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On a Fiber-wise Homogeneous Deformation of the Sasaki Metric

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Abstract. In this paper, we firstly determine a new deformed Sasaki type lift of a metric from a Riemannian manifold to its coframe bundle and investigate a few special (1.1)-tensor structures (i.e. almost Hermit structures) in the coframe bundle equipped with this type lift.

1. Introduction

Inspired by the work of Sasaki [10] some authors continued investigations on natural lifts of metrics, i.e. on deformed Sasaki type lifts in different bundles (see for example [1-4, 6-9, 11]). It is well known that any vector bundle (tangent, cotangent and tensor bundles) one always has the global zero section. But there is the other situation, that of the coframe bundle, which is a $GL(n,R)$-principal bundle without zero section. Using this property we define a homogeneous type deformed Sasaki metric in the coframe bundle. This paper is devoted to the investigation of this lift in the coframe bundle. In Section 2 we briefly describe the definitions and results that are needed later, after which a homogeneous type deformed Sasaki lift (metric) $\tilde{g}$ of a Riemannian metric g to coframe bundle $F^*(M_n)$ is constructed in Section 3. The Levi-Civita connection of $\tilde{g}$ is studied in Section 4. A few special (1,1)-tensor structures, i.e. almost Hermit structures in the linear co-frame bundle equipped with the lift $\tilde{g}$ of a Riemannian metric g is investigated in Section 5.

2. Preliminaries

In this section we shall summarize briefly the basic definitions and results which will be used later. Let M_n be an $n-$dimensional differentiable manifold of class C^∞, and $F^*(M_n) = \{(x,u^*)\,|x \in M_n\,,\ u^* :$ coframe for a dual space $T^*_x(M_n)\}$ be the linear coframe bundle over M_n. We denote by π the natural projection of $F^*(M_n)$ on M_n defined by $\pi(x,u^*) = x$. If $(U; x^1, x^2, ..., x^n)$ is a system of local coordinates in M_n, then a coframe $u^* = (X^\alpha) = (X^1, X^2, ..., X^n)$ for $T^*_x(M_n)$ can be expressed uniquely in the form $X^\alpha = X^\alpha_i (dx^i)_x$ and hence

$$\left(\pi^{-1}(U); x^1, x^2, ..., x^n, X^1_1, X^1_2, ..., X^n_n\right)$$

is a system of local coordinates in $F^*(M_n)$. The indices $i, j, k, ..., \alpha, \beta, \gamma, ...$ have range in $\{1, 2, ..., n\}$, while indices $A, B, C, ...$ have range in

$$\left\{1, ..., n, n+1, ..., n+n^2\right\}.$$

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We put $h_\alpha = \alpha \cdot n + h$ ($h_\alpha = n+1, n+2, ..., n+n^2$). Summation over repeated indices is always implied.

We denote by $\Im^r_s(M_n)$ the set of all differentiable tensor fields of type (r,s) on M_n. We consider a symmetric linear connection ∇ on M_n with components Γ^k_{ij}. It is known that $T(F^*(M_n)) = H(F^*(M_n)) \oplus V(F^*(M_n))$, where $H(F^*(M_n))$ and $V(F^*(M_n))$ are the horizontal and the vertical distributions of a linear coframe bundle $F^*(M_n)$, respectively. Hence every $X \in \Im^1_0(F^*(M_n))$ has the unique decomposing $X = {}^H\tilde{X} + {}^V\tilde{X}$, ${}^H\tilde{X} \in H(F^*(M_n))$, ${}^V\tilde{X} \in V(F^*(M_n))$.

Let $V = V^i\partial_i$ and $\omega = \omega_i dx^i$ be the local expressions in $U \subset M_n$ of a vector and a covector (1-form) fields $V \in \Im^1_0(M_n)$ and $\omega \in \Im^0_1(M_n)$, respectively. Then the complete and horizontal lifts ${}^CV, {}^HV \in \Im^1_0(F^*(M_n))$ of V and the $\beta-$th vertical lifts ${}^{V_\beta}\omega \in \Im^1_0(F^*(M_n))$ $(\beta = 1,2,...,n)$ of ω are defined by

$$ {}^CV = V^i\partial_i - X^\alpha_m \partial_i V^m \partial_{i_\alpha}, \quad {}^HV = V^i\partial_i + X^\alpha_m \Gamma^m_{ik} V^k \partial_{i_\alpha}, \tag{1} $$

$$ {}^{V_\beta}\omega = \sum_i \delta^\alpha_\beta \omega_i \partial_{i_\alpha} \tag{2} $$

with respect to the natural frame $\{\partial_i, \partial_{i_\alpha}\} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial X^\alpha_i}\right\}$, respectively (see [3] for more details). The vertical lift of a smooth function f on M_n is a function ${}^Vf$ on $F^*(M_n)$ defined by ${}^Vf = f \circ \pi$.

Let (U, x^i) be a coordinate system in M_n. In $U \subset M_n$, we put

$$ X_{(i)} = \frac{\partial}{\partial x^i}, \theta^{(i)} = dx^i, i = 1,2,...,n. $$

Taking into account (1) and (2), we easily see that the components of ${}^HX_{(i)}$ and ${}^{V_\alpha}\theta^{(i)}$ are given by

$$ {}^HX_{(i)} = \left(A_i{}^H\right) = \begin{pmatrix} \delta^h_i \\ X^\alpha_j \Gamma^j_{ih} \end{pmatrix}, \tag{3} $$

$$ {}^{V_\alpha}\theta^{(i)} = \left(A_{i_\alpha}{}^H\right) = \begin{pmatrix} 0 \\ \delta^\beta_\alpha \delta^i_h \end{pmatrix} \tag{4} $$

with respect to the natural frame $\{\partial_j, \partial_{j_\beta}\}$, respectively, where δ^h_i are the Kronecker symbols. This $n+n^2$ vector fields are linearly independent and generate, respectively, the horizontal distribution of linear connection ∇ and vertical distribution of the linear bundle $F^*(M_n)$. The set $\{{}^HX_{(i)}, {}^{V_\alpha}\theta^{(i)}\}$ is called the frame adapted to the linear connection ∇ on $\pi^{-1}(U) \subset F^*(M_n)$. By setting

$$ D_i = {}^HX_{(i)}, \quad D_{i_\alpha} = {}^{V_\alpha}\theta^{(i)}, $$

we write the adapted frame as $\{D_I\} = \{D_i, D_{i_\alpha}\}$. From equations (1)-(4), we see that ${}^HV$ and ${}^{V_\alpha}\omega$ have respectively, components

$$ {}^HV = V^iD_i, {}^HV = \left({}^HV^I\right) = \begin{pmatrix} V^i \\ 0 \end{pmatrix}, \tag{5} $$

$$ {}^{V_\beta}\omega = \sum_i \omega_i \delta^\alpha_\beta D_{i_\alpha}, {}^{V_\beta}\omega = \left({}^{V_\beta}\omega^I\right) = \begin{pmatrix} 0 \\ \delta^\alpha_\beta \omega_i \end{pmatrix} \tag{6} $$

with respect to the adapted frame $\{D_I\}$.

Let us consider the local $1-$forms $\tilde{\eta}^I$ in $\pi^{-1}(U)$ defined by

$$ \tilde{\eta}^I = \bar{A}^I{}_J dx^J, $$

where

$$A^{-1} = (\bar{A}^I{}_J) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{j_\beta} \\ \bar{A}^{i_\alpha}{}_j & \bar{A}^{i_\alpha}{}_{j_\beta} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ -X^\alpha_m \Gamma^m_{ij} & \delta^\alpha_\beta \delta^j_i \end{pmatrix}. \tag{7}$$

The matrix (7) is the inverse of the matrix

$$A = (A_K{}^J) = \begin{pmatrix} A_k{}^j & A_{k_\gamma}{}^j \\ A_k{}^{j_\beta} & A_{k_\gamma}{}^{j_\beta} \end{pmatrix} = \begin{pmatrix} \delta^j_k & 0 \\ X^\beta_m \Gamma^m_{jk} & \delta^\beta_\gamma \delta^k_j \end{pmatrix}$$

of the transformation $D_K = A_K{}^J \partial_J$ (see (3) and (4)). It is easy to establish that the set $\left\{\tilde{\eta}^I\right\}$ is the coframe dual to the adapted frame $\{D_K\}$, i.e.

$$\tilde{\eta}^I(D_K) = \bar{A}^I{}_J A_K{}^J = \delta^I_K.$$

The following theorem holds.

Theorem 2.1. *Let M_n be a Riemannian manifold with metric g, let ∇ be the Levi-Civita connection and let R be the Riemannian curvature tensor. Then the Lie bracket of the linear coframe bundle $F^*(M_n)$ of M_n satisfies the following:*
i)

$$[{}^{V_\beta}\omega, {}^{V_\gamma}\theta] = 0, \tag{8}$$

ii)

$$[{}^H X, {}^{V_\beta}\omega] = {}^{V_\beta}(\nabla_X \omega), \tag{9}$$

ii)

$$[{}^H X, {}^H Y] = {}^H[X, Y] + \gamma(R(X, Y)) \tag{10}$$

for all $X, Y \in \Im^1_0(M_n)$ and $\omega, \theta \in \Im^0_1(M_n)$.

Proof. In the case when $I = i$, by using (2), we see that the left hand side of (8) reduces to

$$[{}^{V_\beta}\omega, {}^{V_\gamma}\theta]^I = [{}^{V_\beta}\omega, {}^{V_\gamma}\theta]^i = {}^{V_\beta}\omega^K \partial_K {}^{V_\gamma}\theta^i - {}^{V_\gamma}\theta^K \partial_K {}^{V_\beta}\omega^i = 0.$$

In the case $I = i_\alpha$ we have

$$[{}^{V_\beta}\omega, {}^{V_\gamma}\theta]^I = [{}^{V_\beta}\omega, {}^{V_\gamma}\theta]^{i_\alpha} = {}^{V_\beta}\omega^K \partial_K {}^{V_\gamma}\theta^{i_\alpha} - {}^{V_\gamma}\theta^K \partial_K {}^{V_\beta}\omega^{i_\alpha}$$
$$= {}^{V_\beta}\omega^k \partial_k {}^{V_\gamma}\theta^{i_\alpha} + {}^{V_\beta}\omega^{k_\sigma} \partial_{k_\sigma} {}^{V_\gamma}\theta^{i_\alpha} - {}^{V_\gamma}\theta^k \partial_k {}^{V_\beta}\omega^{i_\alpha} - {}^{V_\gamma}\theta^{k_\sigma} \partial_{k_\sigma} {}^{V_\beta}\omega^{i_\alpha}$$
$$= \delta^\alpha_\gamma \delta^\sigma_\beta \sum_k \omega_k \partial_{k_\sigma} \theta_i - \delta^\sigma_\gamma \delta^\alpha_\beta \sum_k \theta_k \partial_{k_\sigma} \omega_i = 0.$$

ii) In the case $I = i$, from (1) and (2) we have

$$[{}^H X, {}^{V_\beta}\omega]^I = [{}^H X, {}^{V_\beta}\omega]^i = {}^H X^K \partial_K {}^{V_\beta}\omega^i - {}^{V_\beta}\omega^K \partial_K {}^H X^i$$
$$= -{}^{V_\beta}\omega^k \partial_k {}^H X^i - {}^{V_\beta}\omega^{k_\sigma} \partial_{k_\sigma} {}^H X^i = 0.$$

In the case $I = i_\alpha$ we obtain

$$[{}^H X, {}^{V_\beta}\omega]^I = [{}^H X, {}^{V_\beta}\omega]^{i_\alpha} = {}^H X^K \partial_K {}^{V_\beta}\omega^{i_\alpha} - {}^{V_\beta}\omega^K \partial_K {}^H X^{i_\alpha}$$
$$= {}^{V_\beta}\omega^k \partial_k {}^H X^{i_\alpha} + {}^{V_\beta}\omega^k \partial_k {}^H X^{i_\alpha} - {}^{V_\beta}\omega^{k_\sigma} \partial_{k_\sigma} {}^H X^{i_\alpha} - {}^{V_\beta}\omega^{k_\sigma} \partial_{k_\sigma} {}^H X^{i_\alpha}$$

$$= \delta^\alpha_\beta X^k \partial_k \omega_i - \delta^\sigma_\beta \omega_k \partial_{k_\sigma}(X^\alpha_j \Gamma^j_{il} X^l) = \delta^\alpha_\beta X^k \partial_k \omega_i - \delta^\sigma_\beta \omega_k \delta^k_j \delta^\alpha_\sigma \Gamma^j_{il} X^l$$
$$= \delta^\alpha_\beta (X^k \partial_k \omega_i - X^k \Gamma^l_{ik} \omega_l) = \delta^\alpha_\beta \hat{\nabla}_X \omega_i$$

from which, due to symmetry of connection ∇, it follows that

$$[{}^H X, {}^{V_\beta}\omega] = {}^{V_\beta}(\nabla_X \omega).$$

iii) In the case when $I = i$, by using (1), we see that left hand side of (10) reduces to

$$[{}^H X, {}^H Y]^I = [{}^H X, {}^H Y]^i = {}^H X^K \partial_K {}^H Y^i - {}^H Y^K \partial_K {}^H X^i = {}^H X^k \partial_k {}^H Y^i$$
$$+ {}^H X^{k_\sigma} \partial_{k_\sigma} {}^H Y^i - {}^H Y^k \partial_k {}^H X^i - {}^H Y^{k_\sigma} \partial_{k_\sigma} {}^H X^i = X^k \partial_k Y^i - Y^k \partial_k X^i = [X, Y]^i$$
$$= {}^H [X, Y]^i.$$

In the case $I = i_\alpha$ we have

$$[{}^H X, {}^H Y]^I = [{}^H X, {}^H Y]^{i_\alpha} = {}^H X^K \partial_K {}^H Y^{i_\alpha} - {}^H Y^K \partial_K {}^H X^{i_\alpha} = {}^H X^k \partial_k {}^H Y^{i_\alpha}$$
$$+ {}^H X^{k_\sigma} \partial_{k_\sigma} {}^H Y^{i_\alpha} - {}^H Y^k \partial_k {}^H X^{i_\alpha} - {}^H Y^{k_\sigma} \partial_{k_\sigma} {}^H X^{i_\alpha} = X^k \partial_k (X^\alpha_j \Gamma^j_{il} Y^l)$$
$$+ (X^\sigma_m \Gamma^m_{ks} X^s) \partial_{k_\sigma} (X^\alpha_j \Gamma^j_{il} Y^l) - Y^k \partial_k (X^\alpha_j \Gamma^j_{il} X^l) - (X^\sigma_m \Gamma^m_{ks} Y^s) \partial_{k_\sigma} (X^\alpha_j \Gamma^j_{il} X^l)$$
$$= X^k X^\alpha_j (\partial_k \Gamma^j_{il}) Y^l + X^k X^\alpha_j \Gamma^j_{il} \partial_k Y^l + X^\alpha_m X^s Y^l \Gamma^m_{ks} \Gamma^k_{il} - Y^k X^\alpha_j (\partial_k \Gamma^j_{il}) X^l$$
$$- Y^k X^\alpha_j \Gamma^j_{il} \partial_k X^l - X^\alpha_m Y^s X^l \Gamma^m_{ks} \Gamma^k_{il} = X^\alpha_j \Gamma^j_{il} [X, Y]^l + X^k Y^l X^\alpha_j (\partial_k \Gamma^j_{il} - \partial_l \Gamma^j_{ik}$$
$$+ \Gamma^j_{sk} \Gamma^s_{il} - \Gamma^j_{sl} \Gamma^s_{ik}) = {}^H [X, Y]^{i_\alpha} + X^k Y^l X^\alpha_j R^j_{kli} = {}^H [X, Y]^{i_\alpha} + \delta^\alpha_\beta X^\beta_j R(X, Y)^j_i.$$

Therefore

$$[{}^H X, {}^H Y] = {}^H [X, Y] + \gamma(R(X, Y))$$

and Theorem 2.1 is proved. □

Remark 2.2. Using equality (2), it is easy to establish that a vertical vector field $\gamma(R(X, Y)) \in \mathfrak{I}^1_0(F^*(M_n))$ can be represented as

$$\gamma(R(X, Y)) = \sum_{\beta=1}^{n} {}^{V_\beta}(X^\beta \circ R(X, Y)) . \tag{11}$$

3. Homogeneous Type Deformed Sasaki Metric

Let (M_n, g) be a Riemannian manifold. The diagonal lift (or the Sasaki lift) ${}^D g$ of Riemannian metric g to coframe bundle $F^*(M_n)$ is defined by

$${}^D g = g_{ij} dx^i \otimes dx^j + \delta_{\alpha\beta} g^{ij} \delta X^\alpha_i \otimes \delta X^\beta_j$$

and satisfies the following conditions:

$${}^D g({}^H X, {}^H Y) = {}^V(g(X, Y)) = g(X, Y) \circ \pi,$$
$${}^D g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = \delta_{\alpha\beta} \, {}^V(g^{-1}(\omega, \theta)) = \delta_{\alpha\beta} g^{-1}(\omega, \theta) \circ \pi,$$
$${}^D g({}^H X, {}^{V_\beta}\theta) = 0$$

for all $X, Y \in \Im^1_0(M_n)$ and $\omega, \theta \in \Im^0_1(M_n)$, where $\delta X^\alpha_i = dX^\alpha_i - \Gamma^m_{ki} X^\alpha_m dx^k$ and g^{ij} denote contravariant components of g.

Let us consider the homothety $h_\lambda : (x, u^*) \to (x, \lambda u^*)$, $\lambda \in R_+$ on the fibers of the linear coframe bundle $F^*(M_n)$. Then ${}^D g$ is transformed as follows:

$${}^D g(x, \lambda u^*) = g_{ij} dx^i \otimes dx^j + \delta_{\alpha\beta} \lambda^2 g^{ij} \delta X^\alpha_i \otimes \delta X^\beta_j, \forall \lambda \in R_+.$$

We see, that the metric ${}^D g$ is not homogeneous, i.e.

$${}^D g(x, u^*) \neq {}^D g(x, \lambda u^*).$$

Now, we define a new lift $\tilde{g}$ of a Riemannian metric g to the coframe bundle $F^*(M_n)$ as follows:

$$\tilde{g} = g_{ij} dx^i \otimes dx^j + \tfrac{1}{h} \delta_{\alpha\beta} g^{ij} \delta X^\alpha_i \otimes \delta X^\beta_j,$$

where h is a function defined as

$$h = \sum_{\alpha=1}^{n} \|X^\alpha\|^2 = \sum_{\alpha=1}^{n} g^{ij} X^\alpha_i X^\alpha_j = \sum_{\alpha=1}^{n} g^{-1}(X^\alpha, X^\alpha). \tag{12}$$

It is easy to see that $\tilde{g}$ is homogeneous with respect to X^α_i, i.e.

$$\tilde{g}(x, \lambda u^*) = g_{ij} dx^i \otimes dx^j + \tfrac{\lambda^2}{\lambda^2 \cdot h} \delta_{\alpha\beta} g^{ij} \delta X^\alpha_i \otimes \delta X^\beta_j = \tilde{g}(x, u^*), \forall \lambda \in R_+.$$

Remark 3.1. Since $u^* = (X^1, X^2, ..., X^n) \neq 0$ is a basis of the cotangent space $T^*_x(M_n)$, the condition $h \neq 0$ is fulfilled at each point $x \in M_n$ and in the coframe bundle does not exist zero section. This means that the metric $\tilde{g}$ is defined in the linear coframe bundle $F^*(M_n)$.

We get, without difficulties:

Theorem 3.2. *The following properties hold:*

1°. *The pair $(F^*(M_n), \tilde{g})$ is a Riemannian space, depending only on the metric g.*

2°. *$\tilde{g}$ is homogeneous on the linear coframe bundle $F^*(M_n)$.*

3°. *The distributions H and V are orthogonal with respect to $\tilde{g}$:*

$$\tilde{g}({}^H\tilde{X}, {}^V\tilde{X}) = 0, \forall X, Y \in \Im^1_0(F^*(M_n)).$$

We can write $\tilde{g}$ in the form

$$\tilde{g} = \tilde{g}^H + \tilde{g}^V, \tilde{g}^H = g_{ij} dx^i \otimes dx^j, \tilde{g}^V = \tfrac{1}{h} \delta_{\alpha\beta} g^{ij} \delta X^\alpha_i \otimes \delta X^\beta_j.$$

The metric $\tilde{g}$ has components

$$\left(\tilde{g}_{IJ}\right) = \begin{pmatrix} g_{ij} & 0 \\ 0 & \frac{1}{h}\delta_{\alpha\beta} g^{ij} \end{pmatrix} \tag{13}$$

with respect to the adapted frame $\{D_I\}$ in $F^*(M_n)$. From (13) it easily follows that if g is a Riemannian metric in M_n, then $\tilde{g}$ is a Riemannian metric in $F^*(M_n)$ and it is called a homogeneous type deformed Sasaki metric.

It is easily to verify that the inverse matrix $(\tilde{g}^{IJ})$ of matrix $\left(\tilde{g}_{IJ}\right)$ is as follows:

$$\left(\tilde{g}^{IJ}\right) = \begin{pmatrix} g^{ij} & 0 \\ 0 & h\delta^{\alpha\beta} g_{ij} \end{pmatrix}$$

with respect to the adapted frame $\{D_I\}$ in $F^*(M_n)$.

Also, we can represent the metric $\tilde{g}$ by the following global formulas:

$$\tilde{g}({}^{H}X,{}^{H}Y)=g(X,Y)\circ\pi,$$
$$\tilde{g}({}^{V_\alpha}\omega,{}^{V_\beta}\theta)=\tfrac{1}{h}\delta_{\alpha\beta}g^{-1}(\omega,\theta)\circ\pi, \tag{14}$$
$$\tilde{g}({}^{H}X,{}^{V_\beta}\theta)=0$$

for all vector fields $X,Y\in\Im^1_0(M_n)$ and covector fields (1−forms) $\omega,\theta\in\Im^0_1(M_n)$. We recall that any element $t\in\Im^0_2(F^*(M_n))$ is completely determined by its action on vector fields of type ${}^{H}X$ and ${}^{V_\alpha}\omega$. From this it follows that $\tilde{g}$ is completely determined by (14).

Using (5), (6), (7) and $D_K=A_K{}^{J}\partial_J$, after straightforward computations, we obtain:

$$\begin{aligned}
{}^{H}X(h)&={}^{H}X\left(\sum_{\alpha=1}^{n}g^{-1}(X^\alpha,X^\alpha)\right)=(X^iD_i)\left(\sum_{\alpha=1}^{n}g^{-1}(X^\alpha,X^\alpha)\right)\\
&=X^i\left(\partial_i+\Gamma^m_{ik}X^\gamma_m\partial_{k_\gamma}\right)\left(\sum_{\alpha=1}^{n}g^{-1}(X^\alpha,X^\alpha)\right)=X^i\partial_i\left(\sum_{\alpha=1}^{n}g^{-1}(X^\alpha,X^\alpha)\right)\\
&+\Gamma^m_{ik}X^iX^\gamma_m\partial_{k_\gamma}\left(\sum_{\alpha=1}^{n}g^{-1}(X^\alpha,X^\alpha)\right)=X^i\sum_{\alpha=1}^{n}(\partial_ig^{-1})(X^\alpha,X^\alpha)\\
&+\Gamma^m_{ik}X^iX^\gamma_m\partial_{k_\gamma}\left(\sum_{\alpha=1}^{n}g^{-1}(X^\alpha,X^\alpha)\right)=X^i\sum_{\alpha=1}^{n}(\partial_ig^{rs})X^\alpha_rX^\alpha_s\\
&+\Gamma^m_{ik}X^iX^\gamma_m\partial_{k_\gamma}\left(\sum_{\alpha=1}^{n}g^{rs}X^\alpha_rX^\alpha_s\right)=X^i\sum_{\alpha=1}^{n}(-\Gamma^r_{il}g^{ls}-\Gamma^s_{il}g^{rl})X^\alpha_rX^\alpha_s\\
&+\sum_{\alpha=1}^{n}\Gamma^m_{ik}X^ig^{rs}X^\gamma_m\delta^k_r\delta^\alpha_\gamma X^\alpha_s+\sum_{\alpha=1}^{n}\Gamma^m_{ik}X^ig^{rs}X^\gamma_m\delta^k_s\delta^\alpha_\gamma X^\alpha_r\\
&=-X^i\Gamma^r_{il}\sum_{\alpha=1}^{n}g^{ls}X^\alpha_rX^\alpha_s-X^i\Gamma^s_{il}\sum_{\alpha=1}^{n}g^{rl}X^\alpha_rX^\alpha_s+X^i\Gamma^m_{ir}\sum_{\alpha=1}^{n}g^{rs}X^\alpha_mX^\alpha_s\\
&+X^i\Gamma^m_{is}\sum_{\alpha=1}^{n}g^{rs}X^\alpha_mX^\alpha_r=0,
\end{aligned}\tag{15}$$

and

$$\begin{aligned}
{}^{V_\beta}\omega(h)&=\delta^\beta_\gamma\omega_iD_{i_\gamma}\left(\sum_{\alpha=1}^{n}g^{rs}X^\alpha_rX^\alpha_s\right)=\delta^\beta_\gamma\omega_i\partial_{i_\gamma}\left(\sum_{\alpha=1}^{n}g^{rs}X^\alpha_rX^\alpha_s\right)\\
&=\delta^\beta_\gamma\omega_i\sum_{\alpha=1}^{n}\left(\partial_{i_\gamma}X^\alpha_r\right)g^{rs}X^\alpha_s+\delta^\beta_\gamma\omega_i\sum_{\alpha=1}^{n}\left(\partial_{i_\gamma}X^\alpha_s\right)g^{rs}X^\alpha_r\\
&=\delta^\beta_\gamma\omega_i\sum_{\alpha=1}^{n}\delta^\gamma_\alpha\delta^i_rg^{rs}X^\alpha_s+\delta^\beta_\gamma\omega_i\sum_{\alpha=1}^{n}\delta^\gamma_\alpha\delta^i_sg^{rs}X^\alpha_r=\sum_{\alpha=1}^{n}\delta^\beta_\alpha g^{rs}X^\alpha_s\omega_r\\
&+\sum_{\alpha=1}^{n}\delta^\beta_\alpha g^{rs}X^\alpha_r\omega_s=2\sum_{\alpha=1}^{n}\delta^\beta_\alpha g^{-1}\left(X^\alpha,\omega\right).
\end{aligned}\tag{16}$$

for all $X \in \Im^1_0(M_n)$ and $\omega \in \Im^0_1(M_n)$.

From (15) and (16) it immediately follows that

$$^{H}X(\tfrac{1}{h}) = 0, \tag{17}$$

$$^{V_\alpha}\omega(\tfrac{1}{h}) = \frac{-2\sum_{\sigma=1}^{n}\delta^{\alpha}_{\sigma}g^{-1}(X^{\sigma},\omega)}{h^2}. \tag{18}$$

4. Levi-Civita Connection of $\tilde{g}$

It is well-known that the Levi-Civita connection ∇ of a Riemannian metric g is given by Koszul formula

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z)$$
$$-g([Y,Z],X) + g([Z,X],Y) \tag{19}$$

for all vector fields $X, Y, Z \in \Im^1_0(M_n)$.

Using (5), (6), (11), (14), (17), (18) and (19), we have

Theorem 4.1. *Let M_n be a Riemannian manifold with metric g and $\tilde{\nabla}$ be the Levi-Civita connection of the linear coframe bundle $F^*(M_n)$ equipped with the metric $\tilde{g}$. Then $\tilde{\nabla}$ satisfies:*
i)

$$\tilde{\nabla}_{^H X}{}^{H}Y = {}^{H}(\nabla_X Y) + \frac{1}{2}\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^{\sigma}\circ R(X,Y)),$$

ii)

$$\tilde{\nabla}_{^H X}{}^{V_\alpha}\omega = {}^{V_\alpha}(\nabla_X \omega) + \frac{1}{2h}\sum_{\sigma=1}^{n}\delta_{\alpha\sigma}{}^{H}(X^{\sigma}\circ R(\quad, X)\tilde{\omega}),$$

iii)

$$\tilde{\nabla}_{^{V_\alpha}\omega}{}^{H}Y = \frac{1}{2h}\sum_{\sigma=1}^{n}\delta_{\alpha\sigma}{}^{H}(X^{\sigma}\circ R(\quad, Y)\tilde{\omega}), \tag{20}$$

iv)

$$\tilde{\nabla}_{^{V_\alpha}\omega}{}^{V_\beta}\theta = -\tilde{g}({}^{V_\alpha}\omega, \sum_{\sigma=1}^{n}{}^{V_\sigma}X^{\sigma})^{V_\beta}\theta - \tilde{g}({}^{V_\beta}\theta, \sum_{\sigma=1}^{n}{}^{V_\sigma}X^{\sigma})^{V_\alpha}\omega + \tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}\theta)\sum_{\sigma=1}^{n}{}^{V_\sigma}X^{\sigma},$$

for all $X, Y \in \Im^1_0(M_n)$, $\omega, \theta \in \Im^0_1(M_n)$, where $\tilde{\omega} = g^{-1}\circ\omega \in \Im^1_0(M_n)$, $\tilde{X}^{\alpha} = g^{-1}\circ X^{\alpha} \in \Im^1_0(M_n)$ with respect to the adapted frame $\{D_I\}$.

Proof. i) By help of Koszul formula (19), (10) and (11), we have

$$2\tilde{g}(\tilde{\nabla}_{^H X}{}^{H}Y, {}^{H}Z) = {}^{H}X(\tilde{g}({}^{H}Y, {}^{H}Z)) + {}^{H}Y(\tilde{g}({}^{H}Z, {}^{H}X)) - {}^{H}Z(\tilde{g}({}^{H}X, {}^{H}Y))$$
$$-\tilde{g}({}^{H}X, [{}^{H}Y, {}^{H}Z]) + \tilde{g}({}^{H}Y, [{}^{H}Z, {}^{H}X]) + \tilde{g}({}^{H}Z, [{}^{H}X, {}^{H}Y]) = 2g(\nabla_X Y, Z) \tag{21}$$

and

$$2\tilde{g}(\tilde{\nabla}_{^H X}{}^{H}Y, {}^{V_\alpha}\omega) = {}^{H}X(\tilde{g}({}^{H}Y, {}^{V_\alpha}\omega)) + {}^{H}Y(\tilde{g}({}^{V_\alpha}\omega, {}^{H}X)) - {}^{V_\alpha}\omega(\tilde{g}({}^{H}X, {}^{H}Y))$$
$$-\tilde{g}({}^{H}X, [{}^{H}Y, {}^{V_\alpha}\omega]) + \tilde{g}({}^{H}Y, [{}^{V_\alpha}\omega, {}^{H}X]) + \tilde{g}({}^{V_\alpha}\omega, [{}^{H}X, {}^{H}Y]) = {}^{V_\alpha}\omega(g(X,Y))$$

$$-\tilde{g}({}^{H}X, {}^{V_\alpha}(\nabla_Y\omega)) + \tilde{g}({}^{H}Y, -{}^{V_\alpha}(\nabla_Y\omega)) + \tilde{g}({}^{V_\alpha}\omega, {}^{H}[X,Y] + \gamma(R(X,Y)))$$

$$= \tilde{g}({}^{V_\alpha}\omega, \gamma(R(X,Y))) = \tilde{g}\left({}^{V_\alpha}\omega, \sum_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(X,Y))\right). \tag{22}$$

By combining of (21) and (22), we obtain:

$$\tilde{\nabla}_{{}^{H}X}{}^{H}Y = {}^{H}(\nabla_X Y) + \frac{1}{2}\sum_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(X,Y)).$$

ii) By help of Koszul formula (19) and (9), we get:

$$2\tilde{g}(\tilde{\nabla}_{{}^{H}X}{}^{V_\alpha}\omega, {}^{H}Y) = {}^{H}X(\tilde{g}({}^{V_\alpha}\omega, {}^{H}Y)) + {}^{V_\alpha}\omega(\tilde{g}({}^{H}Y, {}^{H}X)) - {}^{H}Y(\tilde{g}({}^{H}X, {}^{V_\alpha}\omega))$$

$$-\tilde{g}({}^{H}X, [{}^{V_\alpha}\omega, {}^{H}Y]) + \tilde{g}({}^{V_\alpha}\omega, [{}^{H}Y, {}^{H}X]) + \tilde{g}({}^{H}Y, [{}^{H}X, {}^{V_\alpha}\omega])$$

$$= \tilde{g}({}^{V_\alpha}\omega, \gamma(R(Y,X))) = \tilde{g}({}^{V_\alpha}\omega, \sum_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(Y,X)))$$

$$= \tilde{g}\left(\sum_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(Y,X)), {}^{V_\alpha}\omega\right) = \sum_{\sigma=1}^{n} \tilde{g}({}^{V_\sigma}(X^\sigma \circ R(Y,X)), {}^{V_\alpha}\omega)$$

$$= \sum_{\sigma=1}^{n} \frac{1}{h}\delta_{\alpha\sigma} g^{-1}(X^\sigma \circ R(Y,X), \omega).$$

Using

$$g^{-1}(X^\sigma \circ R(Y,X), \omega) = g^{ij}(X^\sigma \circ R(Y,X))_i \omega_j = g^{ij} X^\sigma_s R_{kli}{}^{s} Y^k X^l \omega_j$$

$$= X^\sigma_s R_{kli}{}^{s} Y^k X^l \tilde{\omega}^i = g_{km} X^\sigma_s R^{m}_{\cdot\, li}{}^{s} Y^k X^l \tilde{\omega}^i = g(X^\beta(g^{-1} \circ R(\ \ , X)\tilde{\omega}), Y)$$

$$= \tilde{g}({}^{H}(X^\beta(g^{-1} \circ R(\ \ , X)\tilde{\omega})), {}^{H}Y)$$

and

$${}^{H}X(\tfrac{1}{h}) = 0,$$

we have

$$2\tilde{g}(\tilde{\nabla}_{{}^{H}X}{}^{V_\alpha}\omega, {}^{H}Y) = \frac{1}{h}\sum_{\sigma=1}^{n} \delta_{\alpha\sigma}\tilde{g}({}^{H}(X^\sigma(g^{-1} \circ R(\ \ , X)\tilde{\omega})), {}^{H}Y).$$

On the other hand,

$$2\tilde{g}(\tilde{\nabla}_{{}^{H}X}{}^{V_\alpha}\omega, {}^{V_\beta}\theta) = {}^{H}X(\tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}\theta)) - \tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_X\theta))$$

$$+\tilde{g}({}^{V_\beta}\theta, {}^{V_\alpha}(\nabla_X\omega)) = \tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_X\theta)) + \tilde{g}({}^{V_\beta}\theta, {}^{V_\alpha}(\nabla_X\omega))$$

$$-\tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_X\theta)) + \tilde{g}({}^{V_\beta}\theta, {}^{V_\alpha}(\nabla_X\omega)) = 2\tilde{g}({}^{V_\alpha}(\nabla_X\omega), {}^{V_\beta}\theta).$$

Therefore,

$$\tilde{\nabla}_{{}^{H}X}{}^{V_\alpha}\omega = {}^{V_\alpha}(\nabla_X\omega) + \frac{1}{2h}\sum_{\sigma=1}^{n} \delta_{\alpha\sigma}{}^{H}(X^\sigma(g^{-1} \circ R(\ \ , X)\tilde{\omega})).$$

iii) By calculations analogy to those in ii), we obtain:

$$2\tilde{g}(\tilde{\nabla}_{^{V_\alpha}\omega}{}^{H}Y, {}^{V_\beta}\theta) = {}^{H}Y(\tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}\theta)) - \tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_Y\theta))$$

$$-\tilde{g}({}^{V_\beta}\theta, {}^{V_\alpha}(\nabla_Y\omega)) = \tilde{g}({}^{V_\alpha}(\nabla_Y\omega), {}^{V_\beta}\theta) + \tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_Y\theta))$$

$$-\tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_Y\theta)) - \tilde{g}({}^{V_\beta}\theta, {}^{V_\alpha}(\nabla_Y\omega)) = 0$$

and

$$2\tilde{g}(\tilde{\nabla}_{^{V_\alpha}\omega}{}^{H}Y, {}^{H}Z) = -\tilde{g}({}^{V_\alpha}\omega, \gamma(R(Y,Z))) = -\tilde{g}\left({}^{V_\alpha}\omega, \sum_{\sigma=1}^{n}{}^{V_\sigma}(R(Y,Z)X^\sigma)\right)$$

$$= -\sum_{\sigma=1}^{n}\tilde{g}({}^{V_\sigma}(X^\sigma \circ R(Y,Z)), {}^{V_\alpha}\omega) = -\sum_{\sigma=1}^{n}\tfrac{1}{h}\delta_{\alpha\sigma}g^{-1}(X^\sigma \circ R(Y,Z), \omega)$$

$$= \sum_{\sigma=1}^{n}\tfrac{1}{h}\delta_{\alpha\sigma}\tilde{g}({}^{H}(X^\sigma(g^{-1} \circ R(\quad, Y)\tilde{\omega}), {}^{H}Z).$$

Thus, we have

$$\tilde{\nabla}_{^{V_\alpha}\omega}{}^{H}Y = \tfrac{1}{2h}\sum_{\sigma=1}^{n}\delta_{\alpha\sigma}{}^{H}(X^\sigma(g^{-1} \circ R(\quad, Y)\tilde{\omega})).$$

iv) By using Koszul formula (19), we have

$$2\tilde{g}(\tilde{\nabla}_{^{V_\alpha}\omega}{}^{V_\beta}\theta, {}^{H}Z) = -{}^{H}Z(\tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}\theta)) + \tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_Z\theta))$$

$$+\tilde{g}({}^{V_\beta}\theta, {}^{V_\alpha}(\nabla_Z\omega)) = -\tilde{g}({}^{V_\alpha}(\nabla_Z\omega), {}^{V_\beta}\theta) - \tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_Z\theta))$$

$$+\tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}(\nabla_Z\theta)) + \tilde{g}({}^{V_\beta}\theta, {}^{V_\alpha}(\nabla_Z\omega)) = 0.$$

On the other hand, using (8), (11) (for $R(X, Y) = \delta$) , (14) and

$${}^{V_\alpha}\omega(\tfrac{1}{h}) = \tfrac{-2\sum_{\sigma=1}^{n}\delta_\sigma^\alpha g^{-1}(X^\sigma,\omega)}{h^2},$$

$${}^{V_\alpha}\omega(\tilde{g}({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) = {}^{V_\alpha}\omega(\tfrac{1}{h}\delta_{\beta\gamma}g^{-1}(\theta,\xi)) = \tfrac{-2\sum_{\sigma=1}^{n}\delta_\sigma^\alpha g^{-1}(X^\sigma,\omega)}{h^2}\cdot\delta_{\beta\gamma}g^{-1}(\theta,\xi),$$

$$\tilde{g}({}^{V_\alpha}\omega, \gamma\delta) = \tilde{g}\left({}^{V_\alpha}\omega, \sum_{\sigma=1}^{n}{}^{V_\sigma}X^\sigma\right) = \sum_{\sigma=1}^{n}\tilde{g}({}^{V_\alpha}\omega, {}^{V_\sigma}X^\sigma) = \tfrac{1}{h}\sum_{\sigma=1}^{n}\delta_\sigma^\alpha g^{-1}(\omega, X^\sigma),$$

we have the following

$$h^2(\tilde{g}(\nabla_{^{V_\alpha}\omega}{}^{V_\beta}\theta, {}^{V_\gamma}\xi) = \tfrac{h^2}{2}({}^{V_\alpha}\omega(\tilde{g}({}^{V_\beta}\theta, {}^{V_\gamma}\xi)) + {}^{V_\beta}\theta(\tilde{g}({}^{V_\gamma}\xi, {}^{V_\alpha}\omega))$$

$$-{}^{V_\gamma}\xi(\tilde{g}({}^{V_\alpha}\omega, {}^{V_\beta}\theta))) = -\sum_{\sigma=1}^{n}\delta_\sigma^\alpha g^{-1}(X^\sigma,\omega)\cdot\delta_{\beta\gamma}g^{-1}(\theta,\xi)$$

$$-\sum_{\sigma=1}^{n}\delta_\sigma^\beta g^{-1}(X^\sigma,\theta)\cdot\delta_{\gamma\alpha}g^{-1}(\xi,\omega) + \sum_{\sigma=1}^{n}\delta_\sigma^\gamma g^{-1}(X^\sigma,\xi)\cdot\delta_{\alpha\beta}g^{-1}(\omega,\theta)$$

$$= -h\sum_{\sigma=1}^{n}\delta_\sigma^\alpha g^{-1}(X^\sigma,\omega)\cdot\tilde{g}({}^{V_\beta}\theta, {}^{V_\gamma}\xi) - h\sum_{\sigma=1}^{n}\delta_\sigma^\beta g^{-1}(X^\sigma,\theta)\cdot\tilde{g}({}^{V_\gamma}\xi, {}^{V_\alpha}\omega)$$

$$+h\sum_{\sigma=1}^{n}\delta_{\sigma}^{\gamma}g^{-1}(X^{\sigma},\xi)\cdot\tilde{g}({}^{V_{\alpha}}\omega,{}^{V_{\beta}}\theta)=\tilde{g}\left(-h^{2}\tilde{g}\left({}^{V_{\alpha}}\omega,\sum_{\sigma=1}^{n}{}^{V_{\sigma}}X^{\sigma}\right){}^{V_{\beta}}\theta-\right.$$

$$\left.-h^{2}\tilde{g}\left({}^{V_{\beta}}\theta,\sum_{\sigma=1}^{n}{}^{V_{\sigma}}X^{\sigma}\right){}^{V_{\alpha}}\omega+h^{2}\tilde{g}\left({}^{V_{\alpha}}\omega,{}^{V_{\beta}}\theta\right)\sum_{\sigma=1}^{n}{}^{V_{\sigma}}X^{\sigma},{}^{V_{\gamma}}\xi\right).$$

Thus

$$\tilde{\nabla}_{{}^{V_{\alpha}}\omega}{}^{V_{\beta}}\theta=-\tilde{g}\left({}^{V_{\alpha}}\omega,\sum_{\sigma=1}^{n}{}^{V_{\sigma}}X^{\sigma}\right){}^{V_{\beta}}\theta-\tilde{g}\left({}^{V_{\beta}}\theta,\sum_{\sigma=1}^{n}{}^{V_{\sigma}}X^{\sigma}\right){}^{V_{\alpha}}\omega+$$

$$+\tilde{g}({}^{V_{\alpha}}\omega,{}^{V_{\beta}}\theta)\sum_{\sigma=1}^{n}{}^{V_{\sigma}}X^{\sigma}$$

and the proof of Theorem 4.1 is completed. □

Let

$$\tilde{\nabla}_{D_I}D_J=\tilde{\Gamma}_{IJ}^{K}D_K$$

with respect to the adapted frame $\{D_K\}$ of linear coframe bundle $F^*(M_n)$, where $\tilde{\Gamma}_{IJ}^{K}$ denote the components of the Levi-Civita connection $\tilde{\nabla}$. Then by using the Theorem 4.1, we immediately get following:

Theorem 4.2. *Let (M_n, g) be a Riemannian manifold and $\tilde{\nabla}$ be the Levi-Civita connection of the linear coframe bundle $F^*(M_n)$ equipped with the homogeneous type deformed Sasaki lift $\tilde{g}$ of a Riemannian metric g on M_n. The particular values of $\tilde{\Gamma}_{IJ}^{K}$ for different indices, by taking account of* (20) *are then found to be*

$$\tilde{\Gamma}_{ij}^{k}=\Gamma_{ij}^{k},\tilde{\Gamma}_{i_{\alpha}j_{\beta}}^{k}=\Gamma_{i_{\alpha}j}^{k_{\gamma}}=0,$$

$$\tilde{\Gamma}_{ij}^{k_{\gamma}}=\frac{1}{2}\sum_{\sigma=1}^{n}\delta_{\sigma}^{\gamma}X_{m}^{\sigma}R_{ijk}^{m},\tilde{\Gamma}_{ij_{\beta}}^{k_{\gamma}}=-\delta_{\beta}^{\gamma}\Gamma_{ik}^{j},$$

$$\tilde{\Gamma}_{i_{\alpha}j}^{k}=\frac{1}{2h}\sum_{\sigma=1}^{n}\delta_{\alpha\sigma}X_{m}^{\sigma}R_{\cdot j\cdot}^{k\;im},\tilde{\Gamma}_{ij_{\beta}}^{k}=\frac{1}{2h}\sum_{\sigma=1}^{n}\delta_{\beta\sigma}X_{m}^{\sigma}R_{\cdot i\cdot}^{k\;jm},$$

$$\tilde{\Gamma}_{i_{\alpha}j_{\beta}}^{k_{\gamma}}=-\frac{1}{h}\left(\delta_{\alpha\varepsilon}g^{im}X_{m}^{\varepsilon}\delta_{\beta}^{\gamma}\delta_{k}^{j}+\delta_{\beta\varepsilon}g^{jm}X_{m}^{\varepsilon}\delta_{\alpha}^{\gamma}\delta_{k}^{i}-\delta_{\alpha\beta}g^{ij}X_{k}^{\gamma}\right)$$

with respect to the adapted frame $\{D_K\}$, where $R_{\cdot i\cdot}^{k\;jm}=g^{kl}g^{js}R_{lis}^{\;\;\;m}$.

5. Almost Hermit Structures on the Coframe Bundle

Various tensor structures of type (1.1) (i.e. (1.1)-tensor structures) on manifolds have been studied by many authors (see for example [5]). Some classes of (1,1)-tensor structures can be an isomorphic representation of certain algebras. Such tensor structures are called algebraic tensor structures. In this section, we define a few specific (1,1)-tensor structures, i.e almost Hermit structures on the linear coframe bundle equipped with a homogeneous type deformed Sasaki lift of the Riemannian metric.

Let (M_n, g) be a Riemannian manifold and let $F^*(M_n)$ be its linear coframe bundle equipped with a lift $\tilde{g}$ of the metric g to $F^*(M_n)$. On linear coframe bundle $F^*(M_n)$ we define the mappings

$$\overset{*}{F}_{\beta},\ \overset{*}{F}_{\beta}:\Im_{0}^{1}(F^{*}(M_{n}))\to\Im_{0}^{1}(F^{*}(M_{n})),\beta=1,2,...,n,$$

as follows:

$$\overset{*}{F}_\beta(D_i) = \overset{*}{F}_\beta(\tfrac{\delta}{\delta x^i}) = \sqrt{h} g_{ij}\ {}^{V_\beta}(dx^j) = \sum_j \sqrt{h}\, g_{ij} D_{j_\beta} = \sum_j \sqrt{h}\, g_{ij} \frac{\partial}{\partial X_j^\beta}, \tag{23}$$

$$\overset{*}{F}_\beta(D_{i_\alpha}) = \overset{*}{F}_\beta(\tfrac{\partial}{\partial X_i^\alpha}) = -\tfrac{1}{\sqrt{h}} \delta_{\alpha\beta} g^{ij H}\left(\tfrac{\partial}{\partial x^j}\right) = -\tfrac{1}{\sqrt{h}}\ \delta_{\alpha\beta}\, g^{ij} D_j, \tag{24}$$

where $\{D_I\} = \{D_i, D_{i_\alpha}\}$ is the adapted frame of the linear frame bundle $F^*(M_n)$ and h is a function defined by (12).

It is not difficult to prove:

Theorem 5.1. *For each $\beta = 1, 2, ..., n$, $\overset{*}{F}_\beta$ has the following properties:*

1°. *$\overset{*}{F}_\beta$ is a (1,1)-tensor structure on linear coframe bundle $F^*(M_n)$;*

2°. *$\overset{*}{F}_\beta$ depends only on the metric g;*

3°. *$\overset{*}{F}_\beta$ is homogeneous on the fibers of the linear coframe bundle $F^*(M_n)$.*

We denote by Π the (1,1)-tensor structure $\left\{\overset{*}{F}_\beta\right\}, \beta = 1, 2, ..., n$, defined by (23) and (24) on the linear coframe bundle $F^*(M_n)$. Using (23) and (24), we have

Theorem 5.2. *The (1.1)-tensor structure $\Pi = \left\{\overset{*}{F}_\beta\right\}, \beta = 1, 2, ..., n$, satisfies the relations:*

$$\overset{*}{F}_\beta{}^2 = -I,\ \beta = 1, 2, ..., n,$$

$$\overset{*}{F}_\beta \circ \overset{*}{F}_\gamma = \mathrm{O},\ \beta \neq \gamma,$$

where I and O are the identity and zero tensor fields on $F^*(M_n)$, respectively.

Proof. From (23) and (24) we obtain

$$\overset{*}{F}_\beta{}^2(D_i) = \overset{*}{F}_\beta\left(\overset{*}{F}_\beta(D_i)\right) = \overset{*}{F}_\beta\left(\sum_j \sqrt{h}\, g_{ij} D_{j_\beta}\right) = \sqrt{h} \sum_j g_{ij} \overset{*}{F}_\beta(D_{j_\beta})$$

$$= -\sqrt{h} \cdot \tfrac{1}{\sqrt{h}} \delta_{\beta\beta} g_{ij} g^{jk} D_k = -\delta_i^k D_k = -D_i$$

$$\overset{*}{F}_\beta{}^2(D_{i_\alpha}) = \overset{*}{F}_\beta(\overset{*}{F}_\beta(D_{i_\alpha})) = \overset{*}{F}_\beta\left(-\tfrac{1}{\sqrt{h}} g^{ij} \delta_{\beta\alpha}\, D_j\right) = -\tfrac{1}{\sqrt{h}} \delta_{\beta\alpha} g^{ij}\, \overset{*}{F}_\beta(D_j)$$

$$-\tfrac{1}{\sqrt{h}} \sqrt{h} \cdot \delta_{\beta\alpha} g^{ij} g_{jk} D_{k_\beta} = -\delta_k^i D_{k_\alpha} = -D_{i_\alpha},$$

from which it follows that

$$\overset{*}{F}_\beta{}^2 = -I,\ \beta = 1, 2, ..., n.$$

Similarly, we have

$$(\overset{*}{F}_\beta \circ \overset{*}{F}_\gamma)(D_i) = \overset{*}{F}_\beta(\overset{*}{F}_\gamma(D_i)) = \overset{*}{F}_\beta\left(\sum_j \sqrt{h}\, g_{ij} D_{j_\gamma}\right) = \sqrt{h} \sum_j g_{ij}\, \overset{*}{F}_\beta(D_{j_\gamma})$$

$$= -\sqrt{h} \cdot \tfrac{1}{\sqrt{h}} g_{ij} g^{jk} \delta_\gamma^\beta D_k = -\delta_\gamma^\beta \delta_i^k D_k = -\delta_\gamma^\beta D_i = 0 \quad (\beta \neq \gamma),$$

and

$$(\overset{*}{F}_\beta \circ \overset{*}{F}_\gamma)(D_{i_\alpha}) = \overset{*}{F}_\beta\,(\overset{*}{F}_\gamma(D_{i_\alpha})) = \overset{*}{F}_\beta\left(-\frac{1}{\sqrt{h}}\, g^{ij}\delta_{\alpha\gamma}D_j\right) = -\frac{1}{\sqrt{h}}\delta_{\alpha\gamma}g^{ij}\overset{*}{F}_\beta(D_j)$$

$$= -\frac{1}{\sqrt{h}}\cdot\ \sqrt{h}\delta_{\alpha\gamma}g^{ij}g_{jk}D_{k_\beta} = -\delta_{\alpha\gamma}\delta^{\alpha\beta}\delta^i_k D_{k_\alpha} = -\delta^\beta_\gamma D_{i_\alpha} = 0 \quad (\beta \neq \gamma).$$

Thus

$$\overset{*}{F}_\beta \circ \overset{*}{F}_\gamma = O$$

for all $\beta \neq \gamma$ and Theorem 5.2 is proved. □

On the linear coframe bundle $F^*(M_n)$ we introduce the (1,1)-tensor structure $\tilde{\Pi} = \left\{\varphi_{\bar\beta}\right\}, \bar\beta = 1, 2, ..., n, n+1$ as follows

$$\varphi_{\bar\beta} = \begin{cases} I,\ if\ \bar\beta = 1, \\ \overset{*}{F}_{\bar\beta-1},\ if\ \bar\beta = 2, 3, ..., n, n+1, \end{cases}$$

where I is the identity (1,1)-tensor field on $F^*(M_n)$. Such tensor structures play an important role in the theory of algebraic structures. From Theorem 5.2 we have

Corollary 5.3. *The (1,1)-tensor structure $\tilde{\Pi} = \left\{\varphi_{\bar\beta}\right\}, \bar\beta = 1, 2, ..., n, n+1$ satisfies the conditions*

$$\varphi_{\bar\alpha} \circ \varphi_{\bar\beta} = C^{\bar\gamma}_{\bar\alpha\bar\beta}\varphi_{\bar\gamma},\ \ \bar\alpha, \bar\beta, \bar\gamma, ... = 1, 2, ..., n+1,$$

$$C^1_{11} = C^2_{12} = \cdots = C^{n+1}_{1,n+1} = 1, C^1_{22} = C^1_{33} = \cdots = C^1_{n+1,n+1} = -1,$$

all the other coefficients are zero.

The following theorem holds.

Theorem 5.4. *The homogeneous type deformed Sasaki lift $\tilde g$ of a Riemannian metric g to the coframe bundle $F^*(M_n)$ is compatible with the (1,1)-tensor structure $\Pi = \left\{\overset{*}{F}_\beta\right\}, \beta = 1, 2, ..., n$, i.e.,*

$$\tilde g(\overset{*}{F}_\beta X, \overset{*}{F}_\beta Y) = \tilde g(X, Y), \beta = 1, 2, ..., n,$$

for all $X, Y \in \Im^1_0(F^*(M_n))$.

Proof. Since the matrix (g^{ij}) is the inverse of the matrix (g_{ij}), from (23), (24) and (13), it follows that for each $\beta = 1, 2, ..., n$,

$$\tilde g(\overset{*}{F}_\beta(D_i), \overset{*}{F}_\beta(D_j)) = \tilde g(D_i,\ D_j),$$

$$\tilde g(\overset{*}{F}_\beta(D_{i_\alpha}), \overset{*}{F}_\beta(D_{j_\gamma})) = \tilde g(D_{i_\alpha},\ D_{j_\gamma}),$$

$$\tilde g(\overset{*}{F}_\beta(D_{i_\alpha}), \overset{*}{F}_\beta(D_j)) = \tilde g(D_{i_\alpha},\ D_j) = 0.$$

Hence

$$\tilde g(\overset{*}{F}_\beta X, \overset{*}{F}_\beta Y) = \tilde g(X, Y)$$

for all $X, Y \in \Im^1_0(F^*(M_n))$ and $\beta = 1, 2, ..., n$. Thus, Theorem 5.4 is proved. □

From Theorem 5.4 it follows

Corollary 5.5. *The triple $(F^*(M_n), \tilde g, \overset{*}{F}_\beta)$ is an almost Hermit manifold for any $\beta = \overline{1, n}$.*

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