Filomat 35:12 (2021), 4197–4208
https://doi.org/10.2298/FIL2112197K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Positive Solutions for Second Order Impulsive Differential Equations with Integral Boundary Conditions on the Real Line

Ilkay Yaslan Karaca[a], **Sezgi Aksoy**[a]

[a] *Department of Mathematics, Ege University, 35100 Bornova, Izmir, Turkey*

Abstract. This paper is denoted to study the existence of impulsive differential equations involving integral boundary conditions on the whole line by means of the Leray-Schauder Nonlinear Alternative theorem. An example is demonstrated the effectiveness of the our main result.

1. Introduction

The theory of impulsive differential equation is sufficient mathematical models for description of evolution processes. Those mathematical models whose evolution processes are characterised by combination of a continuous and jump change of their states. Impulsive differential equations arise often in recent studies and have been applied many fields, for example, in physics, natural science, chemical technology, economics, biotecnology, endustrial robotics, population dynamics etc. [1, 5, 9, 10, 14–17, 22–25]. There has been a substantial development in impulsive differential equations theory with fixed moments[3].

At the same time the existence and multiplicity of positive solutions for linear and nonlinearly second-order impulsive dynamics equations have been extensively studied, [2, 6–8, 12, 13, 19, 27].

The theory of boundary value problems on infinite intervals frequently seen in physics and applied mathematics, such as, in study of plasma physics, in analyzing the heat transfer in radial flow between circular disks, and in an analysis of the mass transfer on a rotating disk in non-Newtonian fluid, see [4, 21] and the references therein. While boundary value problems with integral boundary conditions are of great importance and arise in different fields such as chemical engineering, thermoelasticity, heat conduction, underground water flow and plasma physics [5, 6, 18, 26]. However, to the best our knowledge, the corresponding theory for the double impulsive integral boundary value problems on real line is not considered till now.

Karaca and Aksoy in [11] considered the following impulsive differential equations with integral boundary conditions on an infinite interval,

2020 *Mathematics Subject Classification*. Primary 34B37; Secondary 34B15, 34B18, 34B40.
Keywords. Impulsive differential equations, positive solutions, integral boundary conditions, Leray-Schauder Nonlinear Alternative Theorem, impulsive boundary value problems, fixed point theorem, real line.
Received: 26 March 2021; Accepted: 16 May 2021
Communicated by Maria Alessandra Ragusa
Email addresses: ilkay.karaca@ege.edu.tr (Ilkay Yaslan Karaca), sezgiaksoy6@gmail.com (Sezgi Aksoy)

$$\begin{cases} \frac{1}{p(t)}(p(t)z'(t))' + f(t,z(t),z'(t)) = 0, \quad \forall t \in J_+^{'} \\ \Delta z|_{t=t_k} = I_k(z(t_k)), \qquad k = 1,2,...,n \\ \Delta z'|_{t=t_k} = -\bar{I}_k(z(t_k)), \quad k = 1,2,...,n \\ \alpha_1 z(0) - \beta_1 \lim_{t\to 0^+} p(t)z'(t) = \int_0^\infty g_1(z(s))\psi(s)ds, \\ \alpha_2 \lim_{t\to\infty} z(t) + \beta_2 \lim_{t\to\infty} p(t)z'(t) = \int_0^\infty g_2(z(s)\psi(s)ds, \end{cases} \tag{1}$$

where $J = [0,\infty)$, $J_+ = (0,\infty)$, $J'_+ = J_+\backslash\{t_1,...,t_n\}$, $0 < t_1 < t_2 < ... < t_n$, $\Delta z|_{t=t_k}$ and $\Delta z'|_{t=t_k}$ denote the jump of $z(t)$ and $z^{'}(t)$ at $t = t_k$, $k = 1,2,...,n$, respectively. The authors showed the existence results of the positive solutions by using the fixed point theorem in cones.

In this article, we aim to investigate the existence of positive solutions for the following second-order impulsive integral boundary value problem (IBVP) with integral boundary conditions on the real line of the form,

$$\begin{cases} \frac{1}{p(t)}(p(t)z'(t))' + f(t,z(t),z'(t)) = 0, \quad \forall t \in J^{'} \\ \Delta z(t)|_{t=t_k} = I_k(z(t_k)), \qquad k = 1,2,... \\ \Delta z^{'}(t)|_{t=t_k} = -\bar{I}_k(z(t_k)), \quad k = 1,2,... \\ \alpha_1 \lim_{t\to-\infty} z(t) - \beta_1 \lim_{t\to-\infty} p(t)z^{'}(t) = \int_{-\infty}^\infty g_1(z(s))\psi(s)ds, \\ \alpha_2 \lim_{t\to\infty} z(t) + \beta_2 \lim_{t\to\infty} p(t)z^{'}(t) = \int_{-\infty}^\infty g_2(z(s))\psi(s)ds. \end{cases} \tag{2}$$

where $J = (-\infty,\infty)$, $J^{'} = J\backslash\{t_1,...,t_n,...\}$, $0 < t_1 < t_2 < ... < t_n$, $\Delta z|_{t=t_k}$ and $\Delta z'|_{t=t_k}$ prescribes the jump of $z(t)$ and $z^{'}(t)$ at each impulsive point $t = t_k$, occur so that,

$$\Delta z|_{t=t_k} = z(t_k^+) - z(t_k^-), \qquad \Delta z'|_{t=t_k} = p(t_k)(z'(t_k^+) - z'(t_k^-)),$$

where $z(t_k^+)$, $z^{'}(t_k^+)$ and $z(t_k^-)$, $z^{'}(t_k^-)$ respent the right-hand limit and left-hand limit of $z(t)$ and $z^{'}(t)$ at $t = t_k$, $k = 1,2,...$, respectively.

Throughout this research work, we assume that the following fundamental conditions hold;

$(H1)$ $\alpha_1,\alpha_2,\beta_1,\beta_2 \in J$ with $D = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(-\infty,\infty) > 0$ in which $B(t,s) = \int_t^s \frac{d\sigma}{p(\sigma)}$.

$(H2)$ $f \in C(J \times [0,\infty) \times J, [0,\infty))$ and also,

$$f(t,z,y) \leq u(t)k(z,y),$$

where $k \in C([0,\infty) \times J, [0,\infty))$ and $u \in L(J,(0,\infty))$ for $t \in J$.

$(H3)$ $g_1, g_2 : J \to [0,\infty)$ are continuous, nondecreasing functions, and for $t \in J$, z in a bounded set, $g_1(z(s)), g_2(z(s))$ are bounded.

$(H4)$ $I_k, \bar{I}_k \in C([0,\infty),[0,\infty))$ are bounded functions where

$$[\beta_2 + \alpha_2 B(t_k,\infty)]\bar{I}_k(z(t_k)) - \frac{\alpha_2}{p(t_k)} I_k(z(t_k)) > 0, \quad (k = 1,2,...).$$

$(H5)$ $\psi : J \to (0, \infty)$ is a continuous function with $\int_{-\infty}^{\infty} \psi(s)ds < \infty$.

$(H6)$ $p \in C(J, (0, \infty)) \cap C^1(J')$ with $p > 0$ on J, and $\int_{-\infty}^{\infty} \frac{ds}{p(s)} < \infty$.

We get the existence results of positive solutions for the impulsive IBVP (2) by means of the Leray-Schauder Nonlinear Alternative theorem in [20].

The plan of the paper is as follows. In Section 2, we give some substantial lemmas that will be used to demonstrate our main result. In Section 3, we give and prove our main result. Finally, in Section 4, we give an example to support our main result.

2. Preliminaries

In this section, we present several lemmas that will be used in the proof of the our main results.

We indicate $\theta(t)$ and $\varphi(t)$ by

$$\begin{aligned} \theta(t) &= \beta_1 + \alpha_1 \int_{-\infty}^{t} \frac{ds}{p(s)}, \\ \varphi(t) &= \beta_2 + \alpha_2 \int_{t}^{\infty} \frac{ds}{p(s)}. \end{aligned} \tag{3}$$

Lemma 2.1. *Assume that* $(H1) - (H6)$ *are satisfied. Then the impulsive IBVP* (2) *has a unique solution*

$$\begin{aligned} z(t) = &\int_{-\infty}^{\infty} G(t,s)p(s)f(s, z(s), z'(s))ds + \frac{\varphi(t)}{D}\int_{-\infty}^{\infty} g_1(z(s))\psi(s)ds \\ &+\frac{\theta(t)}{D}\int_{-\infty}^{\infty} g_2(z(s))\psi(s)ds + \sum_{k=1}^{\infty} G(t,t_k)\bar{I}_k(z(t_k)) + \sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{s=t_k} I_k(z(t_k)), \end{aligned} \tag{4}$$

where G(t,s) is defined by

$$G(t,s) = \frac{1}{D}\begin{cases} \theta(t)\varphi(s), & -\infty < t \le s < \infty, \\ \theta(s)\varphi(t), & -\infty < s \le t < \infty. \end{cases} \tag{5}$$

Remark 2.2. *We can easily obtain the following main properties of* $G(t,s)$*:*

(1) $G(t,s)$ *is continuous on* $J \times J$,

(2) *For each* $s \in J$, $G(t,s)$ *is continuously differentiable on* J *except* $t = s$,

(3) $\frac{\partial G(t,s)}{\partial t}\Big|_{t=s^+} - \frac{\partial G(t,s)}{\partial t}\Big|_{t=s^-} = \frac{1}{p(s)}$,

(4) $G(t,s) \le G(s,s) < \infty$, *and* $G_s(t,s) \le G_s(t,s)\Big|_{t=s} < \infty$,

(5) $|G_t(t,s)| \le \frac{c}{p(t)}G(s,s)$, *and* $|G_{st}(t,s)| \le \frac{c}{p(t)}G_s(t,s)\Big|_{t=s}$,
where

$$c = \frac{\max\{\alpha_1, \alpha_2\}}{\min\{\beta_1, \beta_2\}}, \tag{6}$$

(6) $\overline{G}(s) = \lim_{t\to\infty} G(t,s) = \frac{\beta_2}{D}\,\theta(s) \le G(s,s) < \infty,$

$\underline{G}(s) = \lim_{t\to-\infty} G(t,s) = \frac{\beta_1}{D}\,\varphi(s) \le G(s,s) < \infty,$

(7) $\overline{G}'(s) = \lim_{t\to\infty} G_s(t,s) = \frac{\beta_2}{D}\,\theta'(s) \le G_s(t,s)\big|_{t=s} < \infty,$

$\underline{G}'(s) = \lim_{t\to-\infty} G_s(t,s) = \frac{\beta_1}{D}\,\varphi'(s) \le G_s(t,s)\big|_{t=s} < \infty.$

Set

$PC(J) = \left\{z : J \to [0,\infty) : \ z \in C(J'),\ z(t_k^+) \text{ and } z(t_k^-) \text{ exist and } z(t_k^-) = z(t_k),\ \ 1 \le k \le n\right\}.$

$PC^1(J) = \left\{z \in PC(J) : \ z' \in PC(J'),\ z'(t_k^+) \text{ and } z'(t_k^-) \text{ exist and } z'(t_k^-) = z'(t_k)\right\}.$

$BPC^1(J) = \left\{z \in PC^1(J) : \ \lim_{t\to \mp\infty} z(t) < \infty \ \text{ and } \lim_{t\to\mp\infty} z'(t) < \infty\right\}.$

Then, we consider the Banach space $BPC^1(J)$ equipped with norm

$$\left\|z\right\|_{BPC^1} = \max\left\{\|z\|_{PC^1}, \|z'\|_{PC^1}\right\}$$

occur so that $\|z\|_{PC^1} = \sup_{t\in J}|z(t)|$, $\|z'\|_{PC^1} = \sup_{t\in J}|z'(t)|$. A function $z \in BPC^1(J) \cap C^2(J')$ is called a solution of the impulsive IBVP (2) if it satisfies (2)

Define

$$\begin{aligned}(Tz)(t) &= \int_{-\infty}^{\infty} G(t,s)p(s)f(s,z(s),z'(s))ds \\ &+\frac{\varphi(t)}{D}\int_{-\infty}^{\infty} g_1(z(s))\psi(s)ds + \frac{\theta(t)}{D}\int_{-\infty}^{\infty} g_2(z(s))\psi(s)ds \\ &+\sum_{k=1}^{\infty} G(t,t_k)\bar{I}_k(z(t_k)) + \sum_{k=1}^{\infty} p(t_k)G_s(t,t_k)|_{s=t_k} I_k(z(t_k)),\end{aligned} \tag{7}$$

where G is given by (5).

Obviously, x is a solution of the impulsive IBVP (2) if and only if z is a fixed point of the operator T.

It is necessary to list the following conditions here that,

$(H7)$ $0 < \int_{-\infty}^{\infty} G(s,s)u(s)p(s)ds < \infty.$

$(H8)$ $0 < \sum_{k=1}^{\infty} G(t_k,t_k) < \infty.$

$(H9)$ $0 < \sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} < \infty.$

Theorem 2.3 (Leray-Schauder Nonlinear Alternative Theorem). *([20]) Let E be a convex subset of a Banach space, U be a open subset of E with* $0 \in U$. *Then every completely continuous map* $T : \overline{U} \to E$ *has at least one of the two following properties:*

(i) *There exist an* $u \in \overline{U}$ *such that* $Tu = u$.

(ii) *There exist an* $u \in \partial U$ *and* $\lambda \in (0,1)$ *such that* $u = \lambda Tu$.

Lemma 2.4. *Let conditions (H1)-(H9) are satisfied, then $T : BPC^1(J) \to BPC^1(J)$ is a completely continuous operator.*

Proof. First of all, we show that $T : BPC^1(J) \to BPC^1(J)$ is well defined. Let $z \in BPC^1(J)$. There exists a constant $M > 0$ such that $\|z\|_{BPC^1} \leq M$. From $(H2)$, $(H3)$ and $(H4)$, we have

$$S_M = \sup\{S_1, S_2, S_3, S_4, S_5\}, \tag{8}$$

where

$$\begin{aligned}
S_1 &= \sup\{k(z,y) : -M \leq z \leq M,\ -M \leq y \leq M\} < \infty,\\
S_2 &= \sup\{I_k(z) :\ -M \leq z \leq M\} < \infty,\\
S_3 &= \sup\{\bar{I}_k(z) :\ -M \leq z \leq M\} < \infty,\\
S_4 &= \sup\{g_1(z) :\ -M \leq z \leq M\} < \infty,\\
S_5 &= \sup\{g_2(z) :\ -M \leq z \leq M\} < \infty.
\end{aligned}$$

From $(H5)$, $(H7)$, $(H8)$, $(H9)$, we get

$$\begin{aligned}
(Tz)(t) &= \int_{-\infty}^{\infty} G(t,s)p(s)f(s,z(s),z'(s))ds + \frac{\varphi(t)}{D}\int_{-\infty}^{\infty} g_1(z(s))\psi(s)ds\\
&\quad + \frac{\theta(t)}{D}\int_{-\infty}^{\infty} g_2(z(s))\psi(s)ds + \sum_{k=1}^{\infty} G(t,t_k)\bar{I}_k(z(t_k)) + \sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{s=t_k} I_k(z(t_k))\\
&\leq S_M\Big\{\int_{-\infty}^{\infty} G(s,s)u(s)p(s)ds + \frac{\varphi(-\infty)+\theta(\infty)}{D}\int_{-\infty}^{\infty}\psi(s)ds\\
&\quad + \sum_{k=1}^{\infty} G(t_k,t_k) + \sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Big\}\\
&< \infty.
\end{aligned} \tag{9}$$

So T is well defined. For any $t_1, t_2 \in J,\ t_1 < t_2$, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \big|G(t_1,s) - G(t_2,s)\big| p(s)f(s,z(s),z'(s))ds &\leq 2S_1\int_{-\infty}^{\infty} G(s,s)p(s)u(s)ds < \infty,\\
\sum_{k=1}^{\infty}\big|G(t_1,t_k) - G(t_2,t_k)\big|\bar{I}_k(z(t_k)) &\leq 2S_3\sum_{k=1}^{\infty} G(t_k,t_k) < \infty,\\
\sum_{k=1}^{\infty} p(t_k)\big|G_s(t,s)|_{\substack{t=t_1\\ s=t_k}} - G_s(t,s)|_{\substack{t=t_2\\ s=t_k}}\big| I_k(z(t_k)) &\leq 2S_2\sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} < \infty.
\end{aligned}$$

Hence, by the Lebesgue dominated convergence theorem and the fact that $G(t,s)$ is continuous on $J \times J$, we have for any $t_1, t_2 \in J$, $z \in BPC^1(J)$,

$$\begin{aligned}
\Big|(Tz)(t_1)-(Tz)(t_2)\Big| \quad &\leq \quad \int_{-\infty}^{\infty}\Big|G(t_1,s)-G(t_2,s)\Big|p(s)f(s,z(s),z'(s))ds\\
&+\frac{|\varphi(t_1)-\varphi(t_2)|}{D}\int_{-\infty}^{\infty}g_1(z(s))\psi(s)ds+\frac{|\theta(t_1)-\theta(t_2)|}{D}\int_{-\infty}^{\infty}g_2(z(s))\psi(s)ds\\
&+\sum_{k=1}^{\infty}\Big|G(t_1,t_k)-G(t_2,t_k)\Big|\bar{I}_k(z(t_k))\\
&+\sum_{k=1}^{\infty}\Big|p(t_k)G_s(t,s)|_{\substack{t=t_1\\ s=t_k}}-p(t_k)G_s(t,s)|_{\substack{t=t_2\\ s=t_k}}\Big|I_k(z(t_k))\\
\leq S_M \quad &\Big\{\int_{-\infty}^{\infty}\Big|G(t_1,s)-G(t_2,s)\Big|p(s)u(s)ds\\
&+\Big[\frac{|\varphi(t_1)-\varphi(t_2)|}{D}+\frac{|\theta(t_1)-\theta(t_2)|}{D}\Big]\int_{-\infty}^{\infty}\psi(s)ds\\
&+\sum_{k=1}^{\infty}\Big|G(t_1,t_k)-G(t_2,t_k)\Big|\\
&+\frac{1}{D}\sum_{t_k\leq t_1}p(t_k)\theta'(t_k)\Big|\varphi(t_1)-\varphi(t_2)\Big|\\
&+\frac{1}{D}\sum_{t_2\leq t_k}p(t_k)\Big|\varphi'(t_k)\Big|\Big|\theta(t_1)-\theta(t_2)\Big|\\
&+\frac{1}{D}\sum_{t_1\leq t_k\leq t_2}p(t_k)\Big|\theta(t_1)\varphi'(t_k)-\theta'(t_k)\varphi(t_2)\Big|\Big\}\\
\rightarrow \quad & 0 \;\; as \;\; t_1\rightarrow t_2,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\Big|(Tz)'(t_1)-(Tz)'(t_2)\Big| \quad &\leq \quad S_M\Big\{\frac{\alpha_2}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\int_{-\infty}^{t_1}\theta(s)p(s)u(s)ds+\frac{\alpha_1}{Dp(t_1)}\int_{t_1}^{t_2}\varphi(s)p(s)u(s)ds\\
&+\frac{\alpha_1}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\int_{t_2}^{\infty}\varphi(s)p(s)u(s)ds+\frac{\alpha_2}{Dp(t_2)}\int_{t_1}^{t_2}\theta(s)p(s)u(s)ds\\
&+\frac{\alpha_2}{D}\Big|\frac{1}{p(t_2)}-\frac{1}{p(t_1)}\Big|\int_{-\infty}^{\infty}\psi(s)ds+\frac{\alpha_1}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\int_{-\infty}^{\infty}\psi(s)ds\\
&+\frac{\alpha_1}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\sum_{t_2\leq t_k}\Big[\varphi(t_k)+\alpha_2\Big]+\frac{\alpha_2}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\sum_{t_k\leq t_1}\Big[\theta(t_k)+\alpha_1\Big]\\
&+\frac{\alpha_1}{Dp(t_1)}\sum_{t_1\leq t_k\leq t_2}\Big[\varphi(t_k)+\alpha_2\Big]+\frac{\alpha_2}{Dp(t_2)}\sum_{t_1\leq t_k\leq t_2}\Big[\theta(t_k)+\alpha_1\Big]\Big\}\\
\rightarrow \quad & 0 \;\; as \;\; t_1\rightarrow t_2.
\end{aligned}$$

Thus, $Tz\in PC^1(J)$. We can show that $Tz\in BPC^1(J)$.

Then by $(H5)$, $(H7)$, $(H8)$, $(H9)$, the properties (6), (7) and the Lebegue dominated convergence theorem, we have

$$\begin{aligned}\lim_{t\to\infty}(Tz)(t) &= \int_{-\infty}^{\infty}\overline{G}(s)p(s)f(s,z(s),z^{'}(s))ds+\frac{\varphi(\infty)}{D}\int_{-\infty}^{\infty}g_1(z(s))\psi(s)ds\\ &+\frac{\theta(\infty)}{D}\int_{-\infty}^{\infty}g_2(z(s))\psi(s)ds+\sum_{k=1}^{\infty}\overline{G}(t_k)\bar{I}_k(z(t_k))+\sum_{k=1}^{\infty}p(t_k)\overline{G}^{'}(t_k)I_k(z(t_k))\\ &< \infty, \end{aligned} \tag{11}$$

$$\begin{aligned}\lim_{t\to-\infty}(Tz)(t) &= \int_{-\infty}^{\infty}\underline{G}(s)p(s)f(s,z(s),z^{'}(s))ds+\frac{\varphi(-\infty)}{D}\int_{-\infty}^{\infty}g_1(z(s))\psi(s)ds\\ &+\frac{\theta(-\infty)}{D}\int_{-\infty}^{\infty}g_2(z(s))\psi(s)ds+\sum_{k=1}^{\infty}\underline{G}(t_k)\bar{I}_k(z(t_k))+\sum_{k=1}^{\infty}p(t_k)\underline{G}^{'}(t_k)I_k(z(t_k))\\ &< \infty \end{aligned} \tag{12}$$

and

$$\begin{aligned}\left|(Tz)^{'}(t)\right| &\leq S_M\Big\{\frac{c}{p(t)}\int_{-\infty}^{\infty}G(s,s)p(s)u(s)ds+\frac{\max\{\alpha_1,\alpha_2\}}{Dp(t)}\int_{-\infty}^{\infty}\psi(s)ds\\ &+\frac{c}{p(t)}\sum_{k=1}^{\infty}G(t_k,t_k)+\frac{c}{p(t)}\sum_{k=1}^{\infty}G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Big\}\\ &< \infty. \end{aligned} \tag{13}$$

Therefore, $\lim\limits_{t\to\mp\infty}|(Tz)^{'}(t)|<\infty$.
Hence $T: BPC^1(J)\to BPC^1(J)$.

Next, we prove that T is continuous. Assume that z_n be a sequence in $BPC^1(J)$ such that, then $\|z_n - z\|_{BPC^1}\to 0$ as $n\to\infty$. For this reason, there exists a positive constant r_0 such that

$$\max_{n\in\mathbb{N}-\{0\}}\left\{\|z_n\|_{BPC^1},\|z\|_{BPC^1}\right\}\leq r_0.$$

We will show that $Tz_n\to Tz$. We have

$$\int_{-\infty}^{\infty}G(t,s)p(s)\left|f(s,z_n(s),z_n^{'}(s))-f(s,z(s),z^{'}(s))\right|ds \leq 2S_{r_0}\int_{-\infty}^{\infty}G(s,s)p(s)u(s)ds<\infty,$$

$$\int_{-\infty}^{\infty}G_t(t,s)p(s)\left|f(s,z_n(s),z_n^{'}(s))-f(s,z(s),z^{'}(s))\right|ds \leq 2S_{r_0}\sup_{t\in J}\frac{c}{p(t)}\int_{-\infty}^{\infty}G(s,s)p(s)u(s)ds<\infty,$$

$$\int_{-\infty}^{\infty}\left|g_1(z_n(s))-g_1(z(s))\right|\psi(s)ds\leq 2S_{r_0}\int_{-\infty}^{\infty}\psi(s)ds<\infty,$$

$$\int_{-\infty}^{\infty}\left|g_2(z_n(s))-g_2(z(s))\right|\psi(s)ds\leq 2S_{r_0}\int_{-\infty}^{\infty}\psi(s)ds<\infty,$$

$$\sum_{k=1}^{\infty}G(t,t_k)\left|\bar{I}_k(z_n(t_k))-\bar{I}_k(z(t_k))\right|\leq 2S_{r_0}\sum_{k=1}^{\infty}G(t_k,t_k)<\infty,$$

$$\sum_{k=1}^{\infty} G_t(t,s)|_{s=t_k}\left|\bar{I}_k(z_n(t_k)) - \bar{I}_k(z(t_k))\right| \le 2S_{r_0} \sup_{t\in J} \frac{c}{p(t)} \sum_{k=1}^{\infty} G(t_k,t_k) < \infty,$$

$$\sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{s=t_k}\left|I_k(z_n(t_k)) - I_k(z(t_k))\right| \le 2S_{r_0} \sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} < \infty,$$

$$\sum_{k=1}^{\infty} p(t_k)G_{st}(t,s)|_{s=t_k}\left|I_k(z_n(t_k)) - I_k(z(t_k))\right| \le 2S_{r_0} \sup_{t\in J} \frac{c}{p(t)} \sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} < \infty.$$

Therefore, by the Lebesgue Dominated Converges theorem, we can get

$$\begin{aligned}
\left|(Tz_n)(t) - (Tz)(t)\right| \le\ & \int_{-\infty}^{\infty} G(t,s)p(s)\left|f(s,z_n(s),z_n'(s)) - f(s,z(s),z'(s))\right|ds \\
&+\frac{\varphi(-\infty)}{D}\int_{-\infty}^{\infty}\left|g_1(z_n(s)) - g_1(z(s))\right|\psi(s)ds \\
&+\frac{\theta(\infty)}{D}\int_{-\infty}^{\infty}\left|g_2(z_n(s)) - g_2(z(s))\right|\psi(s)ds \\
&+\sum_{k=1}^{\infty} G(t_k,t_k)\left|\bar{I}_k(z_n(t_k)) - \bar{I}_k(z(t_k))\right| \\
&+\sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\left|I_k(z_n(t_k)) - I_k(z(t_k))\right| \\
\to\ & 0 \ \ as\ \ n\to\infty,
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\left|(Tz_n)'(t) - (Tz)'(t)\right| \le\ & \int_{-\infty}^{\infty} G_t(t,s)p(s)\left|f(s,z_n(s),z_n'(s)) - f(s,z(s),z'(s))\right|ds \\
&+\frac{\alpha_2}{Dp(t)}\int_{-\infty}^{\infty}\left|g_1(z_n(s)) - g_1(z(s))\right|\psi(s)ds \\
&+\frac{\alpha_1}{Dp(t)}\int_{-\infty}^{\infty}\left|g_2(z_n(s)) - g_2(z(s))\right|\psi(s)ds \\
&+\sum_{k=1}^{\infty} G_t(t,s)|_{s=t_k}\left|\bar{I}_k(z_n(t_k)) - \bar{I}_k(z(t_k))\right| \\
&+\sum_{k=1}^{\infty} p(t_k)G_{st}(t,s)|_{s=t_k}\left|I_k(z_n(t_k)) - I_k(z(t_k))\right| \\
\to\ & 0 \ \ as\ \ n\to\infty.
\end{aligned} \tag{15}$$

So we obtain $\|Tz_n - Tz\|_{BPC^1} \to 0$ as $n\to\infty$. Consequently, $T : BPC^1(J) \to BPC^1(J)$ is continuous.

Consequently, we prove that the T is compact provided that it maps bounded sets into relatively compact sets.

Let D be a bounded subset of $BPC^1(J)$, then there exists $M_1 > 0$ such that $\|z\|_{BPC^1} < M_1$ for any $z \in D$. Hence, we have

$$\begin{aligned}\left|(Tz)(t)\right| &\leq S_{M_1}\Big\{\int_{-\infty}^{\infty} G(s,s)p(s)u(s)ds + \frac{\max\{\varphi(-\infty),\theta(\infty)\}}{D}\int_{-\infty}^{\infty}\psi(s)ds \\ &\quad + \sum_{k=1}^{\infty} G(t_k,t_k) + \sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Big\} < \infty. \end{aligned} \tag{16}$$

and

$$\begin{aligned}\left|(Tz)'(t)\right| &\leq S_{M_1}\Big\{\frac{c}{p(t)}\int_{-\infty}^{\infty} G(s,s)p(s)u(s)ds + \frac{\max\{\alpha_1,\alpha_2\}}{Dp(t)}\int_{-\infty}^{\infty}\psi(s)ds \\ &\quad + \frac{c}{p(t)}\sum_{k=1}^{\infty} G(t_k,t_k) + \frac{c}{p(t)}\sum_{k=1}^{\infty} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Big\} < \infty. \end{aligned} \tag{17}$$

It show that TD is uniformly bounded in $BPC^1(J)$.

Using the similar proof as (10), (11), for any $M > 0$, $t_1, t_2 \in [-M, M]$ and $z \in D$ we can get that $|(Tz)(t_1) - (Tz)(t_2)| \to 0$ and $|(Tz)'(t_1) - (Tz)'(t_2)| \to 0$ as $t_1 \to t_2$, i.e., $\|(Tz)(t_1) - (Tz)(t_2)\|_{BPC^1} \to 0$ as $t_1 \to t_2$. Thus $F = \{Tz : z \in D\}$ is equicontinuous on $[-M, M]$. Since $M > 0$ arbitrary, TD is locally equicontinuous on J.

By $(H7) - (H9)$ the properties (6), (7) and the Lebesgue dominated converges theorem, we get

$$\begin{aligned}|(Tz)(\infty) - (Tz)(t)| &\leq S_M\Big\{\int_{-\infty}^{\infty}\left|\overline{G}(s) - G(t,s)\right|p(s)u(s)ds \\ &\quad + \frac{|\varphi(\infty)-\varphi(t)|}{D}\int_{-\infty}^{\infty}\psi(s)ds + \frac{|\theta(\infty)-\theta(t)|}{D}\int_{-\infty}^{\infty}\psi(s)ds \\ &\quad + \sum_{k=1}^{\infty}\left|\overline{G}(t_k) - G(t,s)|_{s=t_k}\right| + \sum_{k=1}^{\infty} p(t_k)\left|\overline{G}'(t_k) - G_s(t,s)|_{s=t_k}\right|\Big\} \\ &\to 0 \ \ as \ \ t \to \infty \end{aligned}$$

and

$$\begin{aligned}|(Tz)'(t) - (Tz)'(\infty)| &\leq S_M\Big\{\frac{\alpha_2}{D}\left|\frac{1}{p(t)} - \frac{1}{p(\infty)}\right|\int_{-\infty}^{t}\theta(s)p(s)u(s)ds \\ &\quad + \frac{\alpha_1}{Dp(t)}\int_{t}^{\infty}\varphi(s)p(s)u(s)ds + \frac{\alpha_2}{Dp(\infty)}\int_{t}^{\infty}\theta(s)p(s)u(s)ds \\ &\quad + \frac{\alpha_2}{D}\left|\frac{1}{p(t)} - \frac{1}{p(\infty)}\right|\int_{-\infty}^{\infty}\psi(s)ds + \frac{\alpha_1}{D}\left|\frac{1}{p(t)} - \frac{1}{p(\infty)}\right|\int_{-\infty}^{\infty}\psi(s)ds \\ &\quad + \frac{\alpha_1}{Dp(t)}\sum_{t\leq t_k}\left[\varphi(t_k) + \alpha_2\right] + \frac{\alpha_2}{D}\left|\frac{1}{p(t)} - \frac{1}{p(\infty)}\right|\sum_{t_k\leq t}\left[\theta(t_k) + \alpha_1\right] \\ &\quad + \frac{\alpha_2}{Dp(\infty)}\sum_{t\leq t_k}\left[\theta(t_k) + \alpha_1\right]\Big\} \\ &\to 0 \ \ as \ \ t \to \infty. \end{aligned} \tag{18}$$

Hence, TD is equiconvergent at ∞. Similarly, we can abtain that TD is equiconvergent at $-\infty$. Therefore, $T : BPC^1(J) \to BPC^1(J)$ is completely continuous. As a result, Lemma 2.4 is proved.

3. Main Result

In this section, we will apply the Theorem 2.3 to get sufficient conditions for the existence of positive solutions for the impulsive IBVP (2).
Define

$$\begin{aligned} M &= \Big[1+\sup_{t\in J}\frac{c}{p(t)}\Big]S_\delta\Big[\int_{-\infty}^{\infty}G(s,s)p(s)u(s)ds+\frac{\max\{\varphi(-\infty),\theta(\infty)\}}{D}\int_{-\infty}^{\infty}\psi(s)ds \\ &\quad +\sum_{k=1}^{\infty}G(t_k,t_k)+\sum_{k=1}^{\infty}p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Big]. \end{aligned}$$

Theorem 3.1. *Assume that the conditions (H1)-(H9) hold and the following condition is satisfied: there exist positive constant δ such that*

$$\frac{\delta}{M}\geq 1 \tag{19}$$

Then the impulsive IBVP (2) *has a positive solution $z=z(t)$ such that*

$$0\leq z(t)\leq\delta,\qquad 0\leq |z^{'}(t)|\leq\delta,\qquad t\in J$$

Proof. Consider the following second-order the impulsive IBVP

$$\begin{cases} \frac{1}{p(t)}(p(t)z^{'}(t))'+\lambda f(t,z(t),z^{'}(t))=0, \quad \forall t\in J^{'} \\ \Delta z(t)|_{t=t_k}=I_k(z(t_k)), \qquad k=1,2,... \\ \Delta z^{'}(t)|_{t=t_k}=-\bar{I}_k(z(t_k)), \quad k=1,2,... \\ \alpha_1\lim\limits_{t\to-\infty}z(t)-\beta_1\lim\limits_{t\to-\infty}p(t)z^{'}(t)=\int_{-\infty}^{\infty}g_1(z(s))\psi(s)ds, \\ \alpha_2\lim\limits_{t\to\infty}z(t)+\beta_2\lim\limits_{t\to\infty}p(t)z^{'}(t)=\int_{-\infty}^{\infty}g_2(z(s))\psi(s)ds. \end{cases} \tag{20}$$

We know that solving the impulsive IBVP (20) is equivalent the solving $z=\lambda Tz$.

Define the open set

$$Z=\Big\{z\in BPC^1(J):\|z\|_{BPC^1}<\delta\Big\}. \tag{21}$$

Now, we show that there is no $z\in\partial Z$ such that $z=\lambda Tz$ for $\lambda\in(0,1)$. If not, then there exist $z\in\partial Z$ and $\lambda\in(0,1)$ such that $z=\lambda Tz$. Then for $\lambda\in(0,1)$, we have

$$\begin{aligned} \|z(t)\|_{PC^1} &= \|\lambda(Tz)(t)\|_{PC^1}=\sup_{t\in J}\lambda\Big|(Tz)(t)\Big|<\sup_{t\in J}|(Tz)(t)| \\ &< S_\delta\Big[\int_{-\infty}^{\infty}G(s,s)p(s)u(s)ds+\frac{\max\{\varphi(-\infty),\theta(\infty)\}}{D}\int_{-\infty}^{\infty}\psi(s)ds \\ &\quad +\sum_{k=1}^{\infty}G(t_k,t_k)+\sum_{k=1}^{\infty}p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Big]. \end{aligned} \tag{22}$$

Similarly we get

$$\begin{aligned}\|z^{'}(t)\|_{PC^1} &= \|\lambda(Tz)^{'}(t)\|_{PC^1} = \sup_{t\in J}\lambda\Big|(Tz)^{'}(t)\Big| < \sup_{t\in J}|(Tz)^{'}(t)|\\ &\le S_\delta\Big[\frac{c}{p(t)}\int_{-\infty}^{\infty}G(s,s)p(s)u(s)ds + \frac{\max\{\alpha_1,\alpha_2\}}{Dp(t)}\int_{-\infty}^{\infty}\psi(s)ds\\ &\quad+\frac{c}{p(t)}\sum_{k=1}^{\infty}G(t_k,t_k) + \frac{c}{p(t)}\sum_{k=1}^{\infty}p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Big]\\ &< \sup_{t\in J}\frac{c}{p(t)}S_\delta\Big[\int_{-\infty}^{\infty}G(s,s)p(s)u(s)ds + \frac{\max\{\varphi(-\infty),\theta(\infty)\}}{D}\int_{-\infty}^{\infty}\psi(s)ds\\ &\quad+\sum_{k=1}^{\infty}G(t_k,t_k) + \sum_{k=1}^{\infty}p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Big].\end{aligned}$$

Therefore $\delta = \|z(t)\|_{BPC^1} = \|\lambda Tz(t)\|_{BPC^1} = \lambda\max\{\|z(t)\|_{PC^1}, \|z^{'}(t)\|_{PC^1}\} < M$. This yields that

$$\frac{\delta}{M} < 1,$$

which is contradiction with (19). Thus, Theorem 2.3 implies that the impulsive IBVP (2) has a positive solution $z = z(t)$ such that

$$0 \le z(t) \le \delta, \quad 0 \le |z^{'}(t)| \le \delta, \quad t \in J.$$

4. Example

To illuminate how our main result can be used in practise we present the following example.
Consider the following boundary value problem:

$$\begin{cases}\frac{1}{1+t^2}((1+t^2)z^{'}(t))' + f(t,z(t),z^{'}(t)) = 0, \quad \forall t\in J^{'}\\ \Delta z(t)|_{t=\tan(\pi-\frac{1}{2^k})} = I_k(z(\tan(\pi-\frac{1}{2^k}))), \quad k=1,2,...\\ \Delta z^{'}(t)|_{t=\tan(\pi-\frac{1}{2^k})} = -\bar{I}_k(z(\tan(\pi-\frac{1}{2^k}))), \quad k=1,2,...\\ \lim_{t\to-\infty}z(t) - \frac{\pi}{2}\lim_{t\to-\infty}(1+t^2)z^{'}(t) = \int_{-\infty}^{\infty}g_1(z(s))\psi(s)ds,\\ \lim_{t\to\infty}z(t) + \frac{\pi}{2}\lim_{t\to\infty}(1+t^2)z^{'}(t) = 0.\end{cases} \tag{23}$$

where $f(t,z(t),z^{'}(t)) = \frac{1}{(1+t^2)^2(\pi^2-\arctan^2 t)}$, $\psi(t) = \frac{1}{1+t^2}$, $I_k(z(t)) = \frac{1}{50^k}\Big(1+\tan^2\frac{1}{2^k}\Big)z(t)$, $\bar{I}_k(z(t)) = \frac{1}{15^k}z(t)$, $g_1(z(t)) = \frac{z(t)}{100}$, $g_2(z(t)) = 0$. Let $k(t) = 1$, $u(t) = \frac{1}{(1+t^2)^2(\pi^2-\arctan^2 t)}$.
By simple computation we get $\int_{-\infty}^{\infty}\psi(t)ds = \pi < \infty$, $\int_{-\infty}^{\infty}\frac{ds}{p(s)} = \pi$, $\int_{-\infty}^{\infty}G(s,s)u(s)p(s)ds = \frac{1}{2} < \infty$, $\sum_{k=1}^{\infty}G(t_k,t_k) = 1 - \frac{1}{6\pi} < \infty$, $\sum_{k=1}^{\infty}p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} = \frac{1}{2\pi} < \infty$. Hence the conditions $(H1)-(H9)$ are satisfied. Choose $\delta = \frac{200}{4+\pi^2}$, then $S_\delta = 1$. Then we get

$$M = \Big(1+\frac{2}{\pi}\Big)S_\delta\Big(\frac{1}{2}+\frac{3\pi}{2}+1+\frac{1}{3\pi}\Big) \approx (10,3406) < \delta \approx (14,42)$$

Then all conditions of Theorem 3.1 are satisfied. Hence by Theorem 3.1, the impulsive IBVP (23) has at least one positive solution $z = z(t)$ such that

$$0 \leq z(t) \leq \delta, \quad 0 \leq |z^{'}(t)| \leq \delta, \quad t \in J.$$

References

[1] A. M. Samoilenko and N. A. Perestyuk, Impulsive Differential Equations, World Scientific Series on Nonlinear Science. Series A: Monographs and Theaties Word Sci., Singapore, (1995).
[2] C. Yu, J. Wang, and Y. Guo, Positive solutions for nonlinear double impulsive differential equations with p-Laplacian on infinite intervals. Bound Value Probl 2015, 147 (2015). https://doi.org/10.1186/s13661-015-0409-2
[3] D. D. Bainov and P. S. Simeonov, Systems with Impulse Effect, Ellis, Horwood, Chichester, (1989).
[4] D.O'Regan, Theory of Singular Boundary Value Problems, World Scientific, River Edge, 2004.
[5] F. Yoruk, N. A. Hamal, Existence results for nonlinear boundary value problems with integral boundary conditions on an infinite interval. Bound Value Probl 2012, 127 (2012). https://doi.org/10.1186/1687-2770-2012-127
[6] I. Yaslan, Positive solution for multi point impulsive boundary value problems on time scales, J. Nonlinear Funct. Anal.(2019), Article ID 5, (2019). doi.org/10.23952/jnfa.2019.5
[7] I. Yaslan, Existence of positive solutions for second-order impulsive boundary value problems on time scales, Mediterr. J. Math. 13, no. 4, 1613–1624, (2016).
[8] I. Y. Karaca and A. Sinanoğlu, Positive solutions of impulsive time-scale boundary value problems with p-Laplacian on the half-line, Filomat 33, no.2, 415–433, (2019).
[9] I. Y. Karaca and F. T. Fen, On positive solutions of nonlinear third-order impulsive boundary value problems on time scales, Mediterr. J. Math. 13, no.6, 4447–4461, (2016).
[10] I. Y. Karaca and F. T. Fen, Existence of positive solutions for a nonlinear nth-order m-point p-Laplacian impulsive boundary value problem, Math. Slovaca 67, no.2, 467–482, (2017).
[11] I. Y. Karaca and S. Aksoy, Positive Solutions for Second Order Impulsive Differential Equations with Integral Boundary Conditions on an Infinite Interval, Miskolc Mathematical Notes (Preprint).
[12] J. Xiao, J. J. Nieto and Z. Luo, Multiple positive solutions of the singular boundary value problem for second-order impulsive differential equations on the half-line, Bound. Value Probl., Article ID 281908, 13 pp., (2010).
[13] J. Li and J. J. Nieto, Existence of positive solutions for multipoint boundary value problem on the half-line with impulses, Bound. Value Probl., Art. ID 834158, (2009).
[14] J. J. Nieto and R. Rodriguez-Lopez, New comparison results for impulsive integro-differential equations and applications, J. Math. Anal. Appl. 328, 1343–1368, (2007).
[15] M. Arhmet, Principles of Discontinuous Dynamic Systems, Springer, New York, (2010).
[16] M. A. Ragusa, Necessary and sufficient condition for a VMO function, Appl. Math. Comput. 218 (24), 11952-11958, (2012).
[17] M. Bechohra, J. Henderson and SK. Ntouyas, Impulsive differential equations and inclusions, Hindawi Publishing Conporation, New York, (2006).
[18] Min Li, Jian-Ping Sun, Ya-Hong Zhao, Existence of positive solution for BVP of nonlinear fractional differential equation with integral boundary conditions. Adv Differ Equ 2020, 177 (2020). https://doi.org/10.1186/s13662-020-02618-9
[19] Panpan Niu, Xiao-Bao Shu And Yonggjin Li, The Existence And Hyers-Ulam Stability For Second Order Random Impulsive Differential Equations, Dynamic Systems and Applications, 28, No. 3 (2019), 673-690, 2019. doi: 10.12732/dsa.v28i3.9
[20] R. P. Agarwal, D.O'Regan and M. Meehan, Fixed Point Theory and Applications, Cambridgen University Press, 2004.
[21] R. P. Agarwal, D.O'Regan, Infinite interval problems for differential, Difference and IntegralEquations, Kluwer Academic Publisher, Netherlands, 2001.
[22] S. Gala, M. A. Ragusa and Z. Zhang, A regularity criterion in terms of pressure for the 3D viscous MHD equations, Bull. Malays. Math. Sci. Soc. 40 (4), 1677-1690, (2017).
[23] S. Gala and M. A. Ragusa, Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices, Appl. Anal. 95 (6), 1271-1279, (2016)
[24] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, World Sci., Singapore, (1989).
[25] W. Jiang, The existence of solutions for impulsive p-Laplacian boundary value problems at resonance on the half line, Bound. Value Probl. 39, 13pp., (2015).
[26] Zhaocai Hao, Tian Wang, "Some Existence Results for High Order Fractional Impulsive Differential Equation on Infinite Interval", Mathematical Problems in Engineering, vol. 2020, Article ID 6018273, 11 pages, 2020. https://doi.org/10.1155/2020/6018273
[27] Zihan Li, Xiao-Bao Shu, Fei Xu. The existence of upper and lower solutions to second order random impulsive differential equation with boundary value problem[J]. AIMS Mathematics, 2020, 5(6): 6189-6210. doi: 10.3934/math.2020398