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On the Difference of Coefficients of Univalent Functions

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Abstract. For $f \in \mathcal{S}$, the class of normalized functions, analytic and univalent in the unit disk $\mathbb{D}$ and given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in \mathbb{D}$, we give an upper bound for the coefficient difference $|a_4| - |a_3|$ when $f \in \mathcal{S}$. This provides an improved bound in the case $n = 3$ of Grinspan's 1976 general bound $||a_{n+1}| - |a_n|| \le 3.61\ldots$. Other coefficients bounds, and bounds for the second and third Hankel determinants when $f \in \mathcal{S}$ are found when either $a_2 = 0$, or $a_3 = 0$.

1. Introduction. preliminaries and definitions

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1}$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$.

Although the famous Bieberbach conjecture $|a_n| \le n$ for $n \ge 2$, was proved by de Branges in 1985 [1], a great many other problems concerning the coefficients a_n remain open. The main aim of this paper (Section 3), is by use of the Grunsky inequalities, to find an upper for the difference of coefficients $|a_4| - |a_3|$ for $f \in \mathcal{S}$, which improves the well-known general bound of Grispan $||a_{n+1}| - |a_n|| \le 3.61\ldots$ [4], when $n = 3$. We also obtain information concerning the initial coefficients of $f(z)$, and of the second and third Hankel determinants when either $a_2 = 0$, or $a_3 = 0$.

For $f \in \mathcal{S}$, the Grunsky coefficients $\omega_{p,q}$ as defined in N. A. Lebedev [6] are given by

$$\log \frac{f(t) - f(z)}{t - z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q} = \omega_{q,p}$, and satisfy the so-called Grunsky inequalities [2, 6]

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$$\sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_p|^2}{p}, \tag{2}$$

where x_p are arbitrary complex numbers such that last series converges.

Further, it is well-known that if f given by (1) belongs to $\mathcal{S}$, then also

$$f_2(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots \tag{3}$$

belongs to $\mathcal{S}$. Thus for the function f_2 we have the appropriate Grunsky coefficients of the form $\omega_{2p-1,2q-1}$, and inequalities (2) take the form

$$\sum_{q=1}^{\infty} (2q-1) \left| \sum_{p=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}. \tag{4}$$

(Note that in this paper, we omit the upper index (2) in $\omega^{(2)}_{2p-1,2q-1}$ in Lebedev's notation).

The following similar inequality follows from the relation (15) on page 57 in [6].

$$\left| \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} x_{2q-1} \right| \leq \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}. \tag{5}$$

Thus for example, from (4) and (5) when $x_{2p-1} = 0$ and $p = 3, 4, \ldots$, we obtain

$$|\omega_{11} x_1 + \omega_{31} x_3|^2 + 3|\omega_{13} x_1 + \omega_{33} x_3|^2 + 5|\omega_{15} x_1 + \omega_{35} x_3|^2 \leq |x_1|^2 + \frac{|x_3|^2}{3} \tag{6}$$

and

$$|\omega_{11} x_1^2 + 2\omega_{13} x_1 x_3 + \omega_{33} x_3^2| \leq |x_1|^2 + \frac{|x_3|^2}{3}, \tag{7}$$

respectively.

It was also shown in [6, p.57], that if $f \in \mathcal{S}$ is given by (1), then the coefficients a_2, a_3, a_4 and a_5 can be expressed in terms of the Grunsky coefficients $\omega_{2p-1,2q-1}$ of the function f_2 given by (3) as follows.

$$\begin{aligned}
a_2 &= 2\omega_{11}, \\
a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\
a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3, \\
a_5 &= 2\omega_{35} + 8\omega_{11}\omega_{33} + 5\omega_{13}^2 + 18\omega_{11}^2\omega_{13} + \frac{7}{3}\omega_{11}^4, \\
0 &= 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33}.
\end{aligned} \tag{8}$$

In this paper we will use these expressions to obtain information concerning the coefficients a_2, a_3, a_4, and a_5 when $f \in \mathcal{S}$.

In recent years a great deal of attention has been given to finding upper bounds for the modulus of the second and third Hankel determinants $H_2(2)$ and $H_3(1)$, defined as follows who's elements are the coefficients of $f \in \mathcal{S}$ (see e.g. [8]).

For $f \in \mathcal{S}$

$$H_2(2) = a_2a_4 - a_3^2$$

and

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \tag{9}$$

Almost all results have concentrated on finding bounds for $|H_2(2)|$ and $|H_3(1)|$ for subclasses of $\mathcal{S}$, and only recently has a significant bound been found for the whole class $\mathcal{S}$ [7] for $|H_2(2)|$ and $|H_3(1)|$. However finding exact sharp bounds remains an open problem.

We begin by using the Grunsky inequalities in (5) to obtain bounds for the modulus of some initial coefficients and $|H_2(2)|$ and $|H_3(1)|$ when $f \in \mathcal{S}$ provided either a_2, or $a_3 = 0$.

2. Coefficient bounds and Hankel determinants

Obtaining sharp bounds for the modulus of the coefficients for odd functions in $\mathcal{S}$ has long been been an open problem. If f_2, given by (3) is an odd function in $\mathcal{S}$, then the only known sharp bounds for $|c_{2n-1}|$ for $n \geq 2$ are $|c_3| \leq 1$, and $|c_5| \leq 1/2 + e^{-2/3} = 1.013\ldots$. In general the best bound to date is $|c_{2n-1}| \leq 1.14$ for $n \geq 2$, (see e.g.[2]).

In our first theorem, we give bounds for $|a_3|$, $|a_4|$ and $|a_5|$ when $f \in \mathcal{S}$ assuming only that only $a_2 = 0$, thus providing bounds for a wider class of functions than the odd functions in $\mathcal{S}$. We also give bounds for $|H_2(2)|$ and $|H_3(1)|$ in this case.

Theorem 2.1. *Let $f \in \mathcal{S}$ and be given by* (1) *with $a_2 = 0$. Then*

(i) $|a_3| \leq 1$,

(ii) $|a_4| \leq \frac{2}{3} = 0.666\ldots$,

(iii) $|a_5| \leq \sqrt{\frac{19}{15}} = 1.67666\ldots$,

(iv) $|H_2(2)| \leq 1$,

(v) $|H_3(1)| \leq \frac{21}{20} = 1.05$.

Proof.

(i) The classical inequality $|a_3 - a_2^2| \leq 1$ for f in $\mathcal{S}$ when $a_2 = 0$, gives $|a_3| \leq 1$, which from (8) gives

$$|\omega_{13}| \leq \frac{1}{2}. \tag{10}$$

(*ii*) Next choose $x_1 = 0$ and $x_3 = 1$ in (7), which gives

$$|\omega_{33}| \leq \frac{1}{3}. \tag{11}$$

Also, since $\omega_{11} = 0$ ($\Leftrightarrow a_2 = 0$), then from (8) and (11) we obtain

$$|a_4| = 2|\omega_{33}| \leq \frac{2}{3} = 0.666\ldots.$$

(*iii*) Again since $\omega_{11} = 0$, from (8) we obtain

$$|a_5| = |2\omega_{35} + 5\omega_{13}^2|. \tag{12}$$

From (6) with $x_1 = 0$ and $x_3 = 1$ we have ($\omega_{11} = 0$)

$$|\omega_{13}|^2 + 3|\omega_{33}|^2 + 5|\omega_{35}|^2 \leq \frac{1}{3}$$

and from here

$$|\omega_{35}| \leq \frac{1}{\sqrt{15}}\sqrt{1 - 3|\omega_{13}|^2}. \tag{13}$$

From (12) and (13) we have

$$|a_5| \leq 2|\omega_{35}| + 5|\omega_{13}|^2 \leq \frac{1}{\sqrt{15}}\sqrt{1 - 3|\omega_{13}|^2} + 5|\omega_{13}|^2 \leq \frac{503}{300} = 1.67666\ldots.$$

(*iv*) Since we are assuming $a_2 = 0$, (*i*) shows that $|H_2(2)| \leq 1$ is trivial.

(*v*) When $\omega_{11} = 0$, from the last relation in (8) we have $\omega_{33} = \omega_{15}$, and from (9),

$$|H_3(1) = |2\omega_{13}^3 + 4\omega_{13}\omega_{35} - 4\omega_{33}^2| \leq 2|\omega_{13}|^3 + 4 + \underbrace{|\omega_{13}\omega_{35} - \omega_{15}^2|}_{E_1}. \tag{14}$$

Now choose $x_1 = -\omega_{15}$, and $x_3 = \omega_{13}$, and since $\omega_{33} = \omega_{15}$, from (6) we obtain

$$|\omega_{13}|^4 + 5E_1^2 \leq |\omega_{15}|^2 + \frac{|\omega_{13}|^2}{3} \leq \frac{1}{5} - \frac{3}{5}|\omega_{13}|^2 + \frac{1}{3}|\omega_{13}|^2,$$

(since by (6) $3|\omega_{13}|^2 + 5|\omega_{15}|^2 \leq 1$ for $x_1 = 1$, $x_3 = 0$ and $\omega_{11} = 0$), which implies $5E_1^2 \leq \frac{1}{5} - \frac{4}{15}|\omega_{13}|^2 - |\omega_{13}|^4$, i.e., $E_1 \leq \frac{1}{5}$.

Finally from (10) and (14), it follows that

$$|H_3(1)| \leq 2 \cdot \frac{1}{8} + 4 \cdot \frac{1}{5} = \frac{21}{20} = 1.05.$$

This completes the proof of Theorem 2.1. □

We next prove a similar result, this time assuming that $a_3 = 0$.

Theorem 2.2. *Let $f \in \mathcal{S}$ and be given by* (1), *with $a_3 = 0$. Then*

(i) $|a_2| \le 1$,

(ii) $|a_4| \le \frac{\sqrt{37}+13}{12} = 1.59023\ldots$,

(iii) $|a_5| \le \frac{1}{4}\sqrt{\frac{757}{15}} + \frac{85}{64} = 3.10412\ldots$,

(iv) $|H_2(2)| \le \frac{13+\sqrt{37}}{12} = 1.59023\ldots$,

(v) $|H_3(1)| \le \frac{24+\sqrt{645}}{30} = 1.64656\ldots$.

Proof.

(i) Since $|a_3 - a_2^2| \le 1$ and $a_3 = 0$, then $|a_2^2| \le 1$, i.e., $|a_2| \le 1$. Also, since by (8), $a_3 = 2\omega_{13} + 3\omega_{11}^2 = 0$, it follows that

$$\omega_{13} = -\frac{3}{2}\omega_{11}^2 \quad \left(\Leftrightarrow\ \omega_{11}^2 = -\frac{2}{3}\omega_{13}\right). \tag{15}$$

Because $|a_2| = |2\omega_{11}| \le 1$, we have

$$|\omega_{11}| \le \frac{1}{2} \quad \text{and} \quad |\omega_{13}| \le \frac{3}{8} \text{ (by (15).} \tag{16}$$

(ii) By using (8) and (15), we obtain

$$\begin{aligned} |a_4| &= \left|2\omega_{33} + 8\omega_{11}\left(-\frac{3}{2}\omega_{11}^2\right) + \frac{10}{3}\omega_{11}^3\right| \\ &= \left|2\omega_{33} - \frac{26}{3}\omega_{11}^3\right| \\ &\le 2|\omega_{33}| + \frac{26}{3}|\omega_{11}|^3. \end{aligned} \tag{17}$$

From (6), using $x_1 = 0$ and $x_3 = 1$, we have

$$|\omega_{13}|^2 + 3|\omega_{33}|^2 \le \frac{1}{3},$$

which implies (with $\omega_{13} = -\frac{3}{2}\omega_{11}^2$, see (15))

$$|\omega_{33}| \le \sqrt{\frac{1}{9} - \frac{3}{4}|\omega_{11}|^4}. \tag{18}$$

Combining (17) and (18) we obtain

$$|a_4| \le 2\sqrt{\frac{1}{9} - \frac{3}{4}|\omega_{11}|^4} + \frac{26}{3}|\omega_{11}|^3 =: \varphi(|\omega_{11}|), \tag{19}$$

where $\varphi(t) = 2\sqrt{\frac{1}{9} - \frac{3}{4}t^4} + \frac{26}{3}t^3$, $0 \le t = |\omega_{11}| \le \frac{1}{2}$ (by (16)). Since φ is increasing function on $[0, 1/2]$,

$$\varphi(t) \le \varphi(1/2) = \frac{\sqrt{37}+13}{12},$$

which, together with (19), gives the desired result.

(iii) From the last relation in (8), using (15) we have $\omega_{33} = \omega_{15} + \frac{11}{6}\omega_{11}^3$, which with the expression for a_5 in (8), gives

$$\begin{aligned} |a_5| &= |2\omega_{35} + 8\omega_{11}\omega_{15} + 5\omega_{13}^2 - 10\omega_{11}^4| \\ &\le \underbrace{2\,|\omega_{35} + 4\omega_{11}\omega_{15}|}_{C_1^*} + \underbrace{5|\omega_{13}|^2 + 10|\omega_{11}|^4}_{C_2^*}. \end{aligned} \tag{20}$$

Once again, using (6) choosing $x_1 = 4\omega_{11}$, $x_3 = 1$ and $\omega_{13} = -\frac{3}{2}\omega_{11}^2$, we have

$$(C_1^*)^2 = |4\omega_{11}\omega_{15} + \omega_{35}|^2 \le -\frac{5}{4}|\omega_{11}|^4 + \frac{16}{5}|\omega_{11}|^2 + \frac{1}{15} \le \frac{757}{64\cdot 15},$$

since $|\omega_{11}| \le \frac{1}{2}$. Thus

$$C_1^* \le \frac{1}{8}\sqrt{\frac{757}{15}}.$$

Next, since $\omega_{13} = -\frac{3}{2}\omega_{11}^2$ and $|\omega_{11}| \le \frac{1}{2}$, we have

$$C_2^* = 5\cdot\frac{9}{4}\cdot|\omega_{11}|^4 + 10|\omega_{11}|^4 = \frac{85}{4}|\omega_{11}|^4 \le \frac{85}{4}\cdot\frac{1}{16} = \frac{85}{64},$$

since $|\omega_{11}| \le \frac{1}{2}$.

Finally from (20) we have

$$|a_5| \le \frac{1}{4}\sqrt{\frac{757}{15}} + \frac{85}{64} = 3.10412\ldots.$$

(iv) By using (9), (8) and (15), we have

$$\begin{aligned} H_2(2) &= 4\omega_{11}\omega_{33} + 4\omega_{11}^2\omega_{13} - 4\omega_{13}^2 - \frac{7}{3}\omega_{11}^4 \\ &= 4\omega_{11}\omega_{33} - \frac{52}{3}\omega_{11}^4 \end{aligned} \tag{21}$$

and from here

$$|H_2(2)| \le 4|\omega_{11}||\omega_{33}| + \frac{52}{3}|\omega_{11}|^4. \tag{22}$$

From (18) and (22) we have

$$|H_2(2)| \le 4|\omega_{11}|\sqrt{\frac{1}{9} - \frac{3}{4}|\omega_{11}|^4} + \frac{52}{3}|\omega_{11}|^4 =: \varphi_1(|\omega_{11}|,$$

where

$$\varphi_1(t) = 4t\sqrt{\frac{1}{9} - \frac{3}{4}t^4} + \frac{52}{3}t^4,$$

with $0 \le t = |\omega_{11}| \le \frac{1}{2}$. Finally, it can be checked that φ_1 is an increasing function on the interval $(0, 1/2)$, and so

$$|H_2(2)| \le \varphi_1(1/2) = \frac{13+\sqrt{37}}{12} = 1.59023\ldots.$$

(v) By using the last relation from (8) with $\omega_{13} = -\frac{3}{2}\omega_{11}^2$, it follows that $\omega_{33} = \omega_{15} + \frac{11}{6}\omega_{11}^3$, and so using (9), after some calculations we obtain

$$H_3(1) = -12\omega_{11}^2\left(\omega_{11}\omega_{15} + \frac{2}{3}\omega_{35}\right) - 4\omega_{15}^2 - 30\omega_{11}^6,$$

which gives

$$|H_3(1)| \leq \underbrace{12|\omega_{11}|^2\left|\omega_{11}\omega_{15} + \frac{2}{3}\omega_{35}\right|}_{D_1} + \underbrace{4|\omega_{15}|^2 + 30|\omega_{11}|^6}_{D_2}. \tag{23}$$

Now choose $x_1 = \omega_{11}$ and $x_3 = \frac{2}{3}$ in (6), then (since $\omega_{13} = -\frac{3}{2}\omega_{11}^2$),

$$\left|\omega_{11}\omega_{15} + \frac{2}{3}\omega_{35}\right| \leq \sqrt{\frac{1}{5}\left(|\omega_{11}|^2 + \frac{4}{27}\right)},$$

and so

$$\begin{aligned} D_1 &\leq 12|\omega_{11}|^2\sqrt{\frac{1}{5}\left(|\omega_{11}|^2 + \frac{4}{27}\right)} \\ &\leq 12\cdot\frac{1}{4}\sqrt{\frac{1}{5}\left(\frac{1}{4} + \frac{4}{27}\right)} = \sqrt{\frac{43}{60}} = \frac{\sqrt{645}}{30} = 0.84656\ldots, \end{aligned} \tag{24}$$

since $|\omega_{11}| \leq \frac{1}{2}$.

Also, as in the proof of (iii), we have

$$5|\omega_{15}|^2 \leq 1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 = 1 - |\omega_{11}|^2 - \frac{27}{4}|\omega_{11}|^4,$$

where we have once again used $\omega_{13} = -\frac{3}{2}\omega_{11}^2$. Now

$$D_2 \leq \frac{4}{5} - \frac{4}{5}|\omega_{11}|^2 - \frac{27}{5}|\omega_{11}|^4 + 30|\omega_{11}|^6 =: \varphi_2(|\omega_{11}|^2),$$

where

$$\varphi_2(t) = \frac{1}{5}\left(4 - 4t - 27t^2 + 150t^3\right),$$

and $0 \leq t = |\omega_{11}|^2 \leq \frac{1}{4}$. Since φ_2 attains its maximum at $t_0 = 0$,

$$D_2 \leq \varphi_2(0) = \frac{4}{5}. \tag{25}$$

Finally, by using (23), (24) and (25) we obtain

$$|H_3(1)| \leq D_1 + D_2 \leq \frac{24 + \sqrt{645}}{30} = 1.64656\ldots.$$

□

3. Coefficient differences for $f \in \mathcal{S}$

A long standing problem in the theory of univalent functions is to find sharp upper and lower bounds for $|a_{n+1}| - |a_n|$, when $f \in \mathcal{S}$. Since the Keobe function has coefficients $a_n = n$, it is natural to conjecture that $||a_{n+1}| - |a_n|| \le 1$. As early as 1933, this was shown to be false even when $n = 2$, when Fekete and Szegö [3] obtained the sharp bounds

$$-1 \le |a_3| - |a_2| \le \frac{3}{4} + e^{-\lambda_0}(2e^{-\lambda_0} - 1) = 1.029\ldots,$$

where λ_0 is the unique value of λ in $0 < \lambda < 1$, satisfying the equation $4\lambda = e^{\lambda}$.

Hayman [5] showed that if $f \in \mathcal{S}$, then $||a_{n+1}| - |a_n|| \le C$, where C is an absolute constant. The exact value of C is unknown, the best estimate to date being $C = 3.61\ldots$ [4], which because of the sharp estimate above when $n = 2$, cannot be reduced to 1.

We now use the methods of this paper to obtain a better upper bound in the case $n = 3$.

Theorem 3.1. *Let $f \in \mathcal{S}$ and be given by* (1). *Then*

$$|a_4| - |a_3| \le 2.1033299\ldots.$$

Proof. By using (8) we have

$$|a_4| - |a_3| \le |a_4| - |\omega_{11}||a_3| \le |a_4 - \omega_{11}a_3| = 2\Big|\underbrace{\omega_{33} + 3\omega_{11}\omega_{33} + \frac{1}{6}\omega_{11}^3}_{B}\Big|.$$

From (7) with $x_1 = \frac{1}{\sqrt{6}}\omega_{11}$ and $x_3 = 1$, we obtain

$$\begin{aligned}
&\left|\omega_{33} + \frac{2}{\sqrt{6}}\omega_{11}\omega_{13} + \frac{1}{6}\omega_{11}^3\right| \le \frac{1}{6}|\omega_{11}|^2 + \frac{1}{3}\\
\Rightarrow\ &\left|B + \left(\frac{2}{\sqrt{6}} - 3\right)\omega_{11}\omega_{13}\right| \le \frac{1}{6}|\omega_{11}|^2 + \frac{1}{3}\\
\Rightarrow\ &|B| \le \left(3 - \frac{\sqrt{6}}{3}\right)|\omega_{11}||\omega_{13}| + \frac{1}{6}|\omega_{11}|^2 + \frac{1}{3}\\
\Rightarrow\ &|B| \le \left(3 - \frac{\sqrt{6}}{3}\right)|\omega_{11}| \cdot \frac{1}{\sqrt{3}}\sqrt{1 - |\omega_{11}|^2} + \frac{1}{6}|\omega_{11}|^2 + \frac{1}{3}\\
\Rightarrow\ &|B| \le \frac{1}{3}\left[(3\sqrt{3} - \sqrt{2})|\omega_{11}|\sqrt{1 - |\omega_{11}|^2} + \frac{1}{2}|\omega_{11}|^2 + 1\right] =: \varphi(|\omega_{11}|),
\end{aligned}$$

where $\varphi(t) = \frac{1}{3}\left[(3\sqrt{3} - \sqrt{2})t\sqrt{1 - t^2} + \frac{1}{2}t^2 + 1\right]$ for $0 \le t \le 1$, and where we have used that $|\omega_{13}| \le \frac{1}{\sqrt{3}}\sqrt{1 - |\omega_{11}|^2}$. Since the function φ attains its maximum at

$$t_0 = \sqrt{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{1}{379}(39 + 8\sqrt{6})}} = 0.75202\ldots,$$

and since $\varphi(t_0) = \frac{1}{12}\left(5 + \sqrt{117 - 24\sqrt{6}}\right)$, it follows that

$$|a_4| - |a_3| \le 2\varphi(t_0) = 2.10495\ldots.$$

□

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