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Stability and Solvability Analysis for a Class of Optimal Control Problems Described by Fractional Differential Equations with Non-Instantaneous Impulses

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Abstract. This paper is intended as an attempt to investigate the existence and stability of solutions for a class of fractional optimal control problems characterized with non-instantaneous impulsive differential equations. By using the method of minimizing sequence and the related conclusions of set-valued mapping, the results of solvability and stability for a class of optimal control problems are obtained in the suitable metric space.

1. Introduction

Impulsive phenomenon are the results of the sudden change in the state of the system due to external interference, which often occur in nature and human activities. According to the instantaneity and continuity of the effects, impulses are divided into instantaneous and non-instantaneous ones. Most of the mathematical models extracted from impulsive phenomena are characterized by impulsive differential equations, which can be classified under two categories in accordance with the types of impulses: non-instantaneous impulsive differential equations[1–8]and instantaneous ones[9–13] .

In view of the reality and significance of the differential equations with non-instantaneous impulses, this paper is intended as an attempt to study a class of optimal control problems described by such equations. For instance, the state change process of some elements during intravenous drug injection, periodic fishing, population survival [14, 15], and criterion for pest management [16] are described by non-instantaneous impulsive differential equations.

In recent years, fractional calculus, a generalization of the traditional calculus, has played an important role in physics, biology, economy, science, engineering, and other fields(see [17–19]). Now, in the real world, quite a considerable number of phenomena and processes are modeled by differential equation of fractional order in consideration of its various applications in various scientific areas, such as control theory, porous media motion and fluid mechanics. There are a lot of researchers are committed to investigating the fractional differential equations, for more general theory of fractional calculus and differential equations of fractional order, we refer readers to the references [17–29] and reference given therein.

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It is shown that the investigation of non-instantaneous impulsive differential equations of fractional order is of great importance to nature and human beings themselves. In the following, we will briefly sketch some existing results about the differential equations of fractional order with non-instantaneous impulse.

In [9], Ravi Agarwal et al. considered an initial value problem of a nonlinear scalar non-instantaneous impulsive fractional differential equation on a closed interval

$$\begin{cases} {}_0^cD^\alpha x(t) = f(t,x), \quad t \in (t_k, s_k], \quad k = 0,1,...,p+1, \\ x(t) = \phi_k(t, x(t), x(s_k - 0)), \quad t \in (s_k, t_{k+1}], \quad k = 0,1,...,p, \\ x(0) = x_0, \end{cases}$$

where $0 < \alpha < 1, p > 0, p \in \mathbb{N}, x, x_0 \in \mathbb{R}, f : \bigcup_{k=0}^{p+1}[t_k, s_k] \times \mathbb{R} \to \mathbb{R},\ \phi_k : [s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

By using iterative technique combined with the method of lower and upper solutions, they established the existence results of solutions for the problem.

In [10], Zhu and Liu studied the following periodic boundary value problem of nonlinear evolution equations of fractional order with non-instantaneous impulses

$$\begin{cases} {}^cD_t^\beta u(t) = A(t)u(t) + f(t, u(t)) + \int_0^t q(t-s)h(s, u(s))ds, \quad t \in (s_i, t_{i+1}], \quad i = 0,1,...,m, \\ u(t) = U_\beta(t, t_i)g_i(t, u(t)), \quad t \in (t_i, s_i], \quad i = 1,...,m, \\ u(0) = u(T), \end{cases}$$

where $\beta \in (0,1]$, $q : [0,T] \to X$ is continuous.

They obtained several sufficient conditions about the existence of mild solutions for the above problem by using non-compactness, the fixed point theorem, and the theory of β-resolvent family.

Optimal control theory originated in the late 1950s and the maximum principle founded by the former Soviet mathematician L. C. Pontryagin marked the beginning of a new stage in its process. Kalman put forward the concept of controllability in 1963 [30], which played an important role in the field of mathematical control theory. In recent years, many researchers have been devoted to the controllability of problems[32, 33, 36–39]. For instance, in [37], K. Balachandran et al. established a set of sufficient conditions for the controllability of nonlinear fractional dynamical system of order $1 < \alpha < 2$ in finite dimensional spaces.

With the further development of computer science and mathematics, the optimal control problems have achieved great progress, and the applications in real life are becoming more and more extensive. A number of scholars have been committed to the study on the optimal control problems (see [31, 34, 35, 40–42]).

In [31], Yu investigated the existence and stability of solutions of the problem

$$J_f(u^*) = \min_{u \in U} J_f(u),$$

where

$$J_f(u) \triangleq h(x(T)) + \int_{t_0}^T g(t, x(t), u(t))dt,$$

h, g are continuous and $x(t)$ satisfies the following differential equation

$$\begin{cases} \dot{x} = f(t, x(t), u(t)), \quad t \in [t_0, T], \\ x(t_0) = x^0. \end{cases}$$

At present, differential equations of fractional order are often used to characterize optimal control problems[34–40].

In [34], H.R. Marzban et al. dealt with existence of solutions for the delay fractional optimal control problems by using a hybrid of block-pulse functions and orthonormal Taylor polynomials. The aim of the paper was to determine the optimal control $U(t)$ by minimizing the cost functional

$$J = \frac{1}{2}X^T(1)SX(1) + \frac{1}{2}\int_0^1 \left(X^T(t)Q(t)X(t) + U^T(t)R(t)U(t)\right)dt,$$

where S and $Q(t)$ are symmetric positive semi-definite matrices, $R(t)$ is a symmetric positive definite matrix, $U(t) \in \mathbb{R}^q$ and $X(t) \in \mathbb{R}^p$ satisfies the following equation

$$\begin{cases} D^\alpha X(t) = A(t)X(t) + B(t)X(t-\tau) + E(t)U(t) + F(t)U(t-\mu), & 0 < \alpha \le 1,\ 0 \le t \le 1, \\ X(0) = X_0, \\ X(t) = \psi_1(t), \quad -\tau \le t < 0, \\ U(t) = \psi_2(t), \quad -\mu \le t < 0, \end{cases}$$

where $A(t), B(t), E(t), F(t)$ are matrices with suitable dimensions, $\psi_1(t)$ and $\psi_2(t)$ are the specified history functions associated with the given system.

In [35], Jean-Daniel Djida et al. discussed a diffusion equation with fractional time derivative with nonsingular Mittag-Leffler kernel in Hilbert spaces and obtained an optimality system, which characterizes the optimal control by using the Euler-Lagrange first-order optimality condition.

In [43], Liu et al. studied the optimal control problem for a new class of non-instantaneous impulsive differential equations and the controllability was proved by constructing a suitable control function. In [44], Achim Ilchmann et al. were concerned with the optimal control problem for regular linear differential-algebraic systems. In their paper, they derived an augmented system as the key to analyze the optimal control problem with tools well known for the optimal control of ordinary differential equations.

So far, we have found that the research findings on non-instantaneous impulsive differential equations of fractional order are still few, and the studies on fractional optimal control problems with non-instantaneous impulse are also less. In view of the widespread use of optimal control problem in industrial and mining enterprises, transportation, power industry and national economic management (see [41, 42]), inspired by [3, 7, 9, 31, 43], we mainly concentrate on the existence and stability of optimal control problem with nonlinear non-instantaneous impulsive differential equations. The problem is as follows.

Problem (P): *Looking for $u^* \in \mathbb{R}^n$ satisfying the equation*

$$J_{f_0,f_1,...,f_{p+1}}(u^*) = \min_{u\in U} J_{f_0,f_1,...,f_{p+1}}(u), \tag{1}$$

where

$$J_{f_0,f_1,...,f_{p+1}}(u) = g(x(0), x(T)) + \int_0^T h(t, x(t), u(t))dt, \tag{2}$$

$p > 0$ is a natural number, $U \subset \mathbb{R}^n$, $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $h : [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, and x satisfies the following differential equation

$$\begin{cases} {}_0^cD^\alpha x(t) = f_k(t, x, u), & t \in (t_k, s_k], \quad k = 0, 1, ..., p+1, \\ x(t) = \phi_k(t, x(t), x(s_k - 0)), & t \in (s_k, t_{k+1}], \quad k = 0, 1, ..., p, \\ x(0) = x_0, \end{cases} \tag{3}$$

where ${}_0^cD^\alpha$ is the Caputo fractional derivative, and $0 < \alpha < 1$, $f_k : [t_k, s_k] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ $(k = 0, 1, ..., p+1)$, $\phi_k : [s_k, t_{k+1}] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ $(k = 0, 1, ..., p)$.

The rest of this article is arranged as follows. In Section 2, we review some standard facts that are necessary for the paper, such as some important definitions and lemmas. In Section 3, it will be shown the existence and uniqueness of the solutions of fractional non-instantaneous impulsive differential equation (3). In Section 4, it is shown that the optimal control problem **(P)** is solvable in the defined space by constructing minimizing sequence. Finally, in Section 5, we discuss the stability of problem **(P)** by using related conclusions on set-valued mapping and an example is given to illustrate this result.

2. Preliminaries

Set two incrementing finite sequences of points $\{t_k\}_{k=0}^{p+1}$ and $\{s_k\}_{k=0}^{p+1}$, where $t_0 = 0 < s_k < t_{k+1} < s_{k+1}, k = 0, 1, ..., p$, $T = s_{p+1}$ and p is a natural number.

For convenience, the norms of all function spaces in the following are uniformly written as the symbol "$\|\cdot\|$" without confusion.

Set

$$PC[0,T] = \left\{ \begin{array}{ll} x|x:[0,T]\to\mathbb{R}, & x\in C\left((t_k,s_k],\mathbb{R}\right), x(t_k+0)<\infty,\ k=0,1,...,p+1, \\ & x\in C\left((s_k,t_{k+1}],\mathbb{R}\right), x(s_k+0)<\infty,\ k=0,1,...,p \end{array} \right\},$$

and

$$PC_m[0,T] = \{x=(x_1,...,x_m)^{\mathrm{T}}\,|\,x_i\in PC[0,T],\ i=1,2,...m\},$$

in which the norm is defined by $\|x\| = \max_{0\le t\le T}\|x(t)\|$, where $x(t)=(x_1(t),x_2(t),...,x_m(t))^{\mathrm{T}}\in PC_m[0,T]$ and $\|x(t)\| = \sqrt{(x_1(t))^2+(x_2(t))^2+...+(x_m(t))^2}$.

It is easy to prove that $(PC_m[0,T],\|\cdot\|)$ is a Banach space.

Now we define a set U and it meets the condition (H_u).

(H_u): *U is a nonempty and closed subset of $C([t_0,T]:R^n)$; U is uniformly bounded, namely there exists a constant $M>0$, such that $\|u\|\le M$ for all $u\in U$; U is equicontinuous, that is, $\forall\varepsilon>0$, there exists $\delta>0$ such that $\forall t_1,t_2\in[0,T]$ with $|t_1-t_2|<\delta$ and any $u\in U$, it holds that $\|u(t_1)-u(t_2)\|<\varepsilon$.*

By virtue of condition (H_u) and Ascoli-Arzela theorem, U is a nonempty compact subset of $C([t_0,T]:R^n)$.

Let

$$B = \{u\in\mathbb{R}^n : \|u\|\le M, M \text{ is the constant in } (H_u)\},$$

$$\mathscr{B}_k = [t_k,s_k]\times\mathbb{R}^m\times B,\ k=0,1,...,p+1.$$

Consider the following conditions($k=0,1,...,p$).

$(H_{w,k})$: $\forall x^1,x^2\in\mathbb{R}^m,\ \forall u\in B,\ \forall\, t\in[t_k,s_k]$,

$$\left\|w(t,x^1,u)-w(t,x^2,u)\right\|\le L_k\left\|x^1-x^2\right\|,$$

and $\sup_{(t,x,u)\in\mathscr{B}_k}\|w(t,x,u)\|\le C_k$, *where $L_k,C_k>0(k=0,1,...,p+1)$ are constants.*

Under the above condition, we define the metric space as follows:

$$F_k = \left\{ \begin{array}{ll} w=(w_1,...,w_m):\mathscr{B}_k\to\mathbb{R}^m| & w_i \text{ is continuous in } \mathscr{B}_k, i=1,...,m, \\ & w \text{ satisfies the condition } (H_{w,k}) \end{array} \right\},$$

with the metric ρ_k defined as

$$\rho_k(w^1,w^2) = \sup_{(t,x,u)\in\mathscr{B}_k}\left\|w^1(t,x,u)-w^2(t,x,u)\right\|,\ \forall w^1,w^2\in F_k.$$

One can demonstrate easily that (F_k,ρ_k) is a complete metric space for each k = 0, 1, ..., p+1.

Let

$$\phi_k : [s_k,t_{k+1}]\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m,\quad k=0,1,...,p,$$

where ϕ_k satisfies the following condition $(H_{\phi,k})$ $(k=0,1,...,p)$.

$(H_{\phi,k})$: *ϕ_k is continuous and $\forall x^1,x^2,y^1,y^2\in\mathbb{R}^m$,*

$$\|\phi_k(t,x^1,y^1)-\phi_k(t,x^2,y^2)\|\le Q_k\|x^1-x^2\|+\overline{Q}_k\|y^1-y^2\|,$$

and $\sup_{(t,x,y)\in[s_k,t_{k+1}]\times\mathbb{R}^m\times\mathbb{R}^m}\left\|\phi_k(t,x,y)\right\|\le D_k$, *where $Q_k,\overline{Q}_k,D_k>0(k=0,1,...,p)$ are constants and $Q_k+\overline{Q}_k<1$.*

Next, two necessary lemmas are given.

Lemma 2.1. *Assuming that U and F_k satisfy the conditions (H_u) and $(H_{w,k})(k=0,...,p+1)$ respectively, then $\forall u\in U,\forall f_k\in F_k$, the differential equation* (3) *is equivalent to the following integral-algebraic one*

$$x(t) = \left\{ \begin{array}{l} x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_0(s,x(s),u(s))ds, \qquad t\in[0,s_0], \\ \phi_k(t,x(t),x(s_k-0)), \qquad t\in(s_k,t_{k+1}],\ k=0,1,...,p, \\ \phi_{k-1}(t_k,x(t_k),x(s_{k-1}-0))+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}f_k(s,x(s),u(s))ds, \\ \qquad t\in(t_k,s_k],k=1,...,p+1. \end{array} \right. \tag{4}$$

Proof. Consider the corresponding initial value problem for the differential equations

$$\begin{cases} {}_{z}^{c}D^{\alpha}x(t) = f_k(t, x, u), & t \in (z, s_k], \quad k = 0, 1, ..., p+1, \\ x(z) = z_0. \end{cases} \tag{5}$$

By a direct calculation, equation (5) is equivalent to the following integral equation

$$x(t) = z_0 + \frac{1}{\Gamma(\alpha)} \int_z^t (t-s)^{\alpha-1} f_0(s, x(s), u(s))ds, \qquad t \in [z, s_k], \quad k = 0, 1, ..., p+1$$

Then, similar to the demonstration of the Lemma 1 of [9], it is easy to obtain this result through simple arguments. □

Lemma 2.2. *(Weakly Singular Gronwall Inequality [45]) Let $\alpha, T, \epsilon_1, \varepsilon_2 \in \mathbb{R}_+$. Moreover, assume that $\delta : [0, T] \to \mathbb{R}$ is a continuous function satisfying the inequality*

$$|\delta(x)| \le \epsilon_1 + \frac{\epsilon_2}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |\delta(t)| dt, \qquad x \in [0, T],$$

then

$$|\delta(x)| \le \epsilon_1 E_\alpha(\epsilon_2 x^\alpha), \qquad x \in [0, T],$$

where $E_\alpha(z)$ is the Mittag-Leffler function, and $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}$.

The follows are some concepts and conclusions related to set-valued mapping and readers can refer to [46, 47] for more details.

Definition 2.1. *([46]) Let U and F be metric spaces, a set-valued mapping $I : F \rightrightarrows U$ is called upper (lower) semi-continuous at $f \in F$ if for each open set $G \subset U$ with $G \supset I(f)(G \cap I(f) \neq \emptyset)$, there exists $\delta > 0$, such that $G \supset I(f')(G \cap I(f') \neq \emptyset)$ for any $f' \in F$ with $\rho(f', f) < \delta$. Furthermore, I is called continuous at F if I is both upper semi-continuous and lower semi-continuous at each $f \in F$.*

Definition 2.2. *([46]) Let U and F be metric spaces, a set-valued mapping $I : F \rightrightarrows U$ is called an usco mapping if I is upper semi-continuous and $I(f)$ is nonempty compact for each $f \in F$.*

Lemma 2.3. *([46]) Let U and F be metric spaces, a set-valued mapping $I : F \rightrightarrows U$ is closed if $Graph(I)$ is closed, where $Graph(I) := \{(f, u) \in F \times U : u \in I(f)\}$ is the graph of I.*

Definition 2.3. *([46]) Let U and F be metric spaces, $I : F \rightrightarrows U$ is a set-valued mapping. For each $f \in F$, $u \in I(f)$ is called an essential solution if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|u - u'\| < \varepsilon$ for any $f' \in F$ with $\rho(f', f) < \delta$.*

Remark 2.1. *([46]) The optimal control problem associated f is called essential if each $u \in I(f)$ is essential.*

Lemma 2.4. *([47]) If a set-valued mapping $I : F \rightrightarrows U$ is closed and U is compact, then I is upper semi-continuous at F.*

Lemma 2.5. *([48, 49]) Let U be a metric space, F be a complete metric space and $I : F \rightrightarrows U$ be an usco mapping. Then there exists a dense residual subset E of F such that I is lower semi-continuous at E.*

Definition 2.4. *([46]) Let (X, d) be a metric space and $\mathcal{A}, \mathcal{B}$ be any two nonempty bounded subsets of X. We call*

$$H(\mathcal{A}, \mathcal{B}) = \inf\{\varepsilon > 0 : \mathcal{A} \subset U(\varepsilon, \mathcal{B}), \mathcal{B} \subset U(\varepsilon, \mathcal{A})\}$$

the Hausdorff metric between $\mathcal{A}$ and $\mathcal{B}$, where

$$U(\varepsilon, \mathcal{A}) = \{x \in X : \exists a \in \mathcal{A}, \text{ such that } d(a, x) < \varepsilon\},$$

$$U(\varepsilon, \mathcal{B}) = \{x \in X : \exists b \in \mathcal{B}, \text{ such that } d(b, x) < \varepsilon\}.$$

3. Existence and uniqueness of solutions for fractional differential equation with non-instantaneous impulses

In this section, it is demonstrated that there exists a unique solution of fractional non-instantaneous impulsive differential equation (3).

Theorem 3.1. *Supposing that the conditions* $(H_u), (H_{w,k}), (H_{\phi,k})$ $(k = 0, 1, ..., p), (H_{w,p+1})$ *are satisfied, then the equation* (3) *has a unique solution.*

Proof. Define an operator:

$$\mathscr{T} : PC_m[0,T] \to PC_m[0,T]$$

where $\forall x \in PC_m[0,T]$,

$$(\mathscr{T}x)(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_0(s, x(s), u(s))ds, & t \in [0, s_0], \\ \phi_k(t, x(t), x(s_k - 0)), & t \in (s_k, t_{k+1}], \quad k = 0, 1, ..., p, \\ \phi_{k-1}(t_k, x(t_k), x(s_{k-1} - 0)) + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1} f_k(s, x(s), u(s))ds, & t \in (t_k, s_k], \ k = 1, ..., p+1. \end{cases}$$

By Lemma 2.1, $\mathscr{T}$ is well defined.

Then, we define another norm in the Banach space $PC_m[0,T]$, that is

$$\|x\|_* = \max_{0 \le t \le T} e^{-\kappa t}\|x\|, \qquad \forall x \in PC_m[0,T],$$

where $\kappa > 0$ is a constant and it satisfies the following conditions (H_κ):

(H_κ): $\frac{L_0}{\kappa^\alpha} < 1$ *and* $\frac{L_k}{\kappa^\alpha} < 1 - Q_{k-1} - \overline{Q}_{k-1}, k = 1, ..., p+1.$

It is easy to verify that $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent norms. In fact,

$$e^{-\kappa t}\|x\| \le \|x\|_* = \max_{0 \le t \le T} e^{-\kappa t}\|x\| \le \|x\|.$$

Furthermore, $\forall x^1, x^2 \in PC_m[0,T]$, $\forall t \in [0,T]$, one has

$$e^{-\kappa t}\|x^1(t) - x^2(t)\| \le e^{-\kappa t}\|x^1 - x^2\| \le \|x^1 - x^2\|_*,$$

then

$$\|x^1(t) - x^2(t)\| \le e^{\kappa t}\|x^1 - x^2\|_*.$$

Next, we use norm $\|\cdot\|_*$ to carry on the related demonstration.

Take two arbitrary functions $x^1, x^2 \in PC_m[0,T]$, then, as shown below, we can get that

(1) for $t \in [0, s_0]$, we have

$$\begin{aligned}
& e^{-\kappa t}\left\|(\mathscr{T}x^1)(t)-(\mathscr{T}x^2)(t)\right\| \\
= \;& e^{-\kappa t}\left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_0(s,x^1(s),u(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_0(s,x^2(s),u(s))ds\right\| \\
\le \;& e^{-\kappa t}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f_0(s,x^1(s),u(s))ds - f_0(s,x^2(s),u(s))\|ds \\
\le \;& e^{-\kappa t}\frac{L_0}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x^1(s)-x^2(s)\|ds \\
= \;& \frac{L_0}{\Gamma(\alpha)}\|x^1-x^2\|_*\int_0^t (t-s)^{\alpha-1}e^{-\kappa(t-s)}ds \\
= \;& \frac{L_0}{\kappa^\alpha\Gamma(\alpha)}\|x^1-x^2\|_*\int_0^t (\kappa\tau)^{\alpha-1}e^{-\kappa\tau}d\kappa\tau \\
= \;& \frac{L_0}{\kappa^\alpha\Gamma(\alpha)}\|x^1-x^2\|_*\int_0^{\kappa t} s^{\alpha-1}e^{-s}ds \\
\le \;& \frac{L_0}{\kappa^\alpha\Gamma(\alpha)}\|x^1-x^2\|_*\int_0^{+\infty} s^{\alpha-1}e^{-s}ds \\
= \;& \frac{L_0}{\kappa^\alpha}\|x^1-x^2\|_*;
\end{aligned}$$

(2) for $t \in (s_k, t_{k+1}], k = 0, 1, ..., p$, it holds that

$$\begin{aligned}
e^{-\kappa t}\|(\mathscr{T}x^1)(t)-(\mathscr{T}x^2)(t)\| &= e^{-\kappa t}\|\phi_k(t,x^1(t),x^1(s_k-0)) - \phi_k(t,x^2(t),x^2(s_k-0))\| \\
&\le e^{-\kappa t}\left(Q_k\|x^1(t)-x^2(t)\| + \overline{Q}_k\|x^1(s_k-0)-x^2(s_k-0)\|\right) \\
&\le e^{-\kappa t}(Q_k+\overline{Q}_k)\|x^1-x^2\| \\
&\le (Q_k+\overline{Q}_k)\|x^1-x^2\|_*;
\end{aligned}$$

(3) for $t \in (t_k, s_k],\ k = 1, ..., p+1$, similar to the processes of parts (1)-(2), we obtain these results like that

$$e^{-\kappa t}\|\phi_{k-1}(t_k,x^1(t_k),x^1(s_{k-1}-0)) - \phi_{k-1}(t_k,x^2(t_k),x^2(s_{k-1}-0))\| \le (Q_{k-1}+\overline{Q}_{k-1})\|x^1-x^2\|_*,$$

and

$$\begin{aligned}
& e^{-\kappa t}\frac{1}{\Gamma(\alpha)}\left\|\int_{t_k}^t (t-s)^{\alpha-1}[f_k(s,x^1(s),u(s))ds - f_k(s,x^2(s),u(s))]ds\right\| \\
\le \;& e^{-\kappa t}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f_k(s,x^1(s),u(s))ds - f_k(s,x^2(s),u(s))\|ds \\
= \;& \frac{L_k}{\kappa^\alpha}\|x^1-x^2\|_*.
\end{aligned}$$

Thus, it yields that

$$\begin{aligned}
& e^{-\kappa t}\|(\mathscr{T}x^1)(t) - (\mathscr{T}x^2)(t)\| \\
= \ & e^{-\kappa t}\left\|\phi_{k-1}(t_k, x^1(t_k), x^1(s_{k-1}-0)) + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f_k(s, x^1(s), u(s))ds\right. \\
& \left. -\phi_{k-1}(t_k, x^2(t_k), x^2(s_{k-1}-0)) - \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f_k(s, x^2(s), u(s))ds\right\| \\
\leq \ & e^{-\kappa t}\|\phi_{k-1}(t_k, x^1(t_k), x^1(s_{k-1}-0)) - \phi_{k-1}(t_k, x^2(t_k), x^2(s_{k-1}-0))\| \\
& + e^{-\kappa t}\frac{1}{\Gamma(\alpha)}\left\|\int_{t_k}^{t}(t-s)^{\alpha-1}[f_k(s, x^1(s), u(s))ds - f_k(s, x^2(s), u(s))]ds\right\| \\
\leq \ & (Q_{k-1} + \overline{Q}_{k-1})\|x^1 - x^2\|_* + \frac{L_k}{\kappa^\alpha}\|x^1 - x^2\|_* \\
= \ & \left(Q_{k-1} + \overline{Q}_{k-1} + \frac{L_k}{\kappa^\alpha}\right)\|x^1 - x^2\|_*.
\end{aligned}$$

Let

$$A = \max_{k=1,\ldots,p+1}\left\{\frac{L_0}{\kappa^\alpha}, Q_{k-1} + \overline{Q}_{k-1} + \frac{L_k}{\kappa^\alpha}\right\}.$$

Consequently, it has $0 < A < 1$ for the conditions $(H_{\phi,k})(k = 0, 1, ..., p)$ and (H_κ).

From parts (1) − (3), it can be concluded that $\forall x^1, x^2 \in PC_m[0, T]$,

$$\|\mathscr{T}x^1 - \mathscr{T}x^2\|_* \leq A\|x^1 - x^2\|_*.$$

According to the Contraction Mapping Principle in Banach spaces, operator $\mathscr{T}$ has a unique fixed point in $PC_m[0, T]$, which indicates that the equation (3) has a unique solution.

This completes the proof. □

4. Solvability for the Optimal Control Problem

In this section, we will investigate the existence of solutions for optimal control problem **(P)** by the method of minimization. A crucial lemma is given as follows firstly.

Lemma 4.1. *Let $\{f_k^q\} \subset F_k(k = 0, ..., p+1), \{u^q\} \subset U$ and ϕ_k^q are functions satisfying the condition $(H_{\phi,k})$ $(k = 0, 1, ..., p)$. Assuming that the conditions (H_u), $(H_{w,k})$ all hold, if $f_k^q \to f_k(q \to +\infty, k = 0, ..., p+1)$, $u^q \to u(q \to +\infty)$ and $\phi_k^q \to \phi_k(q \to +\infty, k = 0, ..., p)$, then $x^q \to x(q \to +\infty)$, where*

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_0(s, x(s), u(s))ds, & t \in [0, s_0], \\ \phi_k(t, x(t), x(s_k - 0)), & t \in (s_k, t_{k+1}], \quad k = 0, ..., p, \\ \phi_{k-1}(t_k, x(t_k), x(s_{k-1}-0)) + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}f_k(s, x(s), u(s))ds, & t \in (t_k, s_k], \ k = 1, ..., p+1, \end{cases}$$

and

$$x^q(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f_0^q(s, x^q(s), u^q(s))ds, & t \in [0, s_0], \\ \phi_k^q(t, x^q(t), x^q(s_k - 0)), & t \in (s_k, t_{k+1}], \quad k = 0, ..., p, \\ \phi_{k-1}^q(t_k, x^q(t_k), x^q(s_{k-1}-0)) + \frac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}f_k^q(s, x^q(s), u^q(s))ds, & t \in (t_k, s_k], \ k = 1, ..., p+1. \end{cases}$$

Proof. In the following, $\varepsilon > 0$ is an arbitrary given number. The argument will be presented step by step.

(1) $\forall t \in [0, s_0]$. On account of $f_0^q \to f_0(q \to +\infty)$, there exists $\widetilde{N}_1 > 0$ such that $\rho_0(f_0^q, f_0) < \frac{\Gamma(\alpha+1)\varepsilon}{2}$ with $q > \widetilde{N}_1$. For a fixed $x \in PC_m[0, T]$, there exists a constant $r > 0$ such that $\|x\| \leq r$. Besides, f_0 is uniformly

continuous in $[0, s_0] \times \{x | x \in \mathbb{R}^m, \|x\| \le r\} \times \{u | u \in \mathbb{R}^n, \|u\| \le M\}$, and $u^q \to u (q \to +\infty)$, then there exists $\widetilde{N}_2 > 0$ such that $\|f_0(t, x, u^q) - f_0(t, x, u)\| < \frac{\Gamma(\alpha+1)\varepsilon}{2}$ with $q > \widetilde{N}_2$. Let $N_0 = \max\{\widetilde{N}_1, \widetilde{N}_2\}$, thus, when $q > N_0$, we have

$$\begin{aligned}
& \|x^q(t) - x(t)\| \\
= & \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_0^q(s, x^q(s), u^q(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_0(s, x(s), u(s)) ds \right\| \\
\le & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f_0^q(s, x^q(s), u^q(s)) - f_0(s, x(s), u(s)) \right\| ds \\
\le & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f_0^q(s, x^q(s), u^q(s)) - f_0(s, x^q(s), u^q(s)) \right\| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f_0(s, x^q(s), u^q(s)) - f_0(s, x(s), u^q(s)) \right\| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f_0(s, x(s), u^q(s)) - f_0(s, x(s), u(s)) \right\| ds \\
= & \frac{\Gamma(\alpha+1)\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{L_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x^q(s) - x(s)\| ds \\
= & t^\alpha \varepsilon + \frac{L_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x^q(s) - x(s)\| ds.
\end{aligned}$$

In consideration of the Gronwall inequality of Lemma 2.2, it holds that

$$\|x^q(t) - x(t)\| \le t^\alpha E_\alpha(L_0 t^\alpha)\varepsilon \le s_0^\alpha E_\alpha(L_0 s_0^\alpha)\varepsilon, \quad q > N_0.$$

(2) $\forall t \in (s_0, t_1]$. For the above $\varepsilon > 0$, there exists $\overline{N}_1 > 0$, such that $\|\phi_0^q - \phi_0\| < \varepsilon$ with $q > \overline{N}_1$ as a result of $\phi_0^q \to \phi_0 (q \to +\infty)$. Then, one has

$$\begin{aligned}
\|x^q(t) - x(t)\| &= \|\phi_0^q(t, x^q(t), x^q(s_0 - 0)) - \phi_0(t, x(t), x(s_0 - 0))\| \\
&\le \|\phi_0^q(t, x^q(t), x^q(s_0 - 0)) - \phi_0(t, x^q(t), x^q(s_0 - 0))\| + \|\phi_0(t, x^q(t), x^q(s_0 - 0)) - \phi_0(t, x(t), x(s_0 - 0))\| \\
&\le \varepsilon + Q_0 \|x^q(t) - x(t)\| + \overline{Q}_0 \|x^q(s_0 - 0) - x(s_0 - 0)\|.
\end{aligned}$$

It is available from (1) that $\|x^q(s_0 - 0) - x(s_0 - 0)\| \le s_0^\alpha E_\alpha(L_0 s_0^\alpha)\varepsilon$ for $q > N_0$. Let $\mathcal{N}_0 = \max\{\overline{N}_1, N_0\}$, therefore if $q > \mathcal{N}_0$, it has

$$\|x^q(t) - x(t)\| \le \frac{1 + s_0^\alpha E_\alpha(L_0 s_0^\alpha)}{1 - Q_0} \varepsilon \triangleq \widetilde{A_0}\varepsilon, \qquad \forall t \in (s_0, t_1].$$

(3) $\forall t \in (t_1, s_1]$. Similar to the process of part (1), there exists a constant $N_1 > 0$ such that

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f_1^q(s, x^q(s), u^q(s)) - f_1(s, x(s), u(s)) \right\| ds \le s_1^\alpha E_\alpha(L_1 s_1^\alpha)\varepsilon, \quad q > N_1.$$

Moreover, part (2) shows that $\|x^q(t_1) - x(t_1)\| \le \widetilde{A}_0\varepsilon$ for $q > \mathcal{N}_0$.

Let $\mathcal{M}_1 = \max\{\mathcal{N}_0, N_1\}$, for $q > \mathcal{M}_1$, it yields that

$$\begin{aligned}
&\|x^q(t) - x(t)\| \\
= &\left\| \phi_0^q(t_1, x^q(t_1), x^q(s_0 - 0)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f_1^q(s, x^q(s), u^q(s))ds \right. \\
&\left. \quad -\phi_0(t_1, x(t_1), x(s_0 - 0)) - \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f_1(s, x(s), u(s))ds \right\| \\
\leq &\left\| \phi_0^q(t_1, x^q(t_1), x^q(s_0 - 0)) - \phi_0(t_1, x(t_1), x(s_0 - 0)) \right\| + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \left\| f_1^q(s, x^q(s), u^q(s)) - f_1(s, x(s), u(s)) \right\| ds \\
\leq &\|x^q(t_1) - x(t_1)\| + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \left\| f_1^q(s, x^q(s), u^q(s)) - f_1(s, x(s), u(s)) \right\| ds \\
\leq &\widetilde{A}_0 \varepsilon + s_1^{\alpha} E_{\alpha}(L_1 s_1^{\alpha}) \varepsilon \\
= &\left(\widetilde{A}_0 + s_1^{\alpha} E_{\alpha}(L_1 s_1^{\alpha}) \right) \varepsilon.
\end{aligned}$$

Repeating the process of parts (2)−(3), we obtain that there exist $\widetilde{A}_k, \mathcal{N}_k, \mathcal{M}_k, \widetilde{A}_0, \mathcal{N}_0, \mathcal{M}_{p+1} > 0$ $(k = 1, ..., p)$, such that

$$\|x^q(t) - x(t)\| \leq \widetilde{A}_k \varepsilon, \quad q > \mathcal{N}_k, \ t \in (s_k, t_{k+1}], \ k = 0, ..., p,$$

and

$$\|x^q(t) - x(t)\| \leq \left(\widetilde{A}_{k-1} + s_k^{\alpha} E_{\alpha}(L_k s_k^{\alpha}) \right) \varepsilon, \quad q > \mathcal{M}_k, \ t \in (t_k, s_k], \ k = 1, ..., p+1.$$

Let

$$N = \max_{1 \leq k \leq p} \left\{ N_0, \widetilde{A}_0, \mathcal{N}_0, \mathcal{M}_{p+1}, \mathcal{N}_k, \mathcal{M}_k \right\},$$

and

$$\widetilde{A} = \max_{1 \leq k \leq p+1} \left\{ s_0^{\alpha} E_{\alpha}(L_0 s_0^{\alpha}), \ \widetilde{A}_{k-1} + s_k^{\alpha} E_{\alpha}(L_k s_k^{\alpha}) \right\},$$

then, $0 < \widetilde{A} < +\infty$.

From the above discussion, we can get the result that $\forall \varepsilon > 0, \forall t \in [0, T]$,

$$\|x^q(t) - x(t)\| \leq \widetilde{A} \varepsilon, \quad q > N.$$

Owing to the arbitrariness of ε and t, it shows that

$$x^q \to x (q \to +\infty).$$

Then the proof is completed. □

The following corollary is a direct consequence of the above lemma.

Corollary 4.1. *Let $f_k \in F_k (k = 0, ..., p+1), \{u^q\} \subset U$ and ϕ_k^q are functions satisfying the condition $(H_{\phi,k})(k = 0, 1, ..., p)$. Assuming that the conditions $(H_u), (H_{w,k})(k = 0, ..., p+1)$ all hold, if $u^q \to u(q \to +\infty)$ and $\phi_k^q \to \phi_k(q \to +\infty, k = 0, ..., p)$, then $x^q \to x$ $(q \to +\infty)$.*

Now, in order to make the study go on wheels, we need to do some proper restrictions on the functions g and h in equation (2), which are set forth below.

(H_{gh}): *$g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ and $h : [0, T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ are both continuous functions.*

For simplicity, in the latter part of this section, we suppose that the conditions $(H_u), (H_{w,k}), (H_{\phi,k})$ $(k = 0, ..., p)$, $(H_{w,p+1})$ and (H_{gh}) are all satisfied.

Lemma 4.2. *Let $\{f_k^q\} \subset F_k (k = 0, ..., p+1)$ and $\{u^q\} \subset U$, if $f_k^q \to f_k (q \to +\infty)$ and $u^q \to u(q \to +\infty)$, then*

$$J_{f_0^q, f_1^q, ..., f_{p+1}^q}(u^q) \to J_{f_0, f_1, ..., f_{p+1}}(u) \ (q \to +\infty).$$

Proof. By the definition of J, we know that

$$J_{f_0^q,f_1^q,\dots,f_{p+1}^q}(u^q) = g(x^q(0), x^q(T)) + \int_0^T h(t, x^q(t), u^q(t))dt,$$

$$J_{f_0,f_1,\dots,f_{p+1}}(u) = g(x(0), x(T)) + \int_0^T h(t, x(t), u(t))dt,$$

where x^q, x are described in Lemma 4.1.

From the hypothesis of this lemma and Lemma 4.1, we immediately obtain that $x^q \to x$.

In view of the continuity of g and h, one can get that

$$g(x^q(0), x^q(T)) \to g(x(0), x(T))\ (q \to +\infty), \tag{6}$$

and

$$h(t, x^q(t), u^q(t)) \to h(t, x(t), u(t))\ (q \to +\infty). \tag{7}$$

Next, we will prove that the solution of equation (3) is bounded step by step.

(1) For $t \in [0, s_0]$,

$$\begin{aligned}
\|x(t)\| &= \left\| x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_0(s, x(s), u(s))ds \right\| \\
&\le \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f_0(s, x(s), u(s))\| ds \\
&\le \|x_0\| + \frac{C_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&\le \|x_0\| + \frac{C_0 T^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

(2) For $t \in (s_k, t_{k+1}],\ k = 0, \dots, p$,

$$\|x(t)\| = \|\phi_k(t, x(t), x(s_k - 0))\| \le D_k.$$

(3) For $t \in (t_k, s_k],\ k = 1, \dots, p+1$,

$$\begin{aligned}
\|x(t)\| &= \left\| \phi_{k-1}(t_k, x(t_k), x(s_{k-1} - 0)) + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} f_k(s, x(s), u(s))ds \right\| \\
&\le \|\phi_{k-1}(t_k, x(t_k), x(s_{k-1} - 0))\| + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} \|f_k(s, x(s), u(s))\| ds \\
&\le D_{k-1} + \frac{C_k T^\alpha}{\Gamma(\alpha+1)},
\end{aligned}$$

where $C_{p+1}, C_k, D_k (k = 0, \dots, p)$ are constants in conditions $(H_{w,p+1}), (H_{w,k}), (H_{\phi,k}) (k = 0, \dots, p)$, respectively.

Let

$$\overline{A} = \max_{1 \le k \le p+1} \left\{ \|x_0\| + \frac{C_0 T^\alpha}{\Gamma(\alpha+1)},\ D_{k-1} + \frac{C_k T^\alpha}{\Gamma(\alpha+1)} \right\}.$$

Steps (1)−(3) indicate that $\|x\| \le \overline{A} < +\infty$. Therefore, $h(t, x(t), u(t))$ is bounded for $t \in [0, T]$ with condition $\|u\| \le M$ in (H_u) and the continuity of h. Then by the Dominated Convergence Theorem, equation (7) leads to the fact that

$$\int_0^T h(t, x^q(t), u^q(t))dt \to \int_0^T h(t, x(t), u(t))dt \qquad (q \to \infty). \tag{8}$$

Combining (6) and (8), it holds that

$$J_{f_0^q,f_1^q,\ldots,f_{p+1}^q}(u^q) \to J_{f_0,f_1,\ldots,f_{p+1}}(u) \qquad (q \to +\infty).$$

Then the proof is finished. □

In the light of Lemma 4.2, two corollaries are obtained as below.

Corollary 4.2. *Let $f_k \in F_k (k = 0, 1, ..., p + 1)$ and $\{u^q\} \subset U$ with $u^q \to u$ $(q \to +\infty)$, then*

$$J_{f_0,f_1,\ldots,f_{p+1}}(u^q) \to J_{f_0,f_1,\ldots,f_{p+1}}(u) \; q \to +\infty.$$

Corollary 4.3. *Let $\{f_k^q\} \subset F_k$ with $f_k^q \to f_k$ $(q \to +\infty, k = 0, ..., p + 1)$ and $u \in U$, then*

$$J_{f_0^q,f_1^q,\ldots,f_{p+1}^q}(u) \to J_{f_0,f_1,\ldots,f_{p+1}}(u) \; q \to +\infty.$$

Now let's demonstrate one of the main results in the paper, which is the existence of the solutions for problem **(P)**.

Theorem 4.1. *Problem **(P)** has at least one solution, that is to say, there exists a $u^* \in U$ satisfying the equation*

$$J_{f_0,f_1,\ldots,f_{p+1}}(u^*) = \min_{u \in U} J_{f_0,f_1,\ldots,f_{p+1}}(u).$$

Proof. The proof of Lemma 4.2 shows that $g(x(0), x(T))$, $\int_0^T h(t, x(t), u(t))dt$ are bounded, therefore

$$\begin{aligned} J_{f_0,f_1,\ldots,f_{p+1}}(u) &\geq -|J_{f_0,f_1,\ldots,f_{p+1}}(u)| \\ &= -\left| g(x(0), x(T)) + \int_0^T h(t, x(t), u(t))dt \right| \\ &\geq -|g(x(0), x(T))| - \int_0^T |h(t, x(t), u(t))|dt \\ &> -\infty, \end{aligned}$$

namely, $J_{f_0,f_1,\ldots,f_{p+1}}(u)$ is bounded below.

Take a minimizing sequence $\{u^j\}_{j=1}^{+\infty} \subset U$ such that

$$J_{f_0,f_1,\ldots,f_{p+1}}(u^j) \to \inf_{u \in U} J_{f_0,f_1,\ldots,f_{p+1}}(u), \quad j \to +\infty. \tag{9}$$

Since U is a compact subset of $C_n[0, T]$, there exists a convergent subsequence $\{u^{j'}\}_{j=1}^{+\infty} \subset U$ such that

$$u^{j'} \to u^* \in U, \qquad j' \to +\infty.$$

Hence

$$J_{f_0,f_1,\ldots,f_{p+1}}(u^{j'}) \to \inf_{u \in U} J_{f_0,f_1,\ldots,f_{p+1}}(u), \qquad j' \to +\infty. \tag{10}$$

By Corollary 4.2, we obtain that

$$J_{f_0,f_1,\ldots,f_{p+1}}(u^{j'}) \to J_{f_0,f_1,\ldots,f_{p+1}}(u^*), \qquad j' \to +\infty. \tag{11}$$

Combine (10) and (11), it leads to that

$$J_{f_0,f_1,\ldots,f_{p+1}}(u^*) = \min_{u \in U} J_{f_0,f_1,\ldots,f_{p+1}}(u).$$

This finishes the proof. □

5. Stability of Optimal Control Problem

In this section, we will deal with the stability of problem **(P)** and it is characterized by the stability of solution set of equation (1) for any $f_k \in F_k (k = 0, ..., p+1)$.

For convenience's sake, in this section we suppose that the conditions (H_u), $(H_{w,k})$, $(H_{\phi,k})$ $(k = 0, ..., p)$, $(H_{w,p+1})$ and (H_{gh}) are all satisfied.

Set a metric space

$$\mathcal{F} \triangleq F_0 \times F_1 \times \cdots \times F_{p+1},$$

with metric defined as

$$\widetilde{\rho}(f^1, f^2) = \max_{0 \leq j \leq p+1} \left\| f_j^1 - f_j^2 \right\|,$$

for any $f^1 = (f_0^1, ..., f_{p+1}^1) \in \mathcal{F}$ and $f^2 = (f_0^2, ..., f_{p+1}^2) \in \mathcal{F}$.

Apparently, $(\mathcal{F}, \widetilde{\rho})$ is a complete metric space.

Consider a set-valued mapping $I : \mathcal{F} \rightrightarrows U$, where $I(f)$ is the solution set of (1) for each $f \in \mathcal{F}$. Next, we will make a thorough study about the stability of $I(f)$, namely, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $H(I(f'), I(f)) < \varepsilon$ with $\widetilde{\rho}(f', f) < \delta$, where H is the Hausdorff metric induced by the metric on U.

Whereafter, analogous to the investigation and argument in [31], some meaningful conclusions are approached from similar angles in the following.

Theorem 5.1. *$I(f) \neq \emptyset$ for each $f \in \mathcal{F}$.*

Proof. The result can be obtained directly using Theorem 4.1. □

Theorem 5.2. *$I : \mathcal{F} \rightrightarrows U$ is a usco mapping.*

Proof. From the compactness of U and the conclusions of Lemma2.3, Lemma 2.4 and Theorem 5.1, we know that it is only necessary to prove $Graph(I)$ is closed, where

$$Graph(I) = \{(f, u) \in \mathcal{F} \times U : u \in I(f)\}.$$

Let $\{f^q\} \subset \mathcal{F}$ with $f^q \to f \in \mathcal{F}$ and $\{u^q\} \subset I(f)$ with $u^q \to u^* \in U$. In fact, take into account $u^q \in I(f^q)$ for each $q \in N_+$ and it has

$$J_{f_0^q, f_1^q, ..., f_{p+1}^q}(u^q) \leq J_{f_0^q, f_1^q, ..., f_{p+1}^q}(u), \quad \forall u \in U.$$

From the conditions $\{f^q\} \subset \mathcal{F}$ and $\{u^q\} \subset I(f)$ and the results of Lemma 4.2 and Corollary 4.3, it yields that

$$J_{f_0^q, f_1^q, ..., f_{p+1}^q}(u^q) \to J_{f_0, f_1, ..., f_{p+1}}(u^*), \quad \forall u \in U,$$

and

$$J_{f_0^q, f_1^q, ..., f_{p+1}^q}(u) \to J_{f_0, f_1, ..., f_{p+1}}(u), \quad \forall u \in U.$$

Thus, we have

$$J_{f_0, f_1, ..., f_{p+1}}(u^*) \leq J_{f_0, f_1, ..., f_{p+1}}(u), \quad \forall u \in U,$$

which suggests that $u^* \in I(f)$, namely, $Graph(I)$ is closed. This finishes the proof. □

Theorem 5.3. *Set-valued mapping $I : \mathcal{F} \rightrightarrows U$ is lower semi-continuous at $f \in \mathcal{F}$ if and only if problem **(P)** associated with f is essential.*

Proof. First of all, we show that the lower semi-continuity of $I : \mathcal{F} \rightrightarrows U$ at $f \in \mathcal{F}$ leads to the fact that problem **(P)** associated with f is essential.

For any $u \in I(f)$ and any $\varepsilon > 0$, the open neighborhood $V(u, \varepsilon)$ of u satisfies $V(u, \varepsilon) \cap I(f) \neq \emptyset$. If I is lower semi-continuous at f , then there exists $\delta > 0$ such that $V(u, \varepsilon) \cap I(f') \neq \emptyset$ for any $f' \in F$ with $\widetilde{\rho}(f', f) < \delta$. Take $u' \in V(u, \varepsilon) \cap I(f')$ and then $u' \in I(f')$ and $\|u - u'\| < \varepsilon$ as well. Hence, the solution u is essential. This is finished as a result of Remark 2.1.

On the contrary, presuming that problem **(P)** associated with f is essential. For any open set G with $G \cap I(f) \neq \emptyset$, there exists $u \in G \cap I(f)$. Then G is an open neighborhood of u. There exists $\varepsilon > 0$ such that the open neighborhood $V(u, \varepsilon)$ of u satisfies $V(u, \varepsilon) \subset G$. Because u is an essential solution, there exists $\delta > 0$ such that for any $f' \in \mathcal{F}$ with $\widetilde{\rho}(f', f) < \delta$, it has $u' \in I(f')$ with $\|u - u'\| < \varepsilon$. This shows $u' \in (V(u, \varepsilon) \cap I(f')) \subset (G \cap I(f'))$. Hence, $G \cap I(f') \neq \emptyset$ for any $f' \in \mathcal{F}$ with $\widetilde{\rho}(f', f) < \delta$, namely, I is lower semi-continuous at f.Then the proof is ended. □

Remark 5.1. *From the above results, we can easily draw such conclusions that if the optimal control problem associated with $f \in \mathcal{F}$ is essential, then the set-valued mapping $I : \mathcal{F} \rightrightarrows U$ is continuous at f. What's more, $I(f)$ is stable due to the compactness of $I : \mathcal{F} \rightrightarrows U$ and Theorem 17.15 of [47].*

Remark 5.2. *Given Lemma 2.5, there exists a dense residual subset $\mathcal{E}$ of $\mathcal{F}$ such that problem* **(P)** *associated with $f \in \mathcal{E}$ is essential, therefore for most $f \in \mathcal{F}$, the solution set $I(f)$ is stable and every optimal control problem associated $f \in \mathcal{F}$ can be closely approximated arbitrarily by an essential optimal control problem.*

Example 5.1. *Let*

$$U = \{u^\theta : u^\theta(t) = -\frac{1}{\theta}t + t^2,\ t \in [0, 5], \theta = 1, 2, ...\}.$$

Clearly, U is a nonempty compact subset of $C[0, 5]$.

Consider the following optimal control problems: looking $u^* \in U$ satisfying

$$J_{f_0,f_1,f_2}(u^*) = \min_{u\in U} J_{f_0,f_1,f_2}(u),$$

and

$$\begin{cases} {}_0^cD^{\frac{1}{2}}x(t) = f_k(t, x, u^*), & t \in (t_k, s_k],\quad k = 0, 1, 2, \\ x(t) = \sin(k + 1)t + x(s_k - 0), & t \in (s_k, t_{k+1}],\quad k = 0, 1, \\ x(0) = 0, \end{cases}$$

where $0 = t_0 < s_0 < t_1 < s_1 < t_2 < s_2 = 5$ and points $s_k, t_k (k = 0, 1, 2)$ divide interval $[0, 5]$ into five intervals on average. In addition,

$$J_{f_0,f_1,f_2}(u) = g(x(0), x(5)) + \int_0^5 h(t, x(t), u(t))dt,$$
$$g(x, y) = x + y - 5,\quad h(t, x, u) \equiv 1.$$

(1) *Set $f_0 = u^\theta, f_1 = u^\theta + 1, f_2 = u^\theta + t$. It can be obtained by calculation that*

$$x(t) = \begin{cases} -\frac{4}{3\sqrt{\pi}\theta}t^{\frac{3}{2}} + \frac{16}{15\sqrt{\pi}}t^{\frac{5}{2}}, & t \in [0, 1], \\ \sin t + c_1, & t \in (1, 2], \\ \frac{2}{\sqrt{\pi}}(-\frac{2}{\theta} + 5)(t - 2)^{\frac{1}{2}} + \frac{4}{3\sqrt{\pi}}(-\frac{1}{\theta} + 4)(t - 2)^{\frac{3}{2}} + \frac{16}{15\sqrt{\pi}}(t - 2)^{\frac{5}{2}} + c_2, & t \in (2, 3], \\ \sin 2t + c_3, & t \in (3, 4], \\ \frac{2}{\sqrt{\pi}}(-\frac{4}{\theta} + 20)(t - 4)^{\frac{1}{2}} + \frac{4}{3\sqrt{\pi}}(-\frac{1}{\theta} + 9)(t - 4)^{\frac{3}{2}} + \frac{16}{15\sqrt{\pi}}(t - 4)^{\frac{5}{2}} + c_4, & t \in (4, 5], \end{cases}$$

where

$$c_1 = -\frac{4}{3\sqrt{\pi}\theta} + \frac{16}{15\sqrt{\pi}},\quad c_2 = \sin 2 - \frac{4}{3\sqrt{\pi}\theta} + \frac{16}{15\sqrt{\pi}},$$
$$c_3 = \sin 2 - \frac{20}{3\sqrt{\pi}\theta} + \frac{262}{15\sqrt{\pi}},\quad c_4 = \sin 8 + \sin 2 - \frac{20}{3\sqrt{\pi}\theta} + \frac{262}{15\sqrt{\pi}}.$$

Then,

$$J_{f_0,f_1,f_2}(u) = x(0) + x(5) = -\frac{16}{\sqrt{\pi}\theta} + \frac{1058}{15\sqrt{\pi}} + \sin 8 + \sin 2,$$

which shows that $J_{f_0,f_1,f_2}(u)$ reaches the minimum value when $\theta = 1$, i.e. $I(f) = I(f_0, f_1, f_2) = \{u^1\}$.

(2) *Set $f_0^m = u^\theta + \frac{1}{m}, f_1^m = u^\theta + 1 + \frac{1}{m}, f_2^m = u^\theta + t + \frac{1}{m} (m \in \mathbb{N}_+)$. Then,*

$$x(t) = \begin{cases} \frac{2}{\sqrt{\pi}m} t^{\frac{1}{2}} - \frac{4}{3\sqrt{\pi}\theta} t^{\frac{3}{2}} + \frac{16}{15\sqrt{\pi}} t^{\frac{5}{2}}, & t \in [0,1], \\ \sin t + c_1', & t \in (1,2], \\ \frac{2}{\sqrt{\pi}}(-\frac{2}{\theta} + 5 + \frac{1}{m})(t-2)^{\frac{1}{2}} + \frac{4}{3\sqrt{\pi}}(-\frac{1}{\theta} + 4)(t-2)^{\frac{3}{2}} + \frac{16}{15\sqrt{\pi}}(t-2)^{\frac{5}{2}} + c_2', & t \in (2,3], \\ \sin 2t + c_3', & t \in (3,4], \\ \frac{2}{\sqrt{\pi}}(-\frac{4}{\theta} + 20 + \frac{1}{m})(t-4)^{\frac{1}{2}} + \frac{4}{3\sqrt{\pi}}(-\frac{1}{\theta} + 9)(t-4)^{\frac{3}{2}} + \frac{16}{15\sqrt{\pi}}(t-4)^{\frac{5}{2}} + c_4', & t \in (4,5], \end{cases}$$

where

$$c_1' = \frac{2}{\sqrt{\pi}m} - \frac{4}{3\sqrt{\pi}\theta} + \frac{16}{15\sqrt{\pi}}, \quad c_2' = \sin 2 + \frac{2}{\sqrt{\pi}m} - \frac{4}{3\sqrt{\pi}\theta} + \frac{16}{15\sqrt{\pi}},$$

$$c_3' = \frac{4}{\sqrt{\pi}m} + \sin 2 - \frac{20}{3\sqrt{\pi}\theta} + \frac{262}{15\sqrt{\pi}}, \quad c_4' = \sin 8 + \frac{4}{\sqrt{\pi}m} + \sin 2 - \frac{20}{3\sqrt{\pi}\theta} + \frac{262}{15\sqrt{\pi}}.$$

Therefore,

$$J_{f_0^m,f_1^m,f_2^m}(u) = x(0) + x(5) = -\frac{16}{\sqrt{\pi}\theta} + \frac{6}{\sqrt{\pi}m} + \frac{1058}{15\sqrt{\pi}} + \sin 8 + \sin 2.$$

In the same light, $J_{f_0,f_1,f_2}(u)$ achieves the minimum value at $\theta = 1$, i.e. $I(f^m) = I(f_0^m, f_1^m, f_2^m) = \{u^1\}$.

From $(1) - (2)$ *we can see that*

$$\widetilde{\rho}(f^m, f) = \max_{0 \le j \le 2} \left\| f_j^m - f_j \right\| = \frac{1}{m} \to 0, \quad m \to +\infty,$$

and

$$\|u^m - u\| = \|u^1 - u^1\| = 0, \quad \forall m \in \mathbb{N}_+.$$

Consequently, u^1 is essential. Besides,

$$H(I(f^m), I(f)) = H(u^1, u^1) = 0,$$

so the optimal controller u^1 is stable.

6. Conclusions

In this paper, we draw the conclusion that most of the fractional optimal control problem with non-instantaneous impulses of this type are stable in the complete space $\mathcal{F}$, that is, the optimal solution does not be perturbed largely, when some disturbances occur in the function $f \in \mathcal{F}$. In addition, if the initial value condition $x(0) = x_0$ in equation (3) is replaced by the boundary value condition $x(0) = x_0, x(T) = y_0$, or the functions g and h also subject to small perturbation, what stable results will problem **(P)** has? All of these problems are worthy of further study.

7. Declarations

Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

All of the authors contributed equally in writing this paper. They both read and approved the final manuscript.

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