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Investigation of Langevin Equation in Terms of Generalized Proportional Fractional Derivatives with Respect to Another Function

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Abstract. The current work concerns the existence and uniqueness results for a nonlinear Langevin equation involving two generalized proportional fractional operators with respect to another function. The main results are proved by means of Krasnoselskii's fixed point theorem and the Banach contraction principle. An example is set forth to make efficient our main results.

1. Introduction

The theory of fractional differential equations has recently acquired plentiful circulation and great significance because of its rife applications in the fields of science and engineering. For instance, we indicate to the new papers and the books [2, 14, 17, 19, 20, 22–24] and references cited therein.

For many years, the Langevin equation, inspired by P. Langevin [18], was vastly utilized in mathematical physics to describe the dynamical processes revolving in a swing medium like Brownian motion [9]. For the systems in the confused medium, the classical Langevin equation does not tool up the correct description of the dynamics. For this reason, Kubo [15], in 1966, established the generalized Langevin equation, where a fractional memory kernel was incorporated into the Langevin equation to depict the fractal and memory properties. As a result of the rapid progress of fractional calculus, Mainardi and collaborators [20, 21] introduced the fractional Langevin equation at the beginning of the 1990s.

In recent years, there have been various new definitions of fractional derivatives, among these new definitions the so-called fractional conformable derivative, which is introduced by Khalil et al. [13]. Unfortunately, this new definition has a point of weakness as it does not tend to the original function where the order ρ tends to zero. Anderson et al. [3] were able to define the proportional (conformable) derivative of order ρ by

$$^{P}\mathfrak{D}_t^{\rho} h(t) = \chi_1(\rho, t)h(t) + \chi_0(\rho, t)h'(t),$$

where h is differentiable function and $\chi_0, \chi_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ are continuous functions of the variable t and the parameter $\rho \in [0,1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$\lim_{\rho\to 0^+} \chi_0(\rho, t) = 0, \quad \lim_{\rho\to 1^-} \chi_0(\rho, t) = 1, \quad \chi_0(\rho, t) \neq 0, \ \rho \in (0, 1], \tag{1}$$

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$$\lim_{\rho\to 0^+}\chi_1(\rho,t)=1,\quad \lim_{\rho\to 1^-}\chi_1(\rho,t)=0,\ \ \chi_1(\rho,t)\neq 0,\ \rho\in[0,1). \tag{2}$$

The new derivative tends to the initial function as $\rho\to 0$ and hence improving the conformable derivatives. In [11, 12], Jarad et al. proposed more general forms and properties of proportional derivative for function f with respect to another continuous function g. The kernel acquired in their investigation includes an exponential function and is considered as function dependent.

As a result of the emergence of new definitions of fractional derivatives with singular and non-singular kernels, authors who are interested in fractional calculus are struggling to investigate of the Langevin equation that includes these new definitions in several research papers. For example, we address the following brief survey:

In [4], D. Baleanu et al. studied the nonlinear Langevin equation involving Atangana-Baleanu fractional derivatives

$$\begin{cases} {}^{ABD}D^{\beta}\left({}^{ABD}D^{\theta}+\gamma\right)z(t)=h\left(t,z(t)\right),\ \ t\in(0,1),\ \gamma\in\mathbb{R},\\[2ex] z(0)=\alpha_1,\ \ z'(1)=\alpha_2, \end{cases} \tag{3}$$

where $0<\alpha,\beta\leq 1$, and ${}^{ABD}D^{\beta},{}^{ABD}D^{\theta}$ denote the Atangana-Baleanu fractional derivatives. They investigated the existence and uniqueness results by means of the nonlinear alternative of Leray-Schauder type and Banach contraction principle.

In [5], B. Ahmad et al. established sufficient conditions for the existence of solutions for the nonlinear Langevin equation with generalized Liouville-Caputo fractional derivatives

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{a^+}\left({}^{\rho}_{c}D^{\beta}_{a^+}+\lambda\right)x(t)=f\left(t,x(t)\right),\ \ t\in[a,T],\ \lambda\in\mathbb{R},\\[2ex] x(a)=0,\ x(\eta)=0,\ x(T)=\mu\,{}^{\rho}I^{\gamma}_{a^+}x(\xi),\ \ a<\eta<\xi<T, \end{cases} \tag{4}$$

where ${}^{\rho}_{c}D^{\alpha}_{a^+},{}^{\rho}_{c}D^{\beta}_{a^+}$ denote the generalized Liouville-Caputo fractional derivatives of order $1<\alpha\leq 2, 0<\beta<1, \rho>0$, respectively and ${}^{\rho}I^{\gamma}_{a^+}$ denotes the generalized fractional integral of order $\gamma>0$. In view of Krasnoselskii's fixed point theorem and Banach contraction mapping principle, they proved the desired results.

In [10], Rabha W. Ibrahim et al. condidered the following fractional Langevin equation containing two Hilfer-Katugampola fractional derivatives

$$\begin{cases} {}^{\rho}D^{\alpha_1,\beta}\left({}^{\rho}D^{\alpha_2,\beta}+\lambda\right)x(t)=f\left(t,x(t)\right),\ \ t\in(a,b],\ \lambda\in\mathbb{R},\\[2ex] I^{1-\gamma}x(a)=x_a,\ \ \gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta, \end{cases} \tag{5}$$

where ${}^{\rho}D^{\alpha_1,\beta},{}^{\rho}D^{\alpha_2,\beta}$ are the Hilfer-Katugampola fractional derivatives of order α_1,α_2 and type β. They derived the main results by means of Krasnoselskii's fixed point theorem and Banach contraction mapping principle. Also, they discussed the Ulam type stability.

For many interesting contributions relevant to the fractional Langevin equation, we refer the reader to the papers [1, 6–8] and references cited therein.

Motivated by the above papers, we investigate the following Langevin equation with the generalized proportional fractional derivatives with respect to another function

$$\begin{cases} {}_a\mathfrak{D}^{\alpha,\rho,\phi}\left({}_a\mathfrak{D}^{\beta,\rho,\phi}+\lambda\right)w(t)=\mathcal{G}\left(t,w(t)\right), \ t\in[a,b], \ \lambda\in\mathbb{R}, \\ {}_a\mathfrak{I}^{1-\beta,\rho,\phi}w(a)=w_a, \ \ w(b)+\mu w(\xi)=0, \ a<\xi<b, \ w_a\in\mathbb{R}, \end{cases} \tag{6}$$

where $\rho>0$, ${}_a\mathfrak{D}^{\alpha,\rho,\phi}, {}_a\mathfrak{D}^{\beta,\rho,\phi}$ are the generalized proportional fractional derivatives with respect to another continuous function ϕ of order $0<\alpha,\beta\leq 1$, respectively, ${}_a\mathfrak{I}^{1-\beta,\rho,\phi}$ is the generalized proportional fractional integral with respect to another continuous function ϕ of order $1-\beta$, and $\mathcal{G}\in C([a,b]\times\mathbb{R},\mathbb{R})$ is given function.

To the best knowledge of the author, no one has yet been treated with Langevin equation involving the generalized proportional fractional derivative with respect to another function.

2. Preliminaries

Let $\mathfrak{C}=C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|y\|_{\mathfrak{C}}=\sup\{|y(t)|:t\in[a,b]\}.$$

Now, we recall some basic definitions and properties of the fractional proportional derivative and integral of a function with respect to another function. The terms and notations are adopted from [11, 12].

Definition 2.1. *Take $\rho\in[0,1]$. Let $\chi_0,\chi_1:[0,1]\times\mathbb{R}\to[0,\infty)$ be continuous functions such that*

$$\lim_{\rho\to0^+}\chi_1(\rho,t)=1, \quad \lim_{\rho\to0^+}\chi_0(\rho,t)=0, \quad \lim_{\rho\to1^-}\chi_1(\rho,t)=0, \quad \lim_{\rho\to1^-}\chi_0(\rho,t)=1, \quad t\in\mathbb{R},$$

and $\chi_1(\rho,t)\neq0,\rho\in[0,1)$, $\chi_0(\rho,t)\neq0,\rho\in(0,1]$. Let $\phi(t)$ be a strictly increasing continuous function, then the proportional derivative of order ρ of h with respect to ϕ is defined by

$$\mathfrak{D}^{\rho,\phi}h(t)=\chi_1(\rho,t)h(t)+\chi_0(\rho,t)\frac{h'(t)}{\phi'(t)}. \tag{7}$$

For $\chi_1(\rho,t)=1-\rho$ and $\chi_0(\rho,t)=\rho$, the formula (7) becomes

$$\mathfrak{D}^{\rho,\phi}h(t)=(1-\rho)h(t)+\rho\frac{h'(t)}{\phi'(t)}. \tag{8}$$

Definition 2.2. *Take $\rho\in(0,1]$, $\alpha>0$, and $\phi\in C([a,b],\mathbb{R})$, $\phi'(t)>0$. Then thee left-side and right-side fractional integrals of h with respect to ϕ are defined by*

$$\begin{aligned} {}_a\mathfrak{I}^{\alpha,\rho,\phi}h(t) &= \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))}\left(\phi(t)-\phi(s)\right)^{\alpha-1}h(s)\phi'(s)ds, && (9)\\ \mathfrak{I}_b^{\alpha,\rho,\phi}h(t) &= \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_t^b e^{\frac{\rho-1}{\rho}(\phi(s)-\phi(t))}\left(\phi(s)-\phi(t)\right)^{\alpha-1}h(s)\phi'(s)ds, && (10)\end{aligned}$$

respectively.

Definition 2.3. *Take $\rho\in(0,1]$ and $\alpha>0$. Then the left-side fractional derivative of h with respect to ϕ is defined by*

$$\begin{aligned} {}_a\mathfrak{D}^{\alpha,\rho,\phi}h(t) &= \mathfrak{D}^{n,\rho,\phi}\,{}_a\mathfrak{I}^{n-\alpha,\rho,\phi}h(t)\\ &= \frac{\mathfrak{D}_t^{n,\rho,\phi}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))}\left(\phi(t)-\phi(s)\right)^{n-\alpha-1}h(s)\phi'(s)ds, && (11)\end{aligned}$$

and the right-side fractional derivative of h with respect to ϕ is defined by

$$\begin{aligned} \mathfrak{D}_b^{\alpha,\rho,\phi}h(t) &= {}_{\ominus}\mathfrak{D}^{n,\rho,\phi}\mathfrak{I}_b^{n-\alpha,\rho,\phi}h(t) \\ &= \frac{{}_{\ominus}\mathfrak{D}^{n,\rho,\phi}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_t^b e^{\frac{\rho-1}{\rho}(\phi(s)-\phi(t))}\left(\phi(s)-\phi(t)\right)^{n-\alpha-1}h(s)\phi'(s)ds, \end{aligned} \tag{12}$$

where $n=[\alpha]+1, \quad \mathfrak{D}^{n,\rho,\phi} \equiv \underbrace{\mathfrak{D}^{\rho,\phi}\mathfrak{D}^{\rho,\phi}\cdots\mathfrak{D}^{\rho,\phi}}_{n\text{-times}}$ *and*

$${}_{\ominus}\mathfrak{D}^{\rho,\phi}h(t) := (1-\rho)h(t)-\rho\frac{h'(t)}{\phi'(t)}, \quad {}_{\ominus}\mathfrak{D}^{n,\rho,\phi} \equiv \underbrace{{}_{\ominus}\mathfrak{D}^{\rho,\phi}{}_{\ominus}\mathfrak{D}^{\rho,\phi}\cdots{}_{\ominus}\mathfrak{D}^{\rho,\phi}}_{n\text{-times}}.$$

Lemma 2.4. *([11]) If $\rho\in(0,1]$ and $\alpha,\beta>0$. Then, for h is continuous and defined for $t\geq a$, one has*

$${}_a\mathfrak{I}^{\alpha,\rho,\phi}\left({}_a\mathfrak{I}^{\beta,\rho,\phi}h\right)(t) = {}_a\mathfrak{I}^{\beta,\rho,\phi}\left({}_a\mathfrak{I}^{\alpha,\rho,\phi}h\right)(t) = \left({}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi}h\right)(t), \tag{13}$$

$$\mathfrak{I}_b^{\alpha,\rho,\phi}\left(\mathfrak{I}_b^{\beta,\rho,\phi}h\right)(t) = \mathfrak{I}_b^{\beta,\rho,\phi}\left(\mathfrak{I}_b^{\alpha,\rho,\phi}h\right)(t) = \left(\mathfrak{I}_b^{\alpha+\beta,\rho,\phi}h\right)(t). \tag{14}$$

Lemma 2.5. *([12]) Let $n\in\mathbb{N}^+$, $\alpha\in(n-1,n)$, $h\in L^1(a,b)$, and $({}_a\mathfrak{I}^{\alpha,\rho,\phi}h)(t)\in AC^n([a,b],\mathbb{R})$. Then*

$${}_a\mathfrak{I}^{\alpha,\rho,\phi}{}_a\mathfrak{D}^{\alpha,\rho,\phi}h(t) = h(t)-e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))}\sum_{j=1}^{n}({}_a\mathfrak{I}^{j-\alpha,\rho,\phi}h)(a^+)\frac{(\phi(t)-\phi(a))^{\alpha-j}}{\rho^{\alpha-j}\Gamma(\alpha+1-j)}. \tag{15}$$

For $0<\alpha<1$, one has

$${}_a\mathfrak{I}^{\alpha,\rho,\phi}{}_a\mathfrak{D}^{\alpha,\rho,\phi}h(t) = h(t)-e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))}({}_a\mathfrak{I}^{1-\alpha,\rho,\phi}h)(a^+)\frac{(\phi(t)-\phi(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}. \tag{16}$$

Lemma 2.6. *([12]) Let $\alpha,\beta>0$. Then, for any $\rho>0$, we have*

1. $\left({}_a\mathfrak{I}^{\alpha,\rho,\phi}e^{\frac{\rho-1}{\rho}\phi(t)}\left(\phi(t)-\phi(a)\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)\, e^{\frac{\rho-1}{\rho}\phi(t)}}{\rho^{\alpha}\Gamma(\beta+\alpha)}\left(\phi(t)-\phi(a)\right)^{\alpha+\beta-1}$;

2. $\left(\mathfrak{I}_b^{\alpha,\rho,\phi}e^{-\frac{\rho-1}{\rho}\phi(t)}\left(\phi(b)-\phi(t)\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)\, e^{-\frac{\rho-1}{\rho}\phi(t)}}{\rho^{\alpha}\Gamma(\beta+\alpha)}\left(\phi(b)-\phi(t)\right)^{\alpha+\beta-1}$;

3. $\left({}_a\mathfrak{D}^{\alpha,\rho,\phi}e^{\frac{\rho-1}{\rho}\phi(t)}\left(\phi(t)-\phi(a)\right)^{\beta-1}\right)(t) = \frac{\rho^{\alpha}\Gamma(\beta)\, e^{\frac{\rho-1}{\rho}\phi(t)}}{\Gamma(\beta-\alpha)}\left(\phi(t)-\phi(a)\right)^{\beta-1-\alpha}$;

4. $\left(\mathfrak{D}_b^{\alpha,\rho,\phi}e^{-\frac{\rho-1}{\rho}\phi(t)}\left(\phi(b)-\phi(t)\right)^{\beta-1}\right)(t) = \frac{\rho^{\alpha}\Gamma(\beta)\, e^{-\frac{\rho-1}{\rho}\phi(t)}}{\Gamma(\beta-\alpha)}\left(\phi(b)-\phi(t)\right)^{\beta-1-\alpha}$.

3. Existence and uniqueness results

For investigating the existence and uniqueness of solutions for the Langevin problem (6), we consider the following auxiliary lemma.

Lemma 3.1. *Let $0<\alpha,\beta\leq1$ and $\sigma\in\mathfrak{C}$. Then the linear problem*

$$\begin{cases} {}_a\mathfrak{D}^{\alpha,\rho,\phi}\left({}_a\mathfrak{D}^{\beta,\rho,\phi}+\lambda\right)w(t)=\sigma(t), \quad t\in[a,b], \\ {}_a\mathfrak{I}^{1-\beta,\rho,\phi}w(a)=w_a, \quad w(b)+\mu w(\xi)=0, \quad a<\xi<b, \end{cases} \tag{17}$$

has a solution given by

$$\begin{aligned} w(t) &= {}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi}\sigma(t) - \lambda\, {}_a\mathfrak{I}^{\beta,\rho,\phi} w(t) \\ &+ \frac{1}{\nu_1}\left(\lambda\nu_2 - \nu_3 - w_a\nu_4\right) e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \\ &+ e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} w_a, \end{aligned} \tag{18}$$

where

$$\left.\begin{aligned} &\nu_1 := e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(a))} \frac{(\phi(b)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} + \mu e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(a))} \frac{(\phi(\xi)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \neq 0, \\ &\nu_2 := \left({}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi} w\right)(b) + \mu\left({}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi} w\right)(\xi), \\ &\nu_3 := \left({}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi} \sigma\right)(b) + \mu\left({}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi} \sigma\right)(\xi), \\ &\nu_4 := e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(a))} \frac{(\phi(b)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} + \mu e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(a))} \frac{(\phi(\xi)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}. \end{aligned}\right\} \tag{19}$$

Proof. Applying the proportional fractional integral operator ${}_a\mathfrak{I}^{\alpha,\rho,\phi}(\cdot)$ on the first equation of (17) and using Lemma 2.5, one has

$$\left({}_a\mathfrak{D}^{\beta,\rho,\phi} + \lambda\right) w(t) = {}_a\mathfrak{I}^{\alpha,\rho,\phi}\sigma(t) + c_1 e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}, \tag{20}$$

where $c_1 := \left(\mathfrak{I}^{1-\alpha,\rho,\phi}\left({}_a\mathfrak{D}^{\beta,\rho,\phi} + \lambda\right) w\right)(a^+)$.

Applying the proportional fractional integral operator ${}_a\mathfrak{I}^{\beta,\rho,\phi}(\cdot)$ on the equation (20), using Lemmas 2.5, 2.6 and the boundary condition ${}_a\mathfrak{I}^{1-\beta,\rho,\phi} w(a) = w_a$, we obtain

$$\begin{aligned} w(t) &= {}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi}\sigma(t) - \lambda\, {}_a\mathfrak{I}^{\beta,\rho,\phi} w(t) + c_1 e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \\ &+ e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} w_a. \end{aligned} \tag{21}$$

Using the boundary condition $w(b) + \mu w(\xi) = 0$ in (21), we get

$$\begin{aligned} &\left[\left({}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi}\sigma\right)(b) + \mu\left({}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi}\sigma\right)(\xi)\right] - \lambda\left[\left({}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi} w\right)(b) + \mu\left({}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi} w\right)(\xi)\right] \\ &c_1\left[e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(a))} \frac{(\phi(b)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} + \mu e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(a))} \frac{(\phi(\xi)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\right] \\ &\left[e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(a))} \frac{(\phi(b)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} + \mu e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(a))} \frac{(\phi(\xi)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}\right] w_a = 0. \end{aligned}$$

Consequently, we deduce that

$$c_1 = \frac{1}{\nu_1}\left(\lambda\nu_2 - \nu_3 - w_a\nu_4\right),$$

where the constants ν_1, ν_2, ν_3 and ν_4 are given in (19).

By substitution the value of c_1 in (21), we get

$$\begin{aligned} w(t) &= {}_a\mathfrak{I}^{\alpha+\beta,\rho,\phi}\sigma(t) - \lambda\, {}_a\mathfrak{I}^{\beta,\rho,\phi} w(t) \\ &+ \frac{1}{\nu_1}\left(\lambda\nu_2 - \nu_3 - w_a\nu_4\right) e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \\ &+ e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} w_a. \end{aligned}$$

This shows that $w(t)$ satisfies (18). This completes the proof. □

Using Lemma 3.1, we deduce that the solution of the Langevin problem (6) is given by

$$\begin{aligned} w(t) &= \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^t e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\alpha+\beta-1} \mathcal{G}(s,w(s))\,\phi'(s)ds \\ &- \frac{\lambda}{\rho^{\beta}\Gamma(\beta)} \int_a^t e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\beta-1} w(s)\phi'(s)ds \\ &+ \frac{1}{\nu_1}\Bigg[\frac{\lambda}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^b e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(s))} (\phi(b)-\phi(s))^{\alpha+\beta-1} w(s)\phi'(s)ds \\ &+ \frac{\lambda\mu}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^\xi e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(s))} (\phi(\xi)-\phi(s))^{\alpha+\beta-1} w(s)\phi'(s)ds \\ &- \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^b e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(s))} (\phi(b)-\phi(s))^{\alpha+\beta-1} \mathcal{G}(s,w(s))\,\phi'(s)ds \\ &- \frac{\mu}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^\xi e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(s))} (\phi(\xi)-\phi(s))^{\alpha+\beta-1} \mathcal{G}(s,w(s))\,\phi'(s)ds - w_a\nu_4\Bigg] \\ &\times e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} + e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} w_a. \end{aligned} \tag{22}$$

For fulfillment the main results, we shall pose the following hypotheses.

(H1) The function $\mathcal{G} : [a,b]\times\mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exists a positive constant $\Xi_{\mathcal{G}}$ such that

$$|\mathcal{G}(t,\nu_1) - \mathcal{G}(t,\nu_2)| \leq \Xi_{\mathcal{G}}|\nu_1-\nu_2|,\ \forall t\in[a,b] \text{ and } \nu_1,\nu_2\in\mathbb{R}.$$

(H3) There exist positive constants Ξ_1 and Ξ_2 such that

$$|\mathcal{G}(t,\nu)| \leq \Xi_1 + \Xi_2|\nu|,\ \forall t\in[a,b] \text{ and } \nu\in\mathbb{R}.$$

We set

$$
\begin{aligned}
\Theta_1 &:= \frac{\Xi_1\left(\phi(b)-\phi(a)\right)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{\left(\phi(b)-\phi(a)\right)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}|w_a| \\
&+ \frac{1}{|\nu_1|}\left[\frac{(|\mu|+1)\Xi_1\left(\phi(b)-\phi(a)\right)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+|w_a||\nu_4|\right]\left(\frac{\left(\phi(b)-\phi(a)\right)^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\right).
\end{aligned} \tag{23}
$$

$$
\Theta_2 := \frac{\Xi_2\left(\phi(b)-\phi(a)\right)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{|\lambda|\left(\phi(b)-\phi(a)\right)^{\beta}}{\rho^{\beta}\Gamma(\beta+1)}+\frac{(|\lambda|+\Xi_2)(|\mu|+1)\left(\phi(b)-\phi(a)\right)^{2\alpha+2\beta-1}}{|\nu_1|\rho^{2\alpha+2\beta-1}\Gamma(\alpha+\beta)\Gamma(\alpha+\beta+1)}. \tag{24}
$$

$$
\Theta_3 := \frac{(|\lambda|+\Xi_{\mathcal{G}})(|\mu|+1)\left(\phi(b)-\phi(a)\right)^{2\alpha+2\beta-1}}{|\nu_1|\rho^{2\alpha+2\beta-1}\Gamma(\alpha+\beta)\Gamma(\alpha+\beta+1)}. \tag{25}
$$

3.1. Existence result via Krasnoselskii's fixed point theorem

Theorem 3.2. [16] *(Krasnoselskii's fixed point theorem) Let $\mathfrak{S}$ be a closed convex and non-empty subset of a Banach space X. Let $\mathbb{T}_1, \mathbb{T}_2$ be the operators from $\mathfrak{S}$ to X such that:*

i. $\mathbb{T}_1 u+\mathbb{T}_2 v \in \mathfrak{S}$, *whenever* $u, v \in \mathfrak{S}$;

ii. $\mathbb{T}_1$ *is compact and continuous;*

iii. $\mathbb{T}_2$ *is a contraction mapping.*

Then, there exists a fixed point $z \in \mathfrak{S}$ such that $z=\mathbb{T}_1 z+\mathbb{T}_2 z$.

By virtue of Lemma 3.1, we define the operator $\mathbb{T}: \mathfrak{C} \to \mathfrak{C}$ by

$$
\begin{aligned}
(\mathbb{T}w)(t) &= \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))}\left(\phi(t)-\phi(s)\right)^{\alpha+\beta-1}\mathcal{G}\left(s,w(s)\right)\phi'(s)ds \\
&- \frac{\lambda}{\rho^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))}\left(\phi(t)-\phi(s)\right)^{\beta-1}w(s)\phi'(s)ds \\
&+ \frac{1}{\nu_1}\left[\frac{\lambda}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^b e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(s))}\left(\phi(b)-\phi(s)\right)^{\alpha+\beta-1}w(s)\phi'(s)ds\right. \\
&+ \frac{\lambda\mu}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^\xi e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(s))}\left(\phi(\xi)-\phi(s)\right)^{\alpha+\beta-1}w(s)g'(s)ds \\
&- \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^b e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(s))}\left(\phi(b)-\phi(s)\right)^{\alpha+\beta-1}\mathcal{G}\left(s,w(s)\right)\phi'(s)ds \\
&- \left.\frac{\mu}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^\xi e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(s))}\left(\phi(\xi)-\phi(s)\right)^{\alpha+\beta-1}\mathcal{G}\left(s,w(s)\right)\phi'(s)ds - w_a\nu_4\right] \\
&\times\ e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))}\frac{\left(\phi(t)-\phi(a)\right)^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}+e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))}\frac{\left(\phi(t)-\phi\phi(a)\right)^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}w_a, \quad t\in[a,b].
\end{aligned} \tag{26}
$$

The Langevin problem (6) will be transformed into the fixed point problem $w=\mathbb{T}w$.

Theorem 3.3. *Assume that the hypotheses **(H1)-(H3)** are satisfied. Then the Langevin problem (6) has at least one solution on $[a,b]$, if*

$$
\Theta_3 < 1, \tag{27}
$$

where Θ_3 is given by (25).

Proof. Let $\Gamma_\zeta = \{w \in \mathfrak{C} : \|w\|_{\mathfrak{C}} \le \zeta\}$ be a closed convex and non-empty subset of $\mathfrak{C}$, where $\zeta \ge \frac{\Theta_1}{1-\Theta_2}$ with $\Theta_2 < 1$, where Θ_1 and Θ_2 are given by (23) and (24).

The operator $\mathbb{T}$ will be split into two operators $\mathbb{T}_1$ and $\mathbb{T}_2$ on Γ_ζ as $\mathbb{T} = \mathbb{T}_1 + \mathbb{T}_2$, where

$$\begin{aligned}(\mathbb{T}_1 w)(t) &= \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^t e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\alpha+\beta-1} \mathcal{G}(s, w(s))\, \phi'(s)ds \\ &\quad - \frac{\lambda}{\rho^{\beta}\Gamma(\beta)} \int_a^t e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))} (\phi(t)-\phi(s))^{\beta-1} w(s)\phi'(s)ds,\end{aligned}$$

and

$$\begin{aligned}(\mathbb{T}_2 w)(t) &= \frac{1}{\nu_1}\Bigg[\frac{\lambda}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^b e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(s))} (\phi(b)-\phi(s))^{\alpha+\beta-1} w(s)\phi'(s)ds \\ &\quad + \frac{\lambda\mu}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^\xi e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(s))} (\phi(\xi)-\phi(s))^{\alpha+\beta-1} w(s)\phi'(s)ds \\ &\quad - \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^b e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(s))} (\phi(b)-\phi(s))^{\alpha+\beta-1} \mathcal{G}(s, w(s))\, \phi'(s)ds \\ &\quad - \frac{\mu}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^\xi e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(s))} (\phi(\xi)-\phi(s))^{\alpha+\beta-1} \mathcal{G}(s, w(s))\, \phi'(s)ds - w_a\nu_4\Bigg] \\ &\quad \times e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} + e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))} \frac{(\phi(t)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} w_a, \quad t \in [a,b].\end{aligned}$$

The proof will be divided into three main steps.

Taking into consideration that $\left|e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))}\right| < 1, \ \forall t > a, \ \rho \in (0,1)$, since ϕ is monotonic increasing function.

Step 1. $\mathbb{T}_1 w_1 + \mathbb{T}_2 w_2 \in \Gamma_\zeta$.

For each $t \in [a,b]$ and $w_1, w_2 \in \Gamma_\zeta$, one has

$$\begin{aligned}&|(\mathbb{T}_1 w_1)(t) + (\mathbb{T}_2 w_2)(t)| \\ &\le \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^t \left|e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))}\right| (\phi(t)-\phi(s))^{\alpha+\beta-1} |\mathcal{G}(s, w_1(s))|\, \phi'(s)ds \\ &+ \frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)} \int_a^t \left|e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(s))}\right| (\phi(t)-\phi(s))^{\beta-1} |w_1(s)|\, \phi'(s)ds \\ &+ \frac{1}{|\nu_1|}\Bigg[\frac{|\lambda|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^b \left|e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(s))}\right| (\phi(b)-\phi(s))^{\alpha+\beta-1} |w_2(s)|\, \phi'(s)ds \\ &+ \frac{|\lambda||\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^\xi \left|e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(s))}\right| (\phi(\xi)-\phi(s))^{\alpha+\beta-1} |w_2(s)|\, \phi'(s)ds \\ &+ \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^b \left|e^{\frac{\rho-1}{\rho}(\phi(b)-\phi(s))}\right| (\phi(b)-\phi(s))^{\alpha+\beta-1} |\mathcal{G}(s, w_2(s))|\, \phi'(s)ds \\ &+ \frac{|\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^\xi \left|e^{\frac{\rho-1}{\rho}(\phi(\xi)-\phi(s))}\right| (\phi(\xi)-\phi(s))^{\alpha+\beta-1} |\mathcal{G}(s, w_2(s))|\, \phi'(s)ds + |w_a||\nu_4|\Bigg] \\ &\times \left|e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))}\right| \frac{(\phi(t)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} + \left|e^{\frac{\rho-1}{\rho}(\phi(t)-\phi(a))}\right| \frac{(\phi(t)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} |w_a|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t(\phi(t)-\phi(s))^{\alpha+\beta-1}(\Xi_1+\Xi_2|w_1(s)|)\phi'(s)ds\\
&+ \frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)}\int_a^t(\phi(t)-\phi(s))^{\beta-1}|w_1(s)|\phi'(s)ds\\
&+ \frac{1}{|\nu_1|}\left[\frac{|\lambda|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^b(\phi(b)-\phi(s))^{\alpha+\beta-1}|w_2(s)|\phi'(s)ds\right.\\
&+ \frac{|\lambda||\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{\xi}(\phi(\xi)-\phi(s))^{\alpha+\beta-1}|w_2(s)|\phi'(s)ds\\
&+ \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^b(\phi(b)-\phi(s))^{\alpha+\beta-1}(\Xi_1+\Xi_2|w_2(s)|)\phi'(s)ds\\
&+ \left.\frac{|\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{\xi}(\phi(\xi)-\phi(s))^{\alpha+\beta-1}(\Xi_1+\Xi_2|w_2(s)|)\phi'(s)ds+|w_a||\nu_4|\right]\\
&\times \frac{(\phi(t)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}+\frac{(\phi(t)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}|w_a|\\
&\leq \frac{(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}(\Xi_1+\Xi_2\zeta)+\frac{|\lambda|\,(\phi(t)-\phi(s))^{\beta}}{\rho^{\beta}\Gamma(\beta+1)}\zeta+\frac{1}{|\nu_1|}\left[\frac{|\lambda|(|\mu|+1)\,(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\zeta\right.\\
&+ \left.\frac{(|\mu|+1)\,(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}(\Xi_1+\Xi_2\zeta)+|w_a||\nu_4|\right]\left(\frac{(\phi(b)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\right)+\frac{(\phi(b)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}|w_a|.
\end{aligned}
$$

In view of (23) and (24), we infer that

$$\|\mathbb{T}_1w_1+\mathbb{T}_2w_2\|_{\mathfrak{C}}\leq\Theta_1+\Theta_2\zeta\leq\zeta,$$

which leads to $\mathbb{T}_1w_1+\mathbb{T}_2w_2\in\Gamma_\zeta$.

Step 2. $\mathbb{T}_1$ is compact and continuous.

First, we shall show that $\mathbb{T}_1$ is continuous; Indeed, let w_n be a sequence such that $w_n\to w$ in $\mathfrak{C}$. For each $t\in[a,b]$, one has

$$
\begin{aligned}
|(\mathbb{T}_1w_n)(t)-(\mathbb{T}_1w)(t)| &\leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t(\phi(t)-\phi(s))^{\alpha+\beta-1}|\mathcal{G}(s,w_n(s))-\mathcal{G}(s,w(s))|\phi'(s)ds\\
&+ \frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)}\int_a^t(\phi(t)-\phi(s))^{\beta-1}|w_n(s)-w(s)|\phi'(s)ds,
\end{aligned}
$$

which leads to

$$\|\mathbb{T}_1w_n-\mathbb{T}_1w\|_{\mathfrak{C}}\leq\left(\frac{\Xi_{\mathcal{G}}\,(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}+\frac{|\lambda|\,(\phi(b)-\phi(a))^{\beta}}{\rho^{\beta}\Gamma(\beta+1)}\right)\|w_n-w\|_{\mathfrak{C}}\to0,\ \ as\ \ n\to\infty.$$

Thus, $\mathbb{T}_1$ is continuous. Also, $\mathbb{T}_1$ is uniformly bounded on Γ_ζ as

$$
\begin{aligned}
|(\mathbb{T}_1w)(t)| &\leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t(\phi(t)-\phi(s))^{\alpha+\beta-1}|\mathcal{G}(s,w(s))|\phi'(s)ds\\
&+ \frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)}\int_a^t(\phi(t)-\phi(s))^{\beta-1}|w(s)|\phi'(s)ds.
\end{aligned}
$$

So, we get

$$\|\mathbb{T}_1 w\|_{\mathfrak{C}} \leq \frac{(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}(\Xi_1+\Xi_2\zeta) + \frac{|\lambda|\,(\phi(b)-\phi(a))^{\beta}}{\rho^{\beta}\Gamma(\beta+1)}\zeta.$$

It remains to show that $\mathbb{T}_1$ is equicontinuos.
Set $\sup_{(t,w)\in[a,b]\times\Gamma_\zeta}|\mathcal{G}(t,w)| = \mathcal{M}_{\mathcal{G}} < \infty$. Then, for $a \leq \vartheta_1 < \vartheta_2 \leq b$ and for any $w \in \Gamma_\zeta$, we obtain that

$$\begin{aligned}
&|(\mathbb{T}_1 w)(\vartheta_2) - (\mathbb{T}_1 w)(\vartheta_1)| \\
&\leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{\vartheta_1}\left|(\phi(\vartheta_2)-\phi(s))^{\alpha+\beta-1} - (\phi(\vartheta_1)-\phi(s))^{\alpha+\beta-1}\right| |\mathcal{G}(s,w(s))|\,\phi'(s)ds \\
&+ \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_{\vartheta_1}^{\vartheta_2}(\phi(\vartheta_2)-\phi(s))^{\alpha+\beta-1}|\mathcal{G}(s,w(s))|\,\phi'(s)ds \\
&+ \frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)}\int_a^{\vartheta_1}\left|(\phi(\vartheta_2)-\phi(s))^{\beta-1} - (\phi(\vartheta_1)-\phi(s))^{\beta-1}\right| |w(s)|\,\phi'(s)ds \\
&+ \frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)}\int_{\vartheta_1}^{\vartheta_2}(\phi(\vartheta_2)-\phi(s))^{\beta-1}|w(s)|\,\phi'(s)ds \\
&\leq \frac{\mathcal{M}_{\mathcal{G}}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\left[2(\phi(\vartheta_2)-\phi(\vartheta_1))^{\alpha+\beta} + \left|(\phi(\vartheta_2)-\phi(a))^{\alpha+\beta} - (\phi(\vartheta_1)-\phi(a))^{\alpha+\beta}\right|\right] \\
&+ \frac{|\lambda|\mathcal{M}_{\mathcal{G}}}{\rho^{\beta}\Gamma(\beta+1)}\left[2(\phi(\vartheta_2)-\phi(\vartheta_1))^{\beta} + \left|(\phi(\vartheta_2)-\phi(a))^{\beta} - (\phi(\vartheta_1)-\phi\phi(a))^{\beta}\right|\right],
\end{aligned}$$

which tends to zero, as $\vartheta_2 \to \vartheta_1$ independently of $w \in \Gamma_\zeta$. Thus, $\mathbb{T}_1$ is equicontinuos and consequently $\mathbb{T}_1$ is relatively compact on Γ_ζ. Hence, by the Arzelà-Ascoli theorem, we conclude that $\mathbb{T}_1$ is compact on Γ_ζ.

Step 3. $\mathbb{T}_2$ is a contraction.

For each $t \in [a,b]$ and any $w_1, w_2 \in \mathfrak{C}$, one has

$$\begin{aligned}
&|(\mathbb{T}_2 w_1)(t) - (\mathbb{T}_2 w_2)(t)| \\
&\leq \frac{1}{|\nu_1|}\left[\frac{|\lambda|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^b(\phi(b)-\phi(s))^{\alpha+\beta-1}|w_1(s)-w_2(s)|\,\phi'(s)ds\right. \\
&+ \frac{|\lambda||\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{\xi}(\phi(\xi)-\phi(s))^{\alpha+\beta-1}|w_1(s)-w_2(s)|\,\phi'(s)ds \\
&+ \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^b(\phi(b)-\phi(s))^{\alpha+\beta-1}|\mathcal{G}(s,w_1(s)) - \mathcal{G}(s,w_2(s))|\,\phi'(s)ds \\
&+ \left.\frac{|\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^{\xi}(\phi(\xi)-\phi(s))^{\alpha+\beta-1}|\mathcal{G}(s,w_1(s)) - \mathcal{G}(s,w_2(s))|\,\phi'(s)ds\right]\frac{(\phi(t)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \\
&\leq \frac{(|\lambda|+\Xi_{\mathcal{G}})(|\mu|+1)\,(\phi(b)-\phi(a))^{2\alpha+2\beta-1}}{|\nu_1|\rho^{2\alpha+2\beta-1}\Gamma(\alpha+\beta)\Gamma(\alpha+\beta+1)}\|w_1-w_2\|_{\mathfrak{C}}.
\end{aligned}$$

Hence, using (25), we get

$$\|\mathbb{T}_2 w_1 - \mathbb{T}_2 w_2\|_{\mathfrak{C}} \leq \Theta_3\|w_1-w_2\|_{\mathfrak{C}}.$$

Using condition (27), we deduce that $\mathbb{T}_1$ is a contraction.
Therefore, by Krasnoselskii's fixed point theorem (Theorem 3.2), we infer that the Langevin problem (6) has at least one solution on $[a,b]$. The proof is completed. □

3.2. Uniqueness result via Banach contraction principle

Theorem 3.4. *If hypotheses **(H1)** and **(H2)** hold. Then the Langevin problem (6) has a unique solution on $[a, b]$, provided*

$$\Upsilon := \frac{\Xi_{\mathcal{G}}\,(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{|\lambda|\,(\phi(b)-\phi(a))^{\beta}}{\rho^{\beta}\Gamma(\beta+1)} + \frac{(|\lambda|+\Xi_{\mathcal{G}})(|\mu|+1)\,(\phi(b)-\phi(a))^{2\alpha+2\beta-1}}{|\nu_1|\rho^{2\alpha+2\beta-1}\Gamma(\alpha+\beta)\Gamma(\alpha+\beta+1)} < 1, \tag{28}$$

and

$$\begin{aligned} 0 < \Lambda := & \frac{\mathcal{M}^*_{\mathcal{G}}\,(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{|w_a|\,(\phi(b)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} \\ + & \frac{1}{|\nu_1|}\left[\frac{(|\mu|+1)\mathcal{M}^*_{\mathcal{G}}\,(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + |w_a||\nu_4|\right]\left(\frac{(\phi(b)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)}\right) < \infty. \end{aligned} \tag{29}$$

Proof. Consider the operator $\mathbb{T} : \mathfrak{C} \to \mathfrak{C}$ as defined in (26) and the set $\Gamma_\delta = \{w \in \mathfrak{C} : \|w\|_{\mathfrak{C}} \leq \delta\}$, where $\delta \geq \frac{\Lambda}{1-\Upsilon}$. Set $\mathcal{M}^*_{\mathcal{G}} = \sup_{t\in[a,b]} |\mathcal{G}(t, 0)|$.

For each $t \in [a, b]$ and $w \in \Gamma_\delta$, we have

$$\begin{aligned} |(\mathbb{T}w)(t)| \leq & \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^t (\phi(t)-\phi(s))^{\alpha+\beta-1}\left(|\mathcal{G}(s,w(s))-\mathcal{G}(s,0)|+|\mathcal{G}(s,0)|\right)\phi'(s)ds \\ + & \frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)}\int_a^t (\phi(t)-\phi(s))^{\beta-1}|w(s)|\,\phi'(s)ds \\ + & \frac{1}{|\nu_1|}\left[\frac{|\lambda|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^b (\phi(b)-\phi(s))^{\alpha+\beta-1}|w(s)|\,\phi'(s)ds\right. \\ + & \frac{|\lambda||\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^\xi (\phi(\xi)-\phi(s))^{\alpha+\beta-1}|w(s)|\,\phi'(s)ds \\ + & \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^b (\phi(b)-\phi(s))^{\alpha+\beta-1}\left(|\mathcal{G}(s,w(s))-\mathcal{G}(s,0)|+|\mathcal{G}(s,0)|\right)\phi'(s)ds \\ + & \left.\frac{|\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_a^\xi (\phi(\xi)-\phi(s))^{\alpha+\beta-1}\left(|\mathcal{G}(s,w(s))-\mathcal{G}(s,0)|+|\mathcal{G}(s,0)|\right)\phi'(s)ds + |w_a||\nu_4|\right] \\ \times & \frac{(\phi(t)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} + \frac{|w_a|\,(\phi(t)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)} \\ \leq & \frac{(\Xi_{\mathcal{G}}\delta+\mathcal{M}^*_{\mathcal{G}})\,(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{|\lambda|\delta\,(\phi(b)-\phi(a))^{\beta}}{\rho^{\beta}\Gamma(\beta+1)} \\ + & \frac{1}{|\nu_1|}\left[\frac{|\lambda|\delta(|\mu|+1)\,(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{(|\mu|+1)(\Xi_{\mathcal{G}}\delta+\mathcal{M}^*_{\mathcal{G}})\,(\phi(b)-\phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + |w_a||\nu_4|\right] \\ \times & \frac{(\phi(b)-\phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} + \frac{|w_a|\,(\phi(b)-\phi(a))^{\beta-1}}{\rho^{\beta-1}\Gamma(\beta)}. \end{aligned}$$

Therefore, by (28) and (29), we get

$$\|\mathbb{T}w\|_{\mathfrak{C}} \leq \Lambda + \Upsilon\delta \leq \delta.$$

Thus, $\mathbb{T}\Gamma_\delta \subset \Gamma_\delta$.

We show that $\mathbb{T}$ is a contraction. For each $t \in [a, b]$ and $w, z \in \mathfrak{C}$, we obtain

$$\begin{aligned}
&|(\mathbb{T}w)(t) - (\mathbb{T}z)(t)| \\
&\leq \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^t (\phi(t) - \phi(s))^{\alpha+\beta-1} |\mathcal{G}(s, w(s)) - \mathcal{G}(s, z(s))| \phi'(s)ds \\
&+ \frac{|\lambda|}{\rho^{\beta}\Gamma(\beta)} \int_a^t (\phi(t) - \phi(s))^{\beta-1} |w(s) - z(s)| \phi'(s)ds \\
&\leq \frac{1}{|\nu_1|} \left[\frac{|\lambda|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^b (\phi(b) - \phi(s))^{\alpha+\beta-1} |w(s) - z(s)| \phi'(s)ds \right. \\
&+ \frac{|\lambda||\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^{\xi} (\phi(\xi) - \phi(s))^{\alpha+\beta-1} |w(s) - z(s)| \phi'(s)ds \\
&+ \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^b (\phi(b) - \phi(s))^{\alpha+\beta-1} |\mathcal{G}(s, w(s)) - \mathcal{G}(s, z(s))| \phi'(s)ds \\
&+ \left. \frac{|\mu|}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)} \int_a^{\xi} (\phi(\xi) - \phi(s))^{\alpha+\beta-1} |\mathcal{G}(s, w(s)) - \mathcal{G}(s, z(s))| \phi'(s)ds \right] \frac{(\phi(t) - \phi(a))^{\alpha+\beta-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} \\
&\leq \left(\frac{\Xi_{\mathcal{G}} (\phi(b) - \phi(a))^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} + \frac{|\lambda| (\phi(b) - \phi(a))^{\beta}}{\rho^{\beta}\Gamma(\beta+1)} \right) \|w - z\|_{\mathfrak{C}} \\
&+ \frac{(|\lambda| + \Xi_{\mathcal{G}})(|\mu| + 1) (\phi(b) - \phi(a))^{2\alpha+2\beta-1}}{|\nu_1|\rho^{2\alpha+2\beta-1}\Gamma(\alpha+\beta)\Gamma(\alpha+\beta+1)} \|w - z\|_{\mathfrak{C}},
\end{aligned}$$

which implies that

$$\|\mathbb{T}w - \mathbb{T}z\|_{\mathfrak{C}} \leq \Upsilon \|w - z\|_{\mathfrak{C}}.$$

Therefor, the condition (28) emphasizes that $\mathbb{T}$ is a contraction. Hence, the Banach contraction principle indicates that $\mathbb{T}$ has a unique fixed point, which matches the unique solution of Langevin problem (6) on $[a, b]$. The proof is finished. □

4. An example

In this section, we will set forth an example to make efficient our main results.

Example 4.1. *Consider the following generalized proportional fractional Langevin equation:*

$$\begin{cases} {}_{0^+}\mathfrak{D}^{\frac{1}{3}, \frac{1}{2}, t^2} \left({}_{0^+}\mathfrak{D}^{\frac{1}{4}, \frac{1}{2}, t^2} + \frac{1}{5} \right) w(t) = \mathcal{G}(t, w(t)), \quad t \in [0, 1], \\ {}_{0^+}\mathfrak{I}^{\frac{3}{4}, \frac{1}{2}, t^2} w(0) = 0, \quad w(1) + \frac{1}{6} w(\frac{1}{3}) = 0, \end{cases} \tag{30}$$

where, $a = 0, b = 1, \alpha = \frac{1}{3}, \beta = \frac{1}{4}, \lambda = \frac{1}{5}, \mu = \frac{1}{6}, \xi = \frac{1}{3}, w_a = 0$ and $\rho = \frac{1}{2}$.

Set $\phi(t) = t^2$, it is clear that ϕ is continuous and monotonic increasing function on $[0, 1]$.

We choose $\mathcal{G}(t, w) = \frac{e^{-t}}{\sqrt{9+t^2}} + \frac{1}{99+e^t} \sin w$.

Let $\chi_1, \chi_2 \in \mathbb{R}$ and $t \in [0, 1]$. Then, we get

$$|\mathcal{G}(t, \nu_1) - \mathcal{G}(t, \nu_2)| \leq \frac{1}{100} |\nu_1 - \nu_2|.$$

Thus, the hypotheses **(H1)** and **(H2)** are satisfied with $\Xi_{\mathcal{G}} = \frac{1}{100}$.

Let $\chi \in \mathbb{R}$ and $t \in [0,1]$. Then, we get

$$|\mathcal{G}(t,\nu)| \leq \frac{1}{3} + \frac{1}{100}|\nu|.$$

Obviously, the hypothesis **(H3)** hold true with $\Xi_1 = \frac{1}{3}$ and $\Xi_2 = \frac{1}{100}$.

Using the above data, we obtain that $\Theta_2 = 0.8351543259 < 1$ and $\Theta_3 = 0.5559508375 < 1$, where Θ_2 and Θ_3 are given by (24) and (25) respectively. Therefore, the condition (27) hold true. Thus, all the hypotheses of Theorem 3.3 are satisfied. Hence, the consequence of Theorem 3.3 carries out to the Langevin problem (30) on $[0,1]$.

Furthermore, we get $\Upsilon = 0.8351543259 < 1$. Thus, the condition (28) also hold true. Therefore, all the hypotheses of Theorem 3.4 are satisfied. Hence, we conclude that the Langevin problem (30) has a unique solution on $[0,1]$.

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