$_\delta(A)$ for every set A, so τ_n-cl$(X-G) = X-G$. Thus, G is τ_n-open for all $n \in \mathbb{N}$. □

We end up this section by an example showing that converse of Proposition 3.6 need not be true in general.

Example 3.7. Consider the (a)space $(\mathbb{Z}, \{\tau_n\})$ on $\mathbb{Z}$, where τ_n is the Digital topology if n is odd and $\tau_n = \{G \subseteq \mathbb{Z} \colon G = \emptyset$ or $3 \in G\}$ is the Point-included topology if n is even.

For any subset $A \subseteq \mathbb{Z}$, we have $\tau_2\text{-cl}(A) = \begin{cases} A, & \text{if } 3 \notin A; \\ \mathbb{Z}, & \text{if } 3 \in A. \end{cases}$

and $\tau_2\text{-int}(A) = \begin{cases} \emptyset, & \text{if } 3 \notin A; \\ A, & \text{if } 3 \in A. \end{cases}$

The set $G = \{3,4,5\}$ is (a)-open in $(\mathbb{Z}, \{\tau_n\})$. We show that G is not (a)-δ-open. Let U be any τ_4-open set containing 3. Then τ_4-int$(\tau_2$-cl$(U)) = \mathbb{Z}$. So τ_4-int$(\tau_2$-cl$(U)) \cap (\mathbb{Z}-G) \neq \emptyset$ for all τ_4-open set U containing 3. Therefore, $3 \in \tau_{(2,4)}$-cl$_\delta(\mathbb{Z}-G)$ and $\tau_{(2,4)}$-cl$_\delta(\mathbb{Z}-G) \neq \mathbb{Z}-G$ which implies that $\mathbb{Z}-G$ is not $(2,4)$-δ-closed. Thus, G is not $(2,4)$-δ-open and hence, G is not (a)-δ-open. □

4. Various Selection Properties

In this section, we discuss various selection properties by using notions of semi-closure, pre-closure, α-closure, β-closure and δ-closure and their respective density in (a)spaces and provide interrelationships between them.

Definition 4.1. In an (a)space X, a subset $A \subseteq X$ is said to be:

1. τ_n-dense in X if A is dense in (X, τ_n).
2. dense in X if A is dense in (X, τ_n) for all $n \in \mathbb{N}$.

3. t-(m, n)-dense if $\tau_{(m,n)}$-$\mathrm{cl}_t(A) = X$, where $t \in \{s, p, \alpha, \beta, \delta\}$.

4. t-(a)-dense if A is t-(m, n)-dense for all $m \neq n$, where $t \in \{s, p, \alpha, \beta, \delta\}$.

For each $t \in \{s, p, \alpha, \beta, \delta\}$, let $D^t(m, n)$, D^t and D be the collection of all t-(m, n)-dense, t-(a)-dense and dense subsets of $(X, \{\tau_n\})$, respectively and $D(n)$ be the collection of all dense subsets of (X, τ_n).

Proposition 4.2. *In an (a)space $(X, \{\tau_n\})$, the following holds.*

1. $D^\beta \subset D^s \subset D^\alpha$
2. $D^\beta \subset D^p \subset D^\alpha$
3. $D^s = D^\alpha = D$
4. $D \subset D^\delta$

Proof. (1) Let S be any subset of X. Since every (m, n)-α-open set is (m, n)-semi-open and every (m, n)-semi-open set is (m, n)-β-open for all $m \neq n$, so $\tau_{(m,n)}$-$\mathrm{cl}_\beta(S) \subset \tau_{(m,n)}$-$\mathrm{cl}_s(S) \subset \tau_{(m,n)}$-$\mathrm{cl}_\alpha(S)$ for all $m \neq n$. Therefore, $D^\beta(m, n) \subset D^s(m, n) \subset D^\alpha(m, n)$ holds for all $m \neq n$. Thus, $D^\beta \subset D^s \subset D^\alpha$.

(2) Let S be any subset of X. Since every (m, n)-α-open set is (m, n)-pre-open and every (m, n)-pre-open set is (m, n)-β-open for all $m \neq n$, so $\tau_{(m,n)}$-$\mathrm{cl}_\beta(S) \subset \tau_{(m,n)}$-$\mathrm{cl}_p(S) \subset \tau_{(m,n)}$-$\mathrm{cl}_\alpha(S)$ for all $m \neq n$. Therefore, $D^\beta(m, n) \subset D^p(m, n) \subset D^\alpha(m, n)$ holds for all $m \neq n$. Thus, $D^\beta \subset D^p \subset D^\alpha$.

(3) By (1), $D^s \subset D^\alpha$. It is enough to show that $D^\alpha \subset D \subset D^s$. For any $m \neq n$, let $A \in D^\alpha(m, n)$. Then $X = \tau_{(m,n)}$-$\mathrm{cl}_\alpha(A) = A \cup \tau_n$-$\mathrm{cl}(\tau_m$-$\mathrm{int}(\tau_n$-$\mathrm{cl}(A)))$. So, $X = A \cup \tau_n$-$\mathrm{cl}(\tau_n$-$\mathrm{cl}(A)) \subset \tau_n$-$\mathrm{cl}(A)$ and hence, $A \in D(n)$. Also, for any $A \in D(n)$, τ_n-$\mathrm{cl}(A) = X$ which gives that $X = A \cup \tau_m$-$\mathrm{int}(\tau_n$-$\mathrm{cl}(A)) = \tau_{(m,n)}$-$\mathrm{cl}_s(A)$. So $A \in D^s(m, n)$. Therefore, $D^\alpha(m, n) \subset D(n) \subset D^s(m, n)$ for all $m \neq n$. Hence, $D^\alpha \subset D \subset D^s$.

(4) The proof follows by Proposition 3.6. □

Remark 4.3. Following results hold for all naturals m, k and n.

1. $D^s(m, n) = D^s(k, n)$.
2. $D^\alpha(m, n) = D^\alpha(k, n)$.
3. $D^s(m, n) = D^\alpha(k, n)$.

Proof. Proof follows as $D^\alpha(m, n) = D(n)$ and $D^s(m, n) = D(n)$ for all naturals m and n. □

Example 4.4. D^β is a proper subset of D^p.
Consider the (a)space $(\mathbb{R}, \{\tau_n\})$ on $\mathbb{R}$, where τ_n is the cocountable topology if n is odd and $\tau_n = \{\emptyset, \mathbb{R}, \{p\}, \{q\}, \{p, q\}\}$: $p, q \in \mathbb{R}$, if n is even. It is observe that the set of all (a)-pre-open sets, that is, $P(X) = \{A \subseteq \mathbb{R}\colon A$ is uncountable and $p, q \in A\}$. Now, consider the set $G = \mathbb{R} - \{q\}$. G is an (a)-β-open set having empty intersection with $\{q\}$. So, $\{q\} \notin D^\beta$. But $\{q\}$ intersect with every (a)-pre-open set, hence, $\{q\} \in D^p - D^\beta$. □

Example 4.5. D^p is a proper subset of D.
In Example 2.7, let $F = \{2, 3\}$. Then the set of even integers, say A, is dense in $(\mathbb{Z}, \{\tau_n\})$ but $\mathbb{Z} - A$ is (a)-pre-open in $(\mathbb{Z}, \{\tau_n\})$. Thus, $A \in D - D^p$.

Example 4.6. D^β is a proper subset of D^s.
Consider the (a)space $(\mathbb{R}, \{\tau_n\})$ on $\mathbb{R}$, where $\tau_n = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ if n is odd and $\tau_n = \{\emptyset, \mathbb{R}, (\mathbb{R} - \mathbb{Q}) \cup \{1\}\}$ if n is even. It is observe that the set of all (a)-semi-open sets, that is, $S(X) = \{A \subseteq \mathbb{R}\colon (\mathbb{R} - \mathbb{Q}) \cup \{1\} \subseteq A\}$. Now, consider the set $G = \mathbb{R} - \{1\}$. G is an (a)-β-open set having empty intersection with $\{1\}$. So, $\{1\} \notin D^\beta$. But $\{1\}$ intersect with every (a)-semi-open set, so $\{1\} \in D^s$. Hence, $\{1\} \in D^s - D^\beta$. □

Example 4.7. D is a proper subset of D^δ.

In Example 4.6, $\mathbb{Q}$ is not dense in $(\mathbb{R}, \tau_1)$ as τ_1-cl$(\mathbb{Q}) = \mathbb{Q}$. So $\mathbb{Q} \notin D(1)$ and therefore, $\mathbb{Q} \notin D$. But $\mathbb{Q} \in D^\delta(m,n)$ for all $m \neq n$. Thus, $\mathbb{Q} \in D^\delta - D$.

Conclusively, we have the following relations:

1. $D^\beta \subsetneq D^p \subsetneq D^\alpha = D \subsetneq D^\delta$.
2. $D^\beta \subsetneq D^s = D^\alpha = D \subsetneq D^\delta$.

The following implications are immediate.

$$S_{fin}(D^p, D^\beta) \to S_{fin}(D^\beta, D^\beta),\ S_{fin}(D^p, D^\beta) \to S_{fin}(D^p, D^p),\ S_{fin}(D^\beta, D^\beta) \to S_{fin}(D^\beta, D^p),$$
$$S_{fin}(D^p, D^p) \to S_{fin}(D^\beta, D^p),\ S_{fin}(D^p, D^\beta) \to S_{fin}(D^\beta, D^p);$$

$$S_{fin}(D^\alpha, D^\beta) \to S_{fin}(D^\beta, D^\beta),\ S_{fin}(D^\alpha, D^\beta) \to S_{fin}(D^\alpha, D^\alpha),\ S_{fin}(D^\beta, D^\beta) \to S_{fin}(D^\beta, D^\alpha),$$
$$S_{fin}(D^\alpha, D^\alpha) \to S_{fin}(D^\beta, D^\alpha),\ S_{fin}(D^\alpha, D^\beta) \to S_{fin}(D^\beta, D^\alpha);$$

$$S_{fin}(D^\delta, D^\beta) \to S_{fin}(D^\beta, D^\beta),\ S_{fin}(D^\delta, D^\beta) \to S_{fin}(D^\delta, D^\delta),\ S_{fin}(D^\beta, D^\beta) \to S_{fin}(D^\beta, D^\delta),$$
$$S_{fin}(D^\delta, D^\delta) \to S_{fin}(D^\beta, D^\delta),\ S_{fin}(D^\delta, D^\beta) \to S_{fin}(D^\beta, D^\delta);$$

$$S_{fin}(D^\alpha, D^p) \to S_{fin}(D^p, D^p),\ S_{fin}(D^\alpha, D^p) \to S_{fin}(D^\alpha, D^\alpha),\ S_{fin}(D^p, D^p) \to S_{fin}(D^p, D^\alpha),$$
$$S_{fin}(D^\alpha, D^\alpha) \to S_{fin}(D^p, D^\alpha),\ S_{fin}(D^\alpha, D^p) \to S_{fin}(D^p, D^\alpha);$$

$$S_{fin}(D^\delta, D^p) \to S_{fin}(D^p, D^p),\ S_{fin}(D^\delta, D^p) \to S_{fin}(D^\delta, D^\delta),\ S_{fin}(D^p, D^p) \to S_{fin}(D^p, D^\delta),$$
$$S_{fin}(D^\delta, D^\delta) \to S_{fin}(D^p, D^\delta),\ S_{fin}(D^\delta, D^p) \to S_{fin}(D^p, D^\delta);$$

$$S_{fin}(D^\delta, D^\alpha) \to S_{fin}(D^\alpha, D^\alpha),\ S_{fin}(D^\delta, D^\alpha) \to S_{fin}(D^\delta, D^\delta),\ S_{fin}(D^\alpha, D^\alpha) \to S_{fin}(D^\alpha, D^\delta),$$
$$S_{fin}(D^\delta, D^\delta) \to S_{fin}(D^\alpha, D^\delta),\ S_{fin}(D^\delta, D^\alpha) \to S_{fin}(D^\alpha, D^\delta);$$

$$S_{fin}(D^s, D^\beta) \to S_{fin}(D^\beta, D^\beta),\ S_{fin}(D^s, D^\beta) \to S_{fin}(D^s, D^s),\ S_{fin}(D^\beta, D^\beta) \to S_{fin}(D^\beta, D^s),$$
$$S_{fin}(D^s, D^s) \to S_{fin}(D^\beta, D^s),\ S_{fin}(D^s, D^\beta) \to S_{fin}(D^\beta, D^s);$$

From above implications we have following implications too.

$$S_{fin}(D^\delta, D^p) \to S_{fin}(D^p, D^\alpha),\ S_{fin}(D^\delta, D^p) \to S_{fin}(D^\beta, D^p),$$
$$S_{fin}(D^\delta, D^p) \to S_{fin}(D^\beta, D^\delta),\ S_{fin}(D^\delta, D^p) \to S_{fin}(D^\alpha, D^\delta);$$

$$S_{fin}(D^\alpha, D^p) \to S_{fin}(D^p, D^\delta),\ S_{fin}(D^\alpha, D^p) \to S_{fin}(D^\beta, D^p),$$
$$S_{fin}(D^\alpha, D^p) \to S_{fin}(D^\beta, D^\alpha),\ S_{fin}(D^\alpha, D^p) \to S_{fin}(D^\alpha, D^\delta);$$

$$S_{fin}(D^p, D^\beta) \to S_{fin}(D^p, D^\alpha),\ S_{fin}(D^p, D^\beta) \to S_{fin}(D^p, D^\delta),$$
$$S_{fin}(D^p, D^\beta) \to S_{fin}(D^\beta, D^\delta),\ S_{fin}(D^p, D^\beta) \to S_{fin}(D^\beta, D^\alpha);$$

$$S_{fin}(D^{\delta}, D^{\beta}) \to S_{fin}(D^{\beta}, D^{\alpha}),\ S_{fin}(D^{\delta}, D^{\beta}) \to S_{fin}(D^{\beta}, D^{p}),$$
$$S_{fin}(D^{\delta}, D^{\beta}) \to S_{fin}(D^{p}, D^{\delta}),\ S_{fin}(D^{\delta}, D^{\beta}) \to S_{fin}(D^{\alpha}, D^{\delta});$$

$$S_{fin}(D^{\alpha}, D^{\beta}) \to S_{fin}(D^{\beta}, D^{\delta}),\ S_{fin}(D^{\alpha}, D^{\beta}) \to S_{fin}(D^{\beta}, D^{p}),$$
$$S_{fin}(D^{\alpha}, D^{\beta}) \to S_{fin}(D^{p}, D^{\alpha}),\ S_{fin}(D^{\alpha}, D^{\beta}) \to S_{fin}(D^{\alpha}, D^{\delta});$$

$$S_{fin}(D^{\delta}, D^{\alpha}) \to S_{fin}(D^{p}, D^{\alpha}),\ S_{fin}(D^{\delta}, D^{\alpha}) \to S_{fin}(D^{\beta}, D^{\alpha}),$$
$$S_{fin}(D^{\delta}, D^{\alpha}) \to S_{fin}(D^{p}, D^{\delta}),\ S_{fin}(D^{\delta}, D^{\alpha}) \to S_{fin}(D^{\beta}, D^{\delta});$$

$$S_{fin}(D^{s}, D^{\beta}) \to S_{fin}(D^{\beta}, D^{\delta}),\ S_{fin}(D^{s}, D^{\beta}) \to S_{fin}(D^{\beta}, D^{\alpha}),\ S_{fin}(D^{s}, D^{\beta}) \to S_{fin}(D^{\beta}, D^{p}).$$

5. *R*-Separability and *M*-Separability in (*a*)Topological Spaces

In this section, we study various selective separability properties using weak and strong forms of open sets. We begin with some definitions we will do with.

Definition 5.1. An (*a*)space $(X, \{\tau_n\})$ is said to be separable if there exists a countable subset of X which is dense in X.

Definition 5.2. An (*a*)space $(X, \{\tau_n\})$ is said to be:

1. (*a*)*R*-separable if $S_1(D(n), D(n))$ holds for all n.
2. (*a*)R^t-separable if $S_1(D^t(m, n), D^t(m, n))$ holds for all $m \neq n$, where $t \in \{s, p, \alpha, \beta, \delta\}$.
3. (*a*)*M*-separable if $S_{fin}(D(n), D(n))$ holds for all n.
4. (*a*)M^t-separable if $S_{fin}(D^t(m, n), D^t(m, n))$ for all $m \neq n$, where $t \in \{s, p, \alpha, \beta, \delta\}$.

It is obvious that (*a*)*R*-separability implies (*a*)*M*-separability and (*a*)R^t-separability implies (*a*)M^t-separability. Also, if X is (*a*)*R*-separable (or separable), then each (X, τ_n) is separable. Following theorem is an analogous result of this.

Theorem 5.3. *If X is (a)R^t-separable, $t \in \{s, p, \alpha, \beta\}$, then (X, τ_n) is separable for all n.*

Proof. Let X be an (*a*)R^t-separable space. Then $S_1(D^t(m, n), D^t(m, n))$ holds for all $m \neq n$. For every sequence $< A_k : k \in \mathbb{N} >$ of elements of $D^t(m, n)$, there is a sequence $< a_k : k \in \mathbb{N} >$ such that for each k, $a_k \in A_k$ and $\{a_k : k \in \mathbb{N}\} \in D^t(m, n)$. So $A = \{a_k : a_k \in A_k, k \in \mathbb{N}\}$ is a countable subset of X such that $A \in D^t(m, n)$. But $D^t(m, n) \subset D(n)$ for all $m, n \in \mathbb{N}$. Therefore, (X, τ_n) is separable for all $n \in \mathbb{N}$. □

Recall that a family $\mathcal{B}$ of non empty open subsets of a topological space (X, τ) is said to be π-base of X if for each non empty open subset, say $G \subseteq X$, there exists a $U \in \mathcal{B}$ such that $U \subseteq G$.

Theorem 5.4. *If (X, τ_n) has a countable π-base for all $n \in \mathbb{N}$, then X is (a)R^p-separable.*

Proof. For each $n \in \mathbb{N}$, let $\{Q_k^n : k \in \mathbb{N}\}$ be a countable π-base of (X, τ_n). For any $m \neq n$, let $< F_k : k \in \mathbb{N} >$ be a sequence of p-(m, n)-dense subsets of X. So $\tau_{(m,n)}\text{-cl}_p(F_k) = X$. Therefore, F_k intersects with every (m, n)-pre-open set for all $k \in \mathbb{N}$. Since every τ_n-open set is (m, n)-pre-open for all $m \in \mathbb{N}$, so Q_k^n is (m, n)-pre-open for all $k \in \mathbb{N}$. Let $x_k \in Q_k^n \cap F_k$ for each $k \in \mathbb{N}$. We claim that $\{x_k : k \in \mathbb{N}\}$ is p-(m, n)-dense in X. Let W be a (m, n)-pre-open set. Then $\tau_n\text{-int}(W)$ is τ_n-open so there exists some $l \in \mathbb{N}$ such that $Q_l^n \subseteq \tau_n\text{-int}(W)$. Then $x_l \in W$. Therefore, every (m, n)-pre-open subset of X intersects with $\{x_k : k \in \mathbb{N}\}$. Thus, $\{x_k : k \in \mathbb{N}\} \in D^p(m, n)$ and $S_1(D^p(m, n), D^p(m, n))$ holds for all $m \neq n$. Hence, X is (*a*)R^p-separable. □

In a similar way the following theorem can be proved.

Theorem 5.5. *If (X, τ_n) has countable π-base for all $n \in \mathbb{N}$, then X is $(a)R^t$-separable for all $t \in \{s, \alpha, \beta\}$.*

Theorem 5.6. *If (X, τ_n) has countable π-base for some $n \in \mathbb{N}$, then X satisfies $S_1(D(n), D^\delta(m, n))$ for all $m \neq n$.*

Proof. Let $\{Q_k : k \in \mathbb{N}\}$ be a countable π-base of (X, τ_n). Let $< F_k : k \in \mathbb{N} >$ be a sequence of elements of $D(n)$. Then $Q_k \cap F_k \neq \emptyset$ for each $k \in \mathbb{N}$. Let $x_k \in Q_k \cap F_k$ for each $k \in \mathbb{N}$. The set $\{x_k : k \in \mathbb{N}\} \in D^\delta(m, n)$ for all $m \neq n$. Indeed, for any (m, n)-δ-open set W, there exists some $l \in \mathbb{N}$ such that $Q_l \subseteq W$ as every (m, n)-δ-open set is τ_n-open. Then $x_l \in W$ which gives that every (m, n)-δ-open set of X intersects with $\{x_k : k \in \mathbb{N}\}$. Thus, $\{x_k : k \in \mathbb{N}\} \in D^\delta(m, n)$ for all $m \neq n$. □

Corollary 5.7. *If (X, τ_n) has countable π-base for all $n \in \mathbb{N}$, then X satisfies $S_1(D^t(m, n), D^\delta(m, n))$ for all $m \neq n$ and for all $t \in \{s, p, \alpha, \beta\}$.*

Proof. For each $t \in \{s, p, \alpha, \beta\}$, $D^t(m, n) \subset D(n)$ for all $m \neq n$, so proof follows by Theorem 5.6. □

Definition 5.8. Let X be an (a)space. Then:

1. X has countable (m, n)-t-fan tightness $(m \neq n)$ if for each $x \in X$ and each sequence $< A_k : k \in \mathbb{N} >$ of subsets of X such that $x \in \tau_{(m,n)}\text{-cl}_t(A_k)$ for each k, there are finite sets $F_k \subseteq A_k$ such that $x \in \tau_{(m,n)}\text{-cl}_t(\cup_{k \in \mathbb{N}} F_k)$.

2. X has countable (m, n)-t-strong fan tightness $(m \neq n)$ if for each $x \in X$ and each sequence $< A_k : k \in \mathbb{N} >$ of subsets of X such that $x \in \tau_{(m,n)}\text{-cl}_t(A_k)$ for each k, there are points $x_k \in A_k$ such that $x \in \tau_{(m,n)}\text{-cl}_t(\{x_k : k \in \mathbb{N}\})$.

Theorem 5.9. *Let X be a separable space. If X has countable (m, n)-t-fan tightness for all $m \neq n$, then X is $(a)M^t$-separable, where $t \in \{s, p, \alpha, \beta\}$.*

Proof. Let $A = \{a_k : k \in \mathbb{N}\}$ be a countable set which is dense in each (X, τ_n) and let $< A_k : k \in \mathbb{N} >$ be a sequence of elements of $D^t(m, n)$ for some $m_0 \neq n_0$, $t \in \{s, p, \alpha, \beta\}$. Consider a partition $\mathbb{N} = M_1 \cup M_2 \cup \ldots$ of $\mathbb{N}$ into pairwise disjoint and infinite sets. Since $\tau_{(m_0,n_0)}\text{-cl}_t(A_k) = X$, for each $n \in \mathbb{N}$, $a_n \in \cap_{k \in M_n} \tau_{(m_0,n_0)}\text{-cl}_t(A_k)$. By countable (m_0, n_0)-t-fan tightness of X, there exists a sequence $< F_k : k \in M_n >$ such that for each $k \in M_n$, $F_k \subseteq A_k$ and $a_n \in \tau_{(m_0,n_0)}\text{-cl}_t(\cup\{F_k : k \in M_n\})$. We claim that $\cup\{F_k : k \in \mathbb{N}\}$ is t-(m_0, n_0)-dense subset of X. Let G be a (m_0, n_0)-t-open subset of X and $V = \tau_{n_0}\text{-int}(G)$. Since A is dense in (X, τ_{n_0}), $a_l \in V$ and hence, $a_l \in G$ for some $l \in \mathbb{N}$ but $a_l \in \tau_{(m_0,n_0)}\text{-cl}_t(\cup\{F_k : k \in M_l\})$. Thus, $G \cap \{\cup F_k : k \in M_l\} \neq \emptyset$. Therefore, $G \cap \{\cup F_k : k \in \mathbb{N}\} \neq \emptyset$ and hence, $X = \tau_{(m_0,n_0)}\text{-cl}_t(\cup\{F_k : k \in \mathbb{N}\}$. □

Theorem 5.10. *Let X be a separable space. If X has countable (m, n)-t-strong fan tightness, then X is $(a)R^t$-separable, where $t \in \{s, p, \alpha, \beta\}$.*

Proof. Let $A = \{a_k : k \in \mathbb{N}\}$ be countable dense in each (X, τ_n) and let $< A_k : k \in \mathbb{N} >$ be a sequence of elements of $D^t(m_0, n_0)$ for some $m_0 \neq n_0$, $t \in \{s, p, \alpha, \beta\}$. Consider a partition $\mathbb{N} = M_1 \cup M_2 \cup \ldots$ of $\mathbb{N}$ into pairwise disjoint and infinite sets. Since $\tau_{(m_0,n_0)}\text{-cl}_t(A_k) = X$, for each $n \in \mathbb{N}$, $a_n \in \cap_{k \in M_n} \tau_{(m_0,n_0)}\text{-cl}_t(A_k)$. By countable (m_0, n_0)-t-strong fan tightness of X, there exists a sequence $< x_k : k \in M_n >$ such that for each $k \in M_n$, $x_k \in A_k$ and $a_n \in \tau_{(m_0,n_0)}\text{-cl}_t(\{x_k : k \in M_n\})$. The set $\{x_k : k \in \mathbb{N}\}$ is t-(m_0, n_0)-dense subset of X. Indeed, let G be a (m_0, n_0)-t-open subset of X. Let $V = \tau_{n_0}\text{-int}(G)$. Since A is dense in (X, τ_{n_0}), $a_l \in V$ and hence, $a_l \in G$ for some $l \in \mathbb{N}$ but $a_l \in \tau_{(m_0,n_0)}\text{-cl}_t(\{x_k : k \in M_l\})$. Thus, $G \cap \{x_k : k \in M_l\} \neq \emptyset$. Therefore, $G \cap \{x_k : k \in \mathbb{N}\} \neq \emptyset$ and hence, $X = \tau_{(m_0,n_0)}\text{-cl}_t(\{x_k : k \in \mathbb{N}\})$. □

Theorem 5.11. *Let X be a separable space. If X has countable (m, n)-δ-strong fan tightness for all $m \neq n$, then X is $(a)R^\delta$-separable.*

Proof. Let $< A_k : k \in \mathbb{N} >$ be a sequence of elements of $D^\delta(m_0, n_0)$ for some $m_0 \neq n_0$. So $\tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(A_k) = X$. Let $A = \{a_n : n \in \mathbb{N}\}$ be a countable set which is dense in each (X, τ_n). Consider a partition $\mathbb{N} = M_1 \cup M_2 \cup \ldots$ of $\mathbb{N}$ into pairwise disjoint and infinite sets. Since $\tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(A_k) = X$, for each $n \in \mathbb{N}$, $a_n \in \cap_{k\in M_n} \tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(A_k)$. By countable (m_0, n_0)-δ-strong fan tightness of X, there exists a sequence $< x_k : k \in M_n >$ such that for each $k \in M_n$, $x_k \in A_k$ and $a_n \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(\{x_k : k \in M_n\})$. The set $\{x_k : k \in \mathbb{N}\}$ is δ-(m_0, n_0)-dense in X. Indeed, for any (m_0, n_0)-δ-open subset G, $a_l \in G$ for some $l \in \mathbb{N}$ as δ-(m_0, n_0)-open set is τ_{n_0}-open and A is dense in (X, τ_{n_0}). But $a_l \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(\{x_k : k \in M_l\})$. Thus, $G \cap \{x_k : k \in M_l\} \neq \emptyset$. Therefore, $G \cap \{x_k : k \in \mathbb{N}\} \neq \emptyset$ and hence, $X = \tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(\{x_k : k \in \mathbb{N}\}$. □

Theorem 5.12. *Let X be a separable space. If X has countable (m, n)-δ-fan tightness for all $m \neq n$, then X is $(a)M^\delta$-separable.*

Proof. Let $< F_k : k \in \mathbb{N} >$ be a sequence of elements of $D^\delta(m_0, n_0)$ for some $m_0 \neq n_0$. So $\tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(F_k) = X$. Let $A = \{a_n : n \in \mathbb{N}\}$ be a countable set which is dense in each (X, τ_n). Consider a partition $\mathbb{N} = M_1 \cup M_2 \cup \ldots$ of $\mathbb{N}$ into pairwise disjoint and infinite sets. Since $\tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(F_k) = X$, for each $n \in \mathbb{N}$, $a_n \in \cap_{k\in M_n} \tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(F_k)$. By countable (m_0, n_0)-δ-strong fan tightness of X, there exists a sequence $< A_k : k \in M_n >$ such that for each $k \in M_n$, $A_k \subseteq F_k$ and $a_n \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(\cup_{k\in M_n} A_k)$. The set $\cup_{k\in\mathbb{N}} A_k$ is δ-(m_0, n_0)-dense in X. Indeed, for any (m_0, n_0)-δ-open subset G, $a_l \in G$ for some $l \in \mathbb{N}$ as δ-(m_0, n_0)-open set is τ_{n_0}-open and A is dense in (X, τ_{n_0}). But $a_l \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta((\cup_{k\in M_l} A_k))$. Thus, $G \cap (\cup_{k\in M_l} A_k) \neq \emptyset$. Therefore, $X = \tau_{(m_0,n_0)}$-$\mathrm{cl}_\delta(\cup_{k\in\mathbb{N}} A_k)$ and hence, $\cup_{k\in\mathbb{N}} A_k \in D^\delta(m_0, n_0)$. So X is $(a)M^\delta$-separable. □

It is shown that separable Fréchet-Urysohn spaces are M-separable [4]. Now we are going to prove the t-version of this result in (a)spaces for each $t \in \{s, p, \alpha, \beta, \delta\}$. For this we define the notion of convergence, Fréchet-Urysohn and Hausdorffness in (a)spaces. A sequence $< x_n >$ in (a)space X is said to be $t_{(m,n)}$-converge to some $x \in X$ if for every (m, n)-t-open set containing x, say U, there exists natural number k such that $x_n \in U$ for all $n \geq k$. If for each $A \subseteq X$ and each $x \in \tau_{(m,n)}$-$\mathrm{cl}_t(A)$ there is a sequence $< x_n : n \in \mathbb{N} >$ in A which $t_{(m,n)}$-converge to x, we call X (m, n)-t-Fréchet-Urysohn. We say X to be (m, n)-t-Hausdorff or (m, n)-t-T_2 space if for every pair of distinct points x and y of X, there exist two disjoint (m, n)-t-open sets containing x and y, respectively.

Definition 5.13. An (a)space X is said to be (m, n)-t-dense in itself if no singelton subset of X is (m, n)-t-open, where $t \in \{s, p, \alpha, \beta, \delta\}$.

Theorem 5.14. *Let X be a separable space such that X is (m, n)-t-dense in itself for all $m \neq n$. If X is (m, n)-t-Fréchet-Urysohn and (m, n)-t-Hausdorff space for all $m \neq n$, then X is $(a)M^t$-separable, where $t \in \{s, p, \alpha, \beta\}$.*

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ be a countable set which is dense in each (X, τ_n). For $t \in \{s, p, \alpha, \beta\}$, let $< A_k : k \in \mathbb{N} >$ be a sequence of elements of $D^t(m_0, n_0)$ for some $m_0 \neq n_0$. Since X is (m_0, n_0)-t-dense in itself, $X - \{a\}$ is not (m_0, n_0)-t-closed. So for every $a \in A$, $a \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_t(X - \{a\}) = X$. Since X is (m_0, n_0)-t-Fréchet-Urysohn, there exists a sequence $< x_n >$ in $X - \{a\}$ such that $< x_n >$ is $t_{(m_0,n_0)}$-converges to a. Further, for each n, $x_n \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_t(A_k)$ for all $k \in \mathbb{N}$. So there exist a sequence $y_n = \ < y_{n,m} >$ in A_n which $t_{(m_0,n_0)}$-converge to x_n. We claim that $a \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_t(\cup_{n\in\mathbb{N}}\ y_n)$. Indeed, for any (m_0, n_0)-t-open-set U containing a, there exist $l \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq l$. In particular, $x_l \in U$ and sequence $< y_{l,m} : m \in \mathbb{N} >$ $t_{(m_0,n_0)}$-converge to x_l. So $y_{l,m} \in U$ for all $m \geq k_1$ for some k_1. Thus, $U \cap (\cup_{n\in\mathbb{N}}\ y_n) \neq \emptyset$ and hence, $a \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_t(\cup_{n\in\mathbb{N}}\ y_n)$. There exists a sequence $z = (z_m)$ in $\cup_{n\in\mathbb{N}}\ y_n$ which $t_{(m_0,n_0)}$-converges to a. Now sequence z and y_n $t_{(m_0,n_0)}$-converges to different points, so $y_n \cap z = F_n \subseteq A_n$ is finite as X is (m_0, n_0)-t-Hausdorff. Since $a \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_t(\{z_m : m \in \mathbb{N}\})$ and $z = \cup F_n$, so we have $a \in \tau_{(m_0,n_0)}$-$\mathrm{cl}_t(\cup F_n)$. This implies that X has countable (m_0, n_0)-t-fan tightness at each point $a \in A$. So X has countable (m, n)-t-fan tightness at each point $a \in A$ for all $m \neq n$. By Theorem 5.9, X is $(a)M^t$-separable. □

Theorem 5.15. *Let $(X, \{\tau_n\})$ be a separable space such that (X, τ_n) is dense in itself for all $n \in \mathbb{N}$. If X is (m, n)-δ-Fréchet-Urysohn and (m, n)-δ-T_2 space for all $m \neq n$, then X is $(a)M^\delta$-separable.*

Proof. Since (X, τ_n) is dense in itself for all $n \in \mathbb{N}$, so X is (m, n)-δ-dense in itself for all $m \neq n$ as every (m, n)-δ-open set is τ_n-open. So proof can be easily done on similar lines of Theorem 5.14. $\square$

In [19], it is shown that every Pytkeev M-separable space is R-separable. We are going to prove the t-version of this result in (a)spaces. An (a)space X is (m, n)-t-Pytkeev if for each $A \subseteq X$ and each $x \in (\tau_{(m,n)}\text{-cl}_t(A)) - A$ there are infinite sets $B_k \subseteq A$, $k \in \mathbb{N}$ such that every (m, n)-t-open set containing x contains some B_k, where $t \in \{s, p, \alpha, \beta, \delta\}$.

Theorem 5.16. *Let X be a separable space such that X is (m, n)-t-Pytkeev for all $m \neq n$, $t \in \{s, p, \alpha, \beta, \delta\}$. If X is $(a)M^t$-separable, then X is $(a)R^t$-separable.*

Proof. Let $< A_k : k \in \mathbb{N} >$ be a sequence of elements of $D^t(m_0, n_0)$ for some $m_0 \neq n_0$, $t \in \{s, p, \alpha, \beta\}$. Let $A = \{a_n : n \in \mathbb{N}\}$ be a countable set which is dense in each (X, τ_n). Fix a partition $\mathbb{N} = M_1 \cup M_2 \cup \ldots$ of $\mathbb{N}$ into pairwise disjoint infinite sets and consider the sequence $< A_m : m \in M_n >$. Since X is $(a)M^t$-separable, there exists finite subsets $F_m \subseteq A_m$, $m \in M_n$ such that $\cup_{m \in M_n} F_m = Y_n$ is (m_0, n_0)-t-dense in X. Let $K_1 = \{n \in \mathbb{N} : a_n \in Y_n\}$ and $K_2 = \mathbb{N} - K_1$. For each $n \in K_2$, $a_n \in (\tau_{(m_0,n_0)}\text{-cl}_t(Y_n)) - Y_n$. Since X is (m_0, n_0)-t-Pytkeev so there exists infinite sets $B_{n,k} \subseteq Y_n$, $k \in \mathbb{N}$ such that for each (m_0, n_0)-t-open set U_{a_n} containing a_n, U_{a_n} contains some $B_{n,k}$. Clearly, each $B_{n,k}$ intersects infinitely with finitely many sets, say $F_{m_1}, F_{m_2}, \ldots, F_{m_k}$. Choose $x_{n,m_k} \in B_{n,k} \cap F_{m_k} \subseteq A_{m_k}$. We define the sequence $< z_n >$ by $z_n = a_n$ if $n \in K_1$, $z_n = x_{n,m_k}$ if $n \in K_2$. Sequence $< z_n >$ is (m_0, n_0)-t-dense in X, so X is $(a)R^t$-separable. $\square$

Let $\{(X_\alpha, \{\tau_{n\alpha}\}_{n \in \mathbb{N}}) : \alpha \in \wedge\}$ be a family of (a)spaces. Let X be the cartesian product of X_α, that is, $X = \prod_{\alpha \in \wedge} X_\alpha$. We define an (a)topology structure $(X, \{\tau_n\})$ on X by taking τ_n as the product topology on X generated by the projections $(\tau_n, \tau_{n\alpha})$ continuous for every $\alpha \in \wedge$. The pair $(X, \{\tau_n\})$ is called the product (a)space.

Theorem 5.17. *Let $(X_1, \{\tau_n\})$ and $(X_2, \{\gamma_n\})$ be two (a)spaces and $t \in \{s, p, \alpha, \beta\}$. If X_1 satisfies $S_1(D(n), D^t(m, n))$ for all $m \neq n$ and (X_2, γ_n) has a countable π-base for all $n \in \mathbb{N}$. Then the product (a)space $(X, \{\sigma_n\})$ satisfies $S_1(D(n), D^t(m, n))$ for all $m \neq n$, where $X = X_1 \times X_2$, σ_n is the product topology on X generated by the continuous projections (σ_n, τ_n) and (σ_n, γ_n) for all n.*

Proof. Let $m_0 \neq n_0$ and let $< F_k : k \in \mathbb{N} >$ be a sequence of σ_{n_0}-dense subsets of X. Let $\{P_k : k \in \mathbb{N}\}$ be a countable π-base for (X_2, γ_{n_0}). Let $\mathbb{N} = M_1 \cup M_2 \cup \ldots \cup M_n \cup \ldots$ be a partition of $\mathbb{N}$ into pairwise disjoint subsets. Fix some $n \in \mathbb{N}$ and consider M_n. Since $X_1 \times P_k \in \sigma_{n_0}$, $(X_1 \times P_k) \cap F_k \neq \emptyset$. Let $L_k = (X_1 \times P_k) \cap F_k \subseteq X$ and consider $A_k = \{x_1 \in X_1 : (x_1, x_2) \in L_k\}$. It is clear that A_k is dense in (X_1, τ_{n_0}) for each $k \in M_n$. Indeed, for any $A \in \tau_{n_0}$, $(A \times P_k) \cap F_k \neq \emptyset$. Let $(y_1, y_2) \in (A \times P_k) \cap F_k$. Therefore, $(y_1, y_2) \in L_k$ which implies that $y_1 \in A_k$. Hence, A_k is dense in (X_1, τ_{n_0}). Since X_1 satisfies $S_1(D(n_0), D^t(m_0, n_0))$ and $< A_k >: k \in M_n\}$ be a sequence of τ_{n_0}-dense subsets of X_1, so there exist $x_{1k} \in A_k$ for all $k \in \mathbb{N}$ such that $\{x_{1k} : k \in \mathbb{N}\}$ is t-(m_0, n_0)-dense in X_1. We claim that the set $F = \{(x_{1k}, x_{2k}) \in L_k\}$ is t-(m_0, n_0)-dense in X. Let G be a (m_0, n_0)-t-open subset of X. Then $W = \sigma_{n_0}\text{-int}(G)$ is a σ_{n_0}-open subset of X. Let $W = U \times V$, $U \in \tau_{n_0}$ and $V \in \gamma_{n_0}$. Since every τ_{n_0}-open set is (m_0, n_0)-t-open, so $U \cap \{x_{1k} : k \in \mathbb{N}\} \neq \emptyset$ which implies that $(U \times V) \cap F \neq \emptyset$. So F is t-(m_0, n_0)-dense in X and hence, X satisfies $S_1(D(n_0), D^t(m_0, n_0))$. $\square$

Corollary 5.18. *Let $(X_1, \{\tau_n\})$ and $(X_2, \{\gamma_n\})$ be two (a)spaces and $t \in \{s, p, \alpha, \beta\}$. If X_1 satisfies $S_1(D(n), D^t(m, n))$ for all $m \neq n$ and (X_2, γ_n) has a countable π-base for all $n \in \mathbb{N}$. Then the product (a)space $(X, \{\sigma_n\})$ satisfies $S_1(D(n), D^\delta(m, n))$ for all $m \neq n$, $X = X_1 \times X_2$, σ_n is the product topology on X generated by the continuous projections (σ_n, τ_n) and (σ_n, γ_n) for all $n \in \mathbb{N}$.*

6. H-Separability and GN-Separability in (*a*)Topological Spaces

In a topological space X, a countable dense subset $A \subseteq X$ is said to be groupable if it can be partitioned as $A = \cup_{k \in \mathbb{N}} A_k$, each A_k non empty finite set and every non empty open set in X intersects all but finitely many k. We denote the collection of all dense sets (resp. groupable sets) in X by D'(resp. D'_{gp}). In an

(a)space $(X, \{\tau_n\})$, a countable dense subset $A \subseteq X$ is said to be (m,n)-t-groupable if it can be partitioned as $A = \cup_{k\in\mathbb{N}} A_k$, each A_k is non empty finite set and every non empty (m,n)-t-open set in X intersects all but finitely many k. We denote the collection of all (m,n)-t-groupable sets in (a)space $(X, \{\tau_n\})$ by $D^t_{gp}(m,n)$.

Definition 6.1. ([5]) A topological space (X, τ) is said to be:

1. H-separable if for each sequence $< A_k : k \in \mathbb{N} >$ of elements of D', one can pick finite $F_k \subseteq A_k$ so that for every non empty open set $G \subseteq X$, the intersection $G \cap F_k$ is non empty for all but finitely many k.

2. GN-separable if $S_1(D', D'_{gp})$ holds.

We define t-version of H-separability and GN-separability in (a)spaces for each $t \in \{s, p, \alpha, \beta, \delta\}$.

Definition 6.2. An (a)space $(X, \{\tau_n\})$ is said to be:

1. $(a)GN^t$-separable if $S_1(D^t(m,n), D^t_{gp}(m,n))$ holds for all $m \neq n$, where $t \in \{s, p, \alpha, \beta, \delta\}$.

2. $(a)H^t$-separable if for every $m \neq n$ and for each sequence $< A_k : k \in \mathbb{N} >$ of elements of $D^t(m,n)$, one can pick finite $F_k \subseteq A_k$ so that for every (m,n)-t-open set, say $G \subseteq X$, the intersection $G \cap F_k$ is non empty for all but finitely many k, $t \in \{s, p, \alpha, \beta, \delta\}$.

It is clear that

$$\begin{array}{ccccc} (a)GN^t\text{-separability} & \Rightarrow & (a)R^t\text{-separability} & \Rightarrow & (a)M^t\text{-separability} \\ & & & & \Uparrow \\ & & & & (a)H^t\text{-separability} \end{array}$$

Lemma 6.3. *Let A and B be subsets of an (a)space $(X, \{\tau_n\})$ with $A \subset B$. If B is countable and $A \in D^t_{gp}(m,n)$, then $B \in D^t_{gp}(m,n)$, where $t \in \{s, p, \alpha, \beta, \delta\}$.*

Proof. Since $A \subset B$ with B countable and $A \in D^t_{gp}(m,n)$, B is a countable dense subset of X. Let $A = \cup_{k\in\mathbb{N}} A_k$ be a partition of A, where each A_k is non empty finite set and every non empty (m,n)-t-open set in X intersects all but finitely many k. Without loss of generality we can assume that all $A_k' s$ are pairwise disjoint. If $B - A$ is a non empty finite set, then $B = (\cup_{k\in\mathbb{N}} A_k) \cup (B - A)$ witness that $B \in D^t_{gp}(m,n)$. If $B - A$ is a countably infinite set, let $B - A = \{b_k : k \in \mathbb{N}\}$. Then $\{B_k : B_k = A_k \cup \{b_k\}, k \in \mathbb{N}\}$ is a partition of non empty finite subsets of X such that every non empty (m,n)-t-open set in X intersects all but finitely many k. Thus, $B \in D^t_{gp}$. □

Theorem 6.4. *An (a)space X is $(a)GN^t$-separable if and only if X is $(a)R^t$-separable and each (m,n)-t-dense subset of X contains an (m,n)-t-groupable set for all $m \neq n$, $t \in \{s, p, \alpha, \beta, \delta\}$.*

Proof. It is obvious that $(a)GN^t$-separability implies $(a)R^t$-separability. Let A be a (m,n)-t-dense subset of X. Consider the constant sequence $< A >$, that is, Sequence $< A_k : A_k = A : k \in \mathbb{N} >$. Since X is $(a)GN^t$-separable, there exists $a_k \in A_k$ such that $\{a_k : k \in \mathbb{N}\}$ is (m,n)-t-groupable. Thus, A contains an (m,n)-t-groupable set.

Conversely, Let $< A_k : k \in \mathbb{N} >$ be a sequence of (m,n)-t-dense subsets of X. By $(a)R^t$-separability of X, for each $k \in \mathbb{N}$ there exists $a_k \in A_k$ such that $\{a_k : k \in \mathbb{N}\}$ is (m,n)-t-dense in X. By hypothesis, there exists a (m,n)-t-groupable set, say B, such that $B \subseteq \{a_k : k \in \mathbb{N}\}$. By Lemma 6.3, $\{a_k : k \in \mathbb{N}\} \in D^t_{gp}(m,n)$ and hence, X is $(a)GN^t$-separable. □

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