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On Ideals Defined by Asymptotic Distribution Functions of Ratio Block Sequences

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Abstract. In this paper we study ratio block sequences possessing an asymptotic distribution function. By means of these distribution functions we define new families of subsets of $\mathbb{N}$ which appear to be admissible ideals. We characterize these ideals using the exponent of convergence and this characterization is useful in decision if a given set belongs to a given ideal of this kind.

1. Introduction

In the whole paper we assume $X = \{x_1 < x_2 < \cdots < x_n < \dots\} \subset \mathbb{N}$ where $\mathbb{N}$ denotes the set of all positive integers.

The following sequence derived from X

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1}$$

is called *the ratio block sequence* of the sequence X.

It is formed by the blocks $X_1, X_2, \dots, X_n, \dots$ where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \quad n = 1, 2, \dots$$

is called the *n-th block*. This kind of block sequences was introduced by O. Strauch and J. T. Tóth [14] and they studied the set $G(X_n)$ of its distribution functions.

In this paper we will be interested in ratio block sequences of type (1) possessing an asymptotic distribution function, i.e. $G(X_n)$ is a singleton (see definitions in the next section). By means of these distribution functions we define new families of subsets of $\mathbb{N}$ which appear to be admissible ideals. We characterize these ideals using the exponent of convergence and this characterization is useful in decision if a given set belongs to a given ideal of this kind.

The rest of our paper is organized as follows. In Section 2 and Section 3 we recall some known definitions, notations and theorems, which will be used and extended. In Section 4 our new results are presented.

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2. Definitions

The following basic definitions are from papers [9], [11], [14] and [15].

- For each $n \in \mathbb{N}$ consider the *step distribution function*
$$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n},$$
for $x \in [0,1)$, and for $x = 1$ we define $F(X_n, 1) = 1$.

- A non-decreasing function $g : [0,1] \to [0,1]$, $g(0) = 0$, $g(1) = 1$ is called a *distribution function* (abbreviated d.f.). We shall identify any two d.f.s coinciding at common points of continuity.

- A d.f. $g(x)$ is a d.f. of the sequence of blocks X_n, $n = 1, 2, \dots$, if there exists an increasing sequence $n_1 < n_2 < \cdots$ of positive integers such that
$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$
a.e. on $[0,1]$. This is equivalent to the weak convergence, i.e., the preceding limit holds for every point $x \in [0,1]$ of continuity of $g(x)$.

- Denote by $G(X_n)$ the set of all d.f.s of X_n, $n = 1, 2, \dots$. The set of distribution functions of ratio block sequences was studied in [1–7, 10–14].
If $G(X_n) = \{g(x)\}$ is a singleton, the d.f. $g(x)$ is also called the *asymptotic distribution function* of X_n.
Especially, if $G(X_n) = \{x\}$, then we say that the sequence of blocks X_n is *uniformly distributed* in $[0,1]$.

- Let the function $\lambda : 2^{\mathbb{N}} \to [0,1]$ defined by
$$\lambda(A) = \inf\Big\{t > 0 : \sum_{a\in A} \frac{1}{a^t} < \infty\Big\}$$
be the *exponent of convergence* of a set $A \subset \mathbb{N}$.
If $q > \lambda(A)$ then $\sum_{a\in A} \frac{1}{a^q} < \infty$ and if $q < \lambda(A)$ then $\sum_{a\in A} \frac{1}{a^q} = \infty$. In the case when $q = \lambda(A)$, the series $\sum_{a\in A} \frac{1}{a^q}$ can be either convergent or divergent.
From ([9], p.26, Exercises 113, 114) it follows that the set of all possible values of λ forms the whole interval $[0,1]$, i.e. $\{\lambda(A) : A \subset \mathbb{N}\} = [0,1]$ and if $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$ then $\lambda(A)$ can be calculated by
$$\lambda(A) = \limsup_{n\to\infty} \frac{\log n}{\log a_n}.$$
Evidently the exponent of convergence λ is a monotone set function, i.e. $\lambda(A) \le \lambda(B)$ for $A \subset B \subset \mathbb{N}$ and also $\lambda(A \cup B) = \max\{\lambda(A), \lambda(B)\}$ holds for all $A, B \subset \mathbb{N}$.

- By means of λ we can define the following sets:
$\mathcal{I}_{<q} = \{A \subset \mathbb{N} : \lambda(A) < q\}$ for $0 < q \le 1$,
$\mathcal{I}_{\le q} = \{A \subset \mathbb{N} : \lambda(A) \le q\}$ for $0 \le q \le 1$ and
$\mathcal{I}_0 = \{A \subset \mathbb{N} : \lambda(A) = 0\}$.
Obviously $\mathcal{I}_{\le 0} = \mathcal{I}_0$ and $\mathcal{I}_{\le 1} = 2^{\mathbb{N}}$.
For a finite set $A \subset \mathbb{N}$ we have $\lambda(A) = 0$. Consequently, $\mathcal{I}_f = \{A \subset \mathbb{N} : A \text{ is finite}\} \subset \mathcal{I}_0$. Families $\mathcal{I}_{<q}, \mathcal{I}_{\le q}$ and the well known family
$$\mathcal{I}_c^{(q)} = \Big\{A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N} : \sum_{n=1}^{\infty} \frac{1}{a_n^q} < \infty\Big\}$$
are related for $0 < q < q' < 1$ by following inclusions (see [15])
$$\mathcal{I}_f \subsetneq \mathcal{I}_0 \subsetneq \mathcal{I}_{<q} \subsetneq \mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_{\le q} \subsetneq \mathcal{I}_{<q'} \subsetneq \mathcal{I}_{<1}. \tag{2}$$

- Let $\mathcal{I} \subset 2^{\mathbb{N}}$. Then $\mathcal{I}$ is called an *admissible ideal* of subsets of positive integers, if $\mathcal{I}$ is additive (if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$), hereditary (if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$), containing all finite subsets of $\mathbb{N}$ and it does not contain $\mathbb{N}$.

3. Overwiew of known results

In this section we mention known results related to the topic of this paper and some other ones we use in the proofs of our theorems. In the whole section in (A1)–(A8) we assume $X = \{x_1 < x_2 < \cdots < x_n < \dots\} \subset \mathbb{N}$.

(A1) We will use step function

$$c_0(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x \leq 1. \end{cases}$$

Assume that $G(X_n)$ is singleton, i.e., $G(X_n) = \{g(x)\}$. Then either $g(x) = c_0(x)$ for $x \in [0,1]$; or $g(x) = x^q$ for $x \in [0,1]$ and some fixed $0 < q \leq 1$. [[14], Th. 8.2]

(A2) Let $0 < q \leq 1$ be a real number. Then $G(X_n) = \{x^q\}$ if and only if for every $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{x_{kn}}{x_n} = k^{\frac{1}{q}}.$$

[[6], Th. 1]

(A3) Let $0 < q \leq 1$ be a real number. If $G(X_n) = \{x^q\}$ then

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1.$$

[[4], Remark 3]

(A4) We have

$$G(X_n) = \{c_0(x)\} \Longleftrightarrow \lim_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = 0.$$

[[14], Th. 7.1]

(A5) We have

$$c_0(x) \in G(X_n) \Longleftrightarrow \liminf_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = 0.$$

[[4], Th. 4]

(A6) Let $0 < q \leq 1$ be a real number. Then

$$G(X_n) = \{x^q\} \Longleftrightarrow \lim_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = \frac{q}{q+1}.$$

[[1], Th. 1]

(A7) Let $G(X_n) = \{c_0(x)\}$. Then

$$\lim_{n\to\infty} \frac{\log n}{\log x_n} = 0.$$

[[1], Th. 2]

(A8) Let $0 < q \leq 1$ be a real number and $G(X_n) = \{x^q\}$. Then

$$\lim_{n\to\infty} \frac{\log n}{\log x_n} = q.$$

[[1], Th. 3]

(A9) Let $0 < q \leq 1$. Then each of the families $\mathcal{I}_0$, $\mathcal{I}_{<q}$ and $\mathcal{I}_{\leq q}$ forms an admissible ideal, except for $\mathcal{I}_{\leq 1}$.

[[15], Th. 1]

(A10) We have

$$\mathcal{I}_0 = \bigcap_{0<q\leq 1} \mathcal{I}_{<q} = \bigcap_{0<q\leq 1} \mathcal{I}_c^{(q)} = \bigcap_{0<q\leq 1} \mathcal{I}_{\leq q}.$$

[[15], Th. 2]

(A11) Let $0 \leq q < 1$ be real, $A \subset \mathbb{N}$ and $A(x) = \#\{a \leq x : a \in A\}$ for $x \geq 1$. Then $A \in \mathcal{I}_{\leq q}$ if and only if for every $\delta > 0$

$$\lim_{x\to\infty} \frac{A(x)}{x^{q+\delta}} = 0.$$

[[15], Th. 3]

(A12) Let $0 < q \leq 1$ be a real number and $A \subset \mathbb{N}$. Then $A \in \mathcal{I}_{<q}$ if and only if there exists $\delta > 0$ such that

$$\lim_{x\to\infty} \frac{A(x)}{x^{q-\delta}} = 0.$$

[[15], Th. 4]

4. Results

The result (A1) provides motivation to introduce the following families of subsets of $\mathbb{N}$:

$$\mathcal{U}(c_0(x)) = \{X \subset \mathbb{N} : G(X_n) = \{c_0(x)\}\},$$

$$\mathcal{I}(c_0(x)) = \{A \subset \mathbb{N} : \exists X \in \mathcal{U}(c_0(x)), A \subset X\},$$

and for $0 < q \leq 1$

$$\mathcal{U}(x^q) = \{X \subset \mathbb{N} : G(X_n) = \{x^q\}\},$$

$$\mathcal{I}(x^q) = \{A \subset \mathbb{N} : \exists X \in \mathcal{U}(x^q), A \subset X\}.$$

Obviously

$$\mathcal{U}(c_0(x)) \subsetneq \mathcal{I}(c_0(x)), \quad \mathcal{U}(x^q) \subsetneq \mathcal{I}(x^q).$$

Sets $X = \{x_1 < x_2 < \dots\}$ from $\mathcal{U}(c_0(x))$ are characterized by (A4) and sets belonging to $\mathcal{U}(x^q)$ are characterized by (A2) and (A6).

In the sequel we will demonstrate some properties of these families and we will characterize $\mathcal{I}(c_0(x))$ and $\mathcal{I}(x^q)$ by means of the exponent of convergence. From these properties follows also that families $\mathcal{I}(c_0(x))$ and $\mathcal{I}(x^q)$ are ideals.

Theorem 1. *The family $\mathcal{U}(c_0(x))$ is additive, i.e. it is closed with respect to finite unions.*

Proof. Let $A, B \in \mathcal{U}(c_0(x))$ and

$$A = \{x_1 < x_2 < \cdots\}, \quad B = \{y_1 < y_2 < \cdots\}.$$

Using (A4) for $k \to \infty$ and $n \to \infty$ we have

$$\frac{1}{kx_k} \sum_{x\in A,\ x\le x_k} x \to 0 \quad \text{and} \quad \frac{1}{ny_n} \sum_{y\in B,\ y\le y_n} y \to 0. \tag{3}$$

Let $A \cup B = \{z_1 < z_2 < \cdots\}$ and $z_m \in A \cup B$.

For $z_m = y_n$ and $x_k \le y_n < x_{k+1}$ we have

$$\frac{1}{mz_m} \sum_{z\in A\cup B,\ z\le z_m} z \le \frac{1}{my_n} \left(\sum_{x\in A,\ x\le x_k} x + \sum_{y\in B,\ y\le y_n} y \right) \le$$

$$\le \frac{k}{m} \frac{1}{kx_k} \sum_{x\in A,\ x\le x_k} x + \frac{n}{m} \frac{1}{ny_n} \sum_{y\in B,\ y\le y_n} y.$$

As $\frac{k}{m} \le 1$, $\frac{n}{m} \le 1$, using (3) we obtain for $m \to \infty$ ($k \to \infty$ and $n \to \infty$)

$$\frac{1}{mz_m} \sum_{z\in A\cup B,\ z\le z_m} z \to 0,$$

and using (A4) again we have $A \cup B \in \mathcal{U}(c_0(x))$. The case when $z_m = x_n$ and $y_k \le x_n < y_{k+1}$ follows in the same way. □

Example 1. *The family $\mathcal{U}(c_0(x))$ does not form an ideal as it is not hereditary, i.e. there exists sets $C \in \mathcal{U}(c_0(x))$ and $B \subset C$ such that $B \notin \mathcal{U}(c_0(x))$.*

Proof. Put $C = A \cup B$ where $A = \{2^n; n \in \mathbb{N}\}$ and

$$B = \bigcup_{n=1}^{\infty} B_n \quad \text{where} \quad B_n = [2^{n!}, 2^{n!} + 2^n) \cap \mathbb{N}, (n = 1, 2, \dots).$$

Then for a block B_n we have

$$\sum_{b\in B_n} b = \frac{(2^{n!} + 2^{n!} + 2^n - 1)2^n}{2} < 2^n 2^{n!} + 2^{2n} \le 2^{n!} 2^{n+1},$$

$$\sum_{b\in B_n} b = \frac{(2^{n!} + 2^{n!} + 2^n - 1)2^n}{2} > 2^{n!} 2^n.$$

We will use these estimates in the rest of proof. We will show that $C \in \mathcal{U}(c_0(x))$ and $B \notin \mathcal{U}(c_0(x))$ by (A4).

1) $C \in \mathcal{U}(c_0(x))$. Let $C = \{c_1 < c_2 < \cdots\}$ and $c_n \in C$ where n is sufficiently large. Then there exists such $k \in \mathbb{N}$, that $2^{k!} \le c_n < 2^{(k+1)!}$. Obviously $n \ge k!$, thus we have

$$\frac{1}{nc_n} \sum_{i=1}^{n} c_i \le \frac{1}{nc_n} \left(\sum_{i:2^i\le c_n} 2^i + \sum_{i=1}^{k} \sum_{b\in B_i} b \right) \le \frac{2c_n}{nc_n} + \frac{1}{k! 2^{k!}} \sum_{i=1}^{k} (2^{i!} 2^{i+1}) \le$$

$$\le \frac{2}{n} + \frac{1}{k! 2^{k!}} \left(2^{k!} \sum_{i=1}^{k} 2^{i+1} \right) \le \frac{2}{n} + \frac{2^{k+2}}{k!}.$$

Thus we have $\frac{1}{nc_n}\sum_{i=1}^{n} c_i \to 0$ for $n \to \infty$, consequently also $k \to \infty$.

2) $B \notin \mathcal{U}(c_0(x))$. Let n_k ($k = 1, 2, \dots$) be such that $b_{n_k} = 2^{k!} + 2^k - 1$. Then $n_k = \sum_{i=1}^{k} 2^i = 2^{k+1} - 2$ and also

$$\frac{1}{n_k b_{n_k}} \sum_{i=1}^{n_k} b_i \geq \frac{1}{(2^{k+1}-2)(2^{k!}+2^k)} \sum_{b \in B_k} b > \frac{2^k 2^{k!}}{2^{k+1}(2^{k!}+2^{k!})} = \frac{1}{4}.$$

□

The following theorem shows a natural extension of $\mathcal{U}(c_0(x))$ to an ideal.

Theorem 2. *The family $\mathcal{I}(c_0(x))$ is an ideal.*

Proof. Its proof follows from the obvious fact that the set of all subsets of an additive family forms an ideal. □

On the other hand we have.

Theorem 3. *The inclusion $\mathcal{I}(c_0(x)) \subset \mathcal{I}_c^{(q)}$ holds for each $0 < q \leq 1$.*

Proof. Let $0 < q \leq 1$ and $B \in \mathcal{I}(c_0(x))$. Then there exists a set $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$ such that $A \in \mathcal{U}(c_0(x))$ and $B \subset A$. Choose $p > 0$ such that $pq > 1$. The relation in the statement (A7) yields existence of $n_0 \in \mathbb{N}$ such that $a_n \geq n^p$ holds for all $n > n_0$. Thus we have

$$\sum_{n=n_0+1}^{\infty} \frac{1}{a_n^q} < \sum_{n=n_0+1}^{\infty} \frac{1}{n^{qp}} < +\infty,$$

implying $\sum\limits_{n=1}^{\infty} \frac{1}{a_n^q} < \infty$, consequently $A \in \mathcal{I}_c^{(q)}$ and also $B \in \mathcal{I}_c^{(q)}$. □

Theorem 3 and the relation (A10) yield

Corollary 1.

$$\mathcal{I}(c_0(x)) \subset \bigcap_{0<q\leq 1} \mathcal{I}_c^{(q)} = \mathcal{I}_0.$$

In order to characterize the families $\mathcal{I}(c_0(x))$ and $\mathcal{I}(x^q)$ the following theorem will be very useful.

Theorem 4. *Let $0 < q \leq 1$, $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$, $Y = \{y_1 < y_2 < \cdots\} \subset \mathbb{N}$, let $g(x) \in \{c_0(x), x^q\}$ be fixed and assume that*

$$Y \in \mathcal{U}(g(x)) \quad \text{and} \quad \lim_{t\to\infty} \frac{X(t)}{Y(t)} = 0. \tag{4}$$

Then

$$X \cup Y \in \mathcal{U}(g(x)).$$

Proof. Let $z_m \in X \cup Y$. If $z_m = y_n$ and $x_k \leq y_n < x_{k+1}$ then $\max\{k, n\} \leq m \leq k + n$. Under assumptions, for every $0 < x \leq 1$ we have

$$g(x) = \lim_{n\to\infty} \frac{\#\left\{i \leq n : \frac{y_i}{y_n} < x\right\}}{n} = \lim_{n\to\infty} \frac{Y(xy_n)}{n} \text{ and } \frac{X(z_m)}{Y(z_m)} = \frac{k}{n} \to 0$$

when $m \to \infty$, i.e. also $k \to \infty$ and $n \to \infty$. Thus $\frac{k}{m} \to 0$ and $\frac{n}{m} \to 1$ for $m \to \infty$. For every $0 < x \leq 1$ we have the following estimation

$$\frac{Y(xy_n)}{n}\frac{n}{m} \leq \frac{X \cup Y(xz_m)}{m} \leq \frac{X(xx_{k+1})}{k+1}\frac{k+1}{m} + \frac{Y(xy_n)}{n}\frac{n}{m}.$$

For $m \to \infty$ we obtain

$$\lim_{m\to\infty} \frac{X \cup Y(xz_m)}{m} = \lim_{n\to\infty} \frac{Y(xy_n)}{n} = g(x), \quad \text{i. e.} \quad X \cup Y \in \mathcal{U}(g(x))\,.$$

The proof in the case $z_m = x_k$ and $y_n \le x_k \le y_{n+1}$ is similar. □

Corollary 2. *Let* $0 < q \le 1$, $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$, $Y = \{y_1 < y_2 < \cdots\} \subset \mathbb{N}$. *Assume that*

$$Y \in \mathcal{U}(x^q) \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = \infty. \tag{5}$$

Then

$$X \cup Y \in \mathcal{U}(x^q).$$

Proof. Let $k \in \mathbb{N}$ and $c > 0$ such that $c > k^{\frac{1}{q}}$. From (5) according the (A2) there exist $n_0 \in \mathbb{N}$ such that

$$\frac{x_n}{y_n} > c \quad \text{and} \quad \frac{y_{kn}}{y_n} < c\,.$$

holds for all positive integer $n \ge n_0$. Let now t be a real number and $t > x_{n_0}$. Then $x_n \le t < x_{n+1}$ for some $n \ge n_0$ and we obtained

$$\frac{X(t)}{Y(t)} \le \frac{X(x_{n+1})}{Y(x_n)} \le \frac{X(x_{n+1})}{Y(cy_n)} \le \frac{X(x_{n+1})}{Y(y_{kn})} = \frac{n+1}{kn}\,.$$

Here we used $n \to \infty$ for $t \to \infty$

$$\limsup_{t\to\infty} \frac{X(t)}{Y(t)} \le \lim_{n\to\infty} \frac{n+1}{kn} = \frac{1}{k}\,.$$

The previous inequality is hold for every $k \in \mathbb{N}$, so

$$\lim_{t\to\infty} \frac{X(t)}{Y(t)} = 0\,.$$

Then by Theorem 4 we have $X \cup Y \in \mathcal{U}(x^q)$. □

The following theorem shows $\mathcal{I}(c_0(x)) = \mathcal{I}_0$, it means that also the reverse inclusion to that in Corollary 1 is valid.

Theorem 5. *Let* $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$. *Then*

$$X \in \mathcal{I}(c_0(x)) \text{ if and only if } X \in \mathcal{I}_0 \text{ i.e. } \lim_{n\to\infty} \frac{\log n}{\log x_n} = 0.$$

Proof. If $X \in \mathcal{I}(c_0(x))$ then there exists a set $X' \in \mathcal{U}(c_0(x))$ such that $X \subset X'$. Put $X' = \{x'_1 < x'_2 < \cdots\}$. Then (A7) yields $\lim_{n\to\infty} \frac{\log n}{\log x'_n} = 0$. As $x_n \ge x'_n$ holds for every $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \frac{\log n}{\log x_n} = 0$ and $\lambda(X) = 0$.

We are going to prove the opposite implication. If $\lim_{n\to\infty} \frac{\log n}{\log x_n} = 0$ then $x_n = n^{f(n)}$ for $n \ge 2$ and function $f : \mathbb{N} \to \mathbb{R}^+$ such that $f(n) \to \infty$ with $n \to \infty$.

Define a function $g : \mathbb{N} \to \mathbb{N}$ such that

(i) $g(1) = 1$ and $g(n) \le \max\{1; \frac{1}{2} f([\sqrt{n}])\}$, $n = 2, 3, \ldots,$
where $[x]$ stands for the integer part of x.

(ii) $g(n)$ is nondecreasing and unbounded, i. e. $\lim_{n\to\infty} g(n) = \infty$.

We use g to construct the set $Y = \{y_1 < y_2 < \cdots\}$ by $y_n = n^{g(n)}$. We will show $Y \in \mathcal{U}(c_0(x))$ and $\frac{X(t)}{Y(t)} \to 0$ for $t \to \infty$. Then an application of Theorem 4 yields $X \cup Y \in \mathcal{U}(c_0(x))$, consequently $X \in \mathcal{I}(c_0(x))$.

I) First we show $Y \in \mathcal{U}(c_0(x))$. By (ii) and the definition of y_n we have

$$1 \geq \frac{\#\{i \leq n : \frac{y_i}{y_n} = \frac{i^{g(i)}}{n^{g(n)}} < x\}}{n} \geq \frac{\#\{i \leq n : \left(\frac{i}{n}\right)^{g(n)} < x\}}{n}.$$

Fix $0 < x \leq 1$. Then for every $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$\frac{\#\{i \leq n : \left(\frac{i}{n}\right)^{g(n)} < x\}}{n} \geq 1 - \varepsilon$$

holds for all $n \geq n_0$. Here we used $g(n) \to \infty$ for $n \to \infty$. As $\varepsilon > 0$ was arbitrary, for $\varepsilon \to 0_+$ we have

$$\lim_{n\to\infty} \frac{\#\{i \leq n : \left(\frac{i}{n}\right)^{g(n)} < x\}}{n} = 1, \text{ i.e. } Y \in \mathcal{U}(c_0(x)).$$

II) Now we prove $\frac{X(t)}{Y(t)} \to 0$ for $t \to \infty$.

For sufficiently large n by (i) we have

$$X(y_n) = \max\left\{k : k^{f(k)} < y_n = n^{g(n)}\right\} \leq \sqrt{n}.$$

Let now t be a sufficiently large real number. Then $y_{n-1} < t \leq y_n$ and $n \to \infty$ if $t \to \infty$ moreover

$$0 \leq \frac{X(t)}{Y(t)} \leq \frac{X(y_n)}{Y(y_{n-1})} \leq \frac{\sqrt{n}}{n-1} \to 0 \quad \text{for } n \to \infty.$$

Consequently

$$\lim_{t\to\infty} \frac{X(t)}{Y(t)} = 0.$$

□

Theorem 6. *Let $0 < q \leq 1$. Then*

$$\mathcal{I}_{<q} \subset \mathcal{I}(x^q) \subset \mathcal{I}_{\leq q}. \tag{6}$$

Proof. We prove the first inclusion. Let $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$ be such that $X \in \mathcal{I}_{<q}$, i. e. $\lambda(X) < q$. Thus

$$\liminf_{n\to\infty} \frac{\log x_n}{\log n} > \frac{1}{q}.$$

Then there exists a real number $r > \frac{1}{q}$ and a positive integer n_0 such that

$$\frac{\log x_n}{\log n} \geq r, \quad \text{i. e.} \quad x_n \geq n^r$$

holds for all $n \geq n_0$. Consider the sequence $y_n = [n^{\frac{1}{q}}], (n = 1, 2, \cdots)$.

As $\frac{1}{q} \geq 1$, the inequality $y_n < y_{n+1}$ holds for every $n \in \mathbb{N}$.

Putting $Y = \{y_1 < y_2 < \dots\} \subset \mathbb{N}$ we have $Y \in \mathcal{U}(x^q)$. Let now t be a real number. Then $n < t \leq n+1$. Moreover, as $\frac{1}{r} - q < 0$, we have

$$0 \leq \frac{X(t)}{Y(t)} \leq \frac{X(n+1)}{Y(n)} \leq \frac{(n+1)^{\frac{1}{r}}}{n^q - 1} = \left(\frac{n+1}{n}\right)^{\frac{1}{r}} \frac{n^{\frac{1}{r}-q}}{1 - \frac{1}{n^q}} \to 0$$

if $n \to \infty$. An application of Theorem 4 yields $X \cup Y \in \mathcal{U}(x^q)$, i. e. $X \in \mathcal{I}(x^q)$.

Now we prove the second inclusion. Let $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$ and $X \in \mathcal{I}(x^q)$. Then there exists $X' \in \mathcal{U}(x^q)$ such that $X \subset X'$. Let $X' = \{x'_1 < x'_2 < \dots\}$. By (A8) we have

$$\lim_{n\to\infty} \frac{\log n}{\log x'_n} = q.$$

Using $x_n \geq x'_n$ we obtain

$$\limsup_{n\to\infty} \frac{\log n}{\log x_n} \leq \limsup_{n\to\infty} \frac{\log n}{\log x'_n} = q,$$

consequently $\lambda(X) \leq q$, i.e. $X \in \mathcal{I}_{\leq q}$. □

Corollary 3. *If $0 < q < q' \leq 1$ then $\mathcal{I}(c_0(x)) \subset \mathcal{I}(x^q) \subset \mathcal{I}(x^{q'})$.*

Proof. Theorem 5, (2) and Theorem 6 implies

$$\mathcal{I}(c_0(x)) = \mathcal{I}_0 \subset \mathcal{I}_{<q} \subset \mathcal{I}(x^q) \subset \mathcal{I}_{\leq q} \subset \mathcal{I}_{<q'} \subset \mathcal{I}(x^{q'}).$$

□

The following lemma gives a useful sufficient condition for a set A to belong to $\mathcal{U}(x^q)$.

Lemma 1. *Let $0 < q \leq 1$, $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$ and let its terms be given by*

$$x_n = [n^{\frac{1}{q}+\alpha_n}],$$

where the sequence (α_n) fulfils

$$\lim_{n\to\infty} \alpha_n = 0$$

and

$$\lim_{n\to\infty} (\alpha_{kn} - \alpha_n) \log n = 0$$

for every $k \in \mathbb{N}$. Then $X \in \mathcal{U}(x^q)$.

Proof. By (A2) it is sufficient to show that under assumptions the relation

$$\lim_{n\to\infty} \frac{x_{kn}}{x_n} = k^{\frac{1}{q}}$$

holds for every positive integer k. Thus calculate

$$\lim_{n\to\infty} \frac{x_{kn}}{x_n} = \lim_{n\to\infty} \frac{[(kn)^{\frac{1}{q}+\alpha_{kn}}]}{[n^{\frac{1}{q}+\alpha_n}]} = \lim_{n\to\infty} k^{\frac{1}{q}+\alpha_{kn}} n^{\alpha_{kn}-\alpha_n} =$$

$$= k^{\frac{1}{q}} \lim_{n\to\infty} e^{(\alpha_{kn}-\alpha_n)\log n} = k^{\frac{1}{q}}$$

and the statement of lemma follows. □

The following theorem provides a nice characterization of the family $\mathcal{I}(x^q)$. It follows that $\mathcal{I}(x^q) = \mathcal{I}_{\leq q}$. In the monography ([8] , p.7, exercise 1.13.) it is noted that $X = \mathbb{N} \in \mathcal{U}(x^q)$ in the case $q = 1$. This means that $\mathbb{N} \in \mathcal{I}(x^1)$, but then $A \in \mathcal{I}(x^1)$ holds for every $A \subset \mathbb{N}$, i. e. $\mathcal{I}(x^1) = 2^{\mathbb{N}} = \mathcal{I}_{\leq 1}$. Thus it is sufficient to prove the equality $\mathcal{I}(x^q) = \mathcal{I}_{\leq q}$ for $0 < q < 1$.

Theorem 7. *Let $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$ and $0 < q \leq 1$. Then*

$$X \in \mathcal{I}(x^q) \text{ if and only if } X \in \mathcal{I}_{\leq q} \text{ i.e. } \limsup_{n\to\infty} \frac{\log n}{\log x_n} \leq q.$$

Proof. Let $X = \{x_1 < x_2 < \dots\}$ and $0 < q < 1$. By virtue of (6) it is sufficient to prove the implication, if $\lambda(X) = q$ then $X \in \mathcal{I}(x^q)$. Thus let us assume $\lambda(X) = q$. To complete the proof, it is sufficient to find a set $Y = \{y_1 < y_2 < \dots\}$ such that

$$X \cup Y \in \mathcal{U}(x^q).$$

Thus we assume that $\lambda(X) = q$, consequently its terms x_n can be expressed by $x_n = n^{\frac{1}{q}+\alpha_n}$ where $\liminf_{n\to\infty} \alpha_n = 0$. To simplify technical manipulations we will start with a modification of parameters of the set X. For every positive integer n put

$$\beta_n = \inf\{\alpha_k;\ k = n, n+1, \dots\} - \frac{1}{\log\log(n+2)}.$$

Then β_n is a nondecreasing sequence of not positive numbers converging to 0 and

$$\beta_n \le \alpha_n - \frac{1}{\log\log(n+2)} \tag{7}$$

holds for all $n \in \mathbb{N}$.

Let $f\colon [1,\infty) \to \mathbb{R}$ be a very slowly increasing unbounded function such that

$$\lim_{x\to\infty} (f(px) - f(x)) \log x = 0$$

holds for every $p \in \mathbb{N}$. As an example of such function can serve $\log\log\log(x+3)$. For every positive integer n put $\delta_n = f(n+1) - f(n)$ and construct the sequence (γ_n) as follows. Let $\gamma_1 = \beta_1$ and for $n \in \mathbb{N}$ put by induction

$$\gamma_{n+1} = \begin{cases} \gamma_n + \delta_n, & \text{if } \gamma_n + \delta_n \le \beta_{n+1}, \\ \gamma_n, & \text{if } \gamma_n + \delta_n > \beta_{n+1}. \end{cases}$$

Then $\gamma_n \le \beta_n$ for all $n \in \mathbb{N}$, thus (7) holds also when β_n is replaced by γ_n for every $n \in \mathbb{N}$. Also

$$\lim_{n\to\infty} \gamma_n = 0,$$

as both sequences (β_n) and (δ_n) converge to 0. Moreover,

$$\lim_{n\to\infty} (\gamma_{pn} - \gamma_n) \log n \le \lim_{n\to\infty} (f(pn) - f(n)) \log n = 0,$$

thus the set of positive integers $Y = \{y_1 < y_2 < \dots\}$, where $y_1 = 1$ and

$$y_{n+1} = \max\{y_n + 1, [(n+1)^{\frac{1}{q}+\gamma_{n+1}}]\}$$

belongs to $\mathcal{U}(x^q)$ by Lemma 1. The reason is that the maximum in the above formula is equal to the second term for all sufficiently large $n \in \mathbb{N}$.

To complete the proof, calculate using $\gamma_n \le \beta_n$ and (7)

$$\lim_{n\to\infty} \frac{x_n}{y_n} \ge \lim_{n\to\infty} \frac{n^{\frac{1}{q}+\alpha_n}}{n^{\frac{1}{q}+\gamma_n}} = \lim_{n\to\infty} n^{\alpha_n-\gamma_n} =$$

$$\lim_{n\to\infty} e^{(\alpha_n-\gamma_n)\log n} \ge \lim_{n\to\infty} e^{(\alpha_n-\beta_n)\log n} \ge \lim_{n\to\infty} e^{\frac{\log n}{\log\log(n+2)}} = \infty$$

and application of Corollary 2 proves $X \cup Y \in \mathcal{U}(x^q)$. $\square$

Corollary 4. *Let $0 < q < 1$ and a decreasing sequence $1 \ge q_1 > q_2 > \dots > q_n > \dots$ converges to q. Then*

$$\bigcap_{n=1}^{\infty} \mathcal{I}(x^{q_n}) = \mathcal{I}(x^q).$$

Proof. Theorem 7 implies

$$\bigcap_{n=1}^{\infty} \mathcal{I}(x^{q_n}) = \bigcap_{n=1}^{\infty} \mathcal{I}_{\leq q_n} = \mathcal{I}_{\leq q} = \mathcal{I}(x^q).$$

□

Remark 1. *In Theorem 5. and Theorem 7 there are characterized sets $A \subset \mathbb{N}$ belonging to ideals $\mathcal{I}(c_0(x))$ and $\mathcal{I}(x^q)$ by means of the exponent of convergence of the corresponding sets, i.e. $\lambda(A) = 0$ or $\lambda(A) \leq q$ what means that $A \in \mathcal{I}_0$ or $A \in \mathcal{I}_{\leq q}$. On the other hand, (A11) contains an alternative characterization of the above theorem saying that for every $\delta > 0$ we have $\lim_{x\to\infty} \frac{A(x)}{x^\delta} = 0$ or $\lim_{x\to\infty} \frac{A(x)}{x^{q+\delta}} = 0$. Let us also note that from Theorem 7 and (A9) follows that also the family $\mathcal{I}(x^q)$ is ideal. From (A8) we obtain also the following interesting inclusion holding for studied families of sets (for characterization of $\mathcal{I}_{<q}$ see (A12))*

$$\mathcal{U}(x^q) \subset \mathcal{I}_{\leq q} \setminus \mathcal{I}_{<q} \text{ wich implies } \mathcal{I}_{<q} \subset \mathcal{I}(x^q) \setminus \mathcal{U}(x^q).$$

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