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Explicit Solutions of the Yang-Baxter-Like Matrix Equation for a Singular Diagonalizable Matrix With Three Distinct Eigenvalues

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Abstract. Let A be a singular diagonalizable complex matrix with three distinct eigenvalues. We derive all explicit solutions X of the Yang-Baxter-like matrix equation $AXA = XAX$, by taking advantage of the Jordan form structure of A. The result generates the formula obtained in Chen et al. (2019) and M. Saeed Ibrahim Adam et al. (2019). We give examples to illustrate the validity of the results obtained in this paper.

1. Introduction

Let A be an $n \times n$ singular diagonalizable complex matrix with three distinct eigenvalues. The quadratic matrix equation

$$AXA = XAX, \tag{1}$$

is often called the *Yang-Baxter-like matrix equation* (also called the star-triangle-like equation in statistical mechanics; see, e.g., in Part III of [1]) because of its connections with the classical Yang-Baxter equation arising in statistical mechanics [2–4].

No systematical study of (1) has appeared in the literature as a purely linear algebra problem although some solutions have been found for Yang-Baxter equation in quantum group theory [5]. One possible reason is that Yang-Baxter-like matrix equation (1) is equivalent to solving a polynomial system of n^2 quadratic equations with n^2 unknowns, which solving this system is a very challenging topic. Almost all the works so far have been toward constructing commuting solutions ($AX = XA$) of the equation; see, e.g., [5–16] and the references therein. Finding all non-commuting solutions of Yang-Baxter-like matrix equation (1) is still a challenging task when A is arbitrary. Up to now, there are only isolated results toward this goal for special classes of the given matrix A, e.g., [17–28]. All solutions have been constructed for rank-1 matrices A in [23], rank-2 matrices A in [24, 25], non-diagonalizable elementary matrices A in [26], idempotent matrices

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A ($A^2 = A$) in [19], $A^2 = I$ in [18, 20], $A^3 = A$ in [21], $A^4 = A$ in [27], and diagonalizable matrices A with two different eigenvalues in [22].

In this paper, we try to solve the Yang-Baxter-like matrix equation (1) to derive all explicit solutions X when the given singular diagonalizable matrix A has three distinct eigenvalues 0, λ, and μ, extending the research for which A is diagonalizable matrix with spectrum contained in the set $\{1, \alpha, 0\}$ [17] and A has the minimal polynomial $g(x) = x^3 - x$ [21], respectively. This is an important step to solve more general matrices. Problem of diagonalizing or block-diagonalizing matrices (or operator matrices) is closely related to solving Sylvester equations. The sufficient conditions for the solvability of the Sylvester equation $AX - XB = -C$ are derived in [29], under the premise that $\sigma(A) \cap \sigma(B) \neq \emptyset$. The singular diagonalizable complex matrices are the perfect candidates for the results obtained in [29]. Our results rely on Jordan forms of the given matrices, which agrees perfectly with the expression in [29].

We first provide some preliminary results. Then we study all solutions for Yang-Baxter-like matrix equation (1) under the conditions that $\lambda^2 - \lambda\mu + \mu^2 = 0$ and $\lambda^2 - \lambda\mu + \mu^2 \neq 0$, respectively. The application to the case that the minimal polynomial of A is $g(x) = x^3 - x$ will be demonstrated in Section 5. Finally, we give some numerical experiments to illustrate the validity of the results obtained in Section 6. We conclude with Section 7.

2. Preliminary results

In this section we give some results and denotations for our further discussion. At first, we give an assumption as follows.

Assumption 2.1. *Let A be an $n \times n$ complex diagonalizable matrix with three distinct eigenvalues 0, λ, and μ. Suppose the rank of A is m and the multiplicity of eigenvalue λ is k. Then there exists a nonsingular $S \in \mathbb{C}^{n\times n}$ such that $A = SJS^{-1}$ in which*

$$J = diag\left(\Lambda, 0_{n-m}\right), \tag{2}$$

and $\Lambda = diag\left(\lambda I_k, \mu I_{m-k}\right)$.

Let $Y = S^{-1}XS$, then the matrix equation $AXA = XAX$ is equivalent to $JYJ = YJY$. According to the structure of J in (2), we partition Y as

$$Y = \begin{pmatrix} K & C \\ D & W \end{pmatrix},\ K \in \mathbb{C}^{m\times m},\ W \in \mathbb{C}^{(n-m)\times(n-m)}. \tag{3}$$

Then we give the following results.

Lemma 2.1. *Suppose that A satisfies Assumption 2.1. Then $AXA = XAX$ holds if and only if the matrices K, C, D, W in (3) satisfy the following equations*

$$\begin{cases} K\Lambda K = \Lambda K\Lambda, \\ \quad K\Lambda C = 0, \\ \quad D\Lambda K = 0, \\ \quad D\Lambda C = 0. \end{cases} \tag{4}$$

Then from the equivalent system (4), we see immediately that W is arbitrary for all of its solutions, so in the remainder of the paper it can be any $(n - m) \times (n - m)$ matrix. The first equation in (4) is also a Yang-Baxter-like matrix equation.

Based on Theorems 4.4 and 4.6 in [22], we need to discuss all solutions of $AXA = XAX$ in two different cases as $\lambda^2 - \lambda\mu + \mu^2 = 0$ and $\lambda^2 - \lambda\mu + \mu^2 \neq 0$. So we give our results in the following two sections.

3. All solutions for $AXA = XAX$ in the case $\lambda^2 - \lambda\mu + \mu^2 = 0$

If $\lambda\mu \neq 0$ and $\lambda^2 - \lambda\mu + \mu^2 = 0$, by Theorem 4.4 in [22], all solutions K of the first equation $K\Lambda K = \Lambda K\Lambda$ of (4) are

$$K = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \left[\begin{array}{cc|cc} \lambda I_{t_1} & 0 & F & 0 \\ 0 & 0_{k-t_1} & 0 & 0 \\ \hline G & 0 & \mu I_{t_2} & 0 \\ 0 & 0 & 0 & 0_{m-k-t_2} \end{array}\right] \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix},$$

in which, $P \in \mathbb{C}^{k\times k}$, $Q \in \mathbb{C}^{(m-k)\times(m-k)}$ are any invertible matrices, $0 \leq t_1 \leq k$, $0 \leq t_2 \leq m-k$, F is an arbitrary $t_1 \times t_2$ matrix, and $G = (I - F^\dagger F)M(I - FF^\dagger)$, M is an arbitrary $t_2 \times t_1$ matrix. Let

$$\widetilde{C} = \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix},$$

where $C_1 \in \mathbb{C}^{t_1\times(n-m)}$, $C_2 \in \mathbb{C}^{(k-t_1)\times(n-m)}$, $C_3 \in \mathbb{C}^{t_2\times(n-m)}$, and $C_4 \in \mathbb{C}^{(m-k-t_2)\times(n-m)}$. According to the second equation $K\Lambda C = 0$ of (4), we have

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \left[\begin{array}{cc|cc} \lambda I_{t_1} & 0 & F & 0 \\ 0 & 0_{k-t_1} & 0 & 0 \\ \hline G & 0 & \mu I_{t_2} & 0 \\ 0 & 0 & 0 & 0_{m-k-t_2} \end{array}\right] \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix} C = 0. \tag{5}$$

since

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix} = \begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix},$$

the equation (5) is equivalent to

$$\left[\begin{array}{cc|cc} \lambda^2 I_{t_1} & 0 & \mu F & 0 \\ 0 & 0_{k-t_1} & 0 & 0 \\ \hline \lambda G & 0 & \mu^2 I_{t_2} & 0 \\ 0 & 0 & 0 & 0_{m-k-t_2} \end{array}\right] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0.$$

Thus

$$\begin{cases} \lambda^2 C_1 + \mu F C_3 = 0, \\ \lambda G C_1 + \mu^2 C_3 = 0. \end{cases} \tag{6}$$

Notice that $G = (I - F^\dagger F)M(I - FF^\dagger)$, we get $GF = 0$ and $FG = 0$. Because $\lambda\mu \neq 0$, from (6), we have $C_1 = 0$ and $C_3 = 0$. Thus

$$C = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \left[\begin{array}{c} 0 \\ C_2 \\ \hline 0 \\ C_4 \end{array}\right]. \tag{7}$$

Let

$$\widetilde{D} = D \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} D_1 & D_2 & D_3 & D_4 \end{bmatrix},$$

where $D_1 \in \mathbb{C}^{(n-m)\times t_1}$, $D_2 \in \mathbb{C}^{(n-m)\times(k-t_1)}$, $D_3 \in \mathbb{C}^{(n-m)\times t_2}$, and $D_4 \in \mathbb{C}^{(n-m)\times(m-k-t_2)}$. According to the third equation $D\Lambda K = 0$ of (4), we have

$$D\begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\left[\begin{array}{cc|cc} \lambda I_{t_1} & 0 & F & 0 \\ 0 & 0_{k-t_1} & 0 & 0 \\ \hline G & 0 & \mu I_{t_2} & 0 \\ 0 & 0 & 0 & 0_{m-k-t_2} \end{array}\right]\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} = 0. \tag{8}$$

since

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix} = \begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix},$$

the equation (8) is equivalent to

$$\begin{bmatrix} D_1 & D_2 & D_3 & D_4 \end{bmatrix}\left[\begin{array}{cc|cc} \lambda^2 I_{t_1} & 0 & \lambda F & 0 \\ 0 & 0_{k-t_1} & 0 & 0 \\ \hline \mu G & 0 & \mu^2 I_{t_2} & 0 \\ 0 & 0 & 0 & 0_{m-k-t_2} \end{array}\right] = 0.$$

Thus

$$\begin{cases} \lambda^2 D_1 + \mu D_3 G = 0, \\ \lambda D_1 F + \mu^2 D_3 = 0. \end{cases} \tag{9}$$

Since $\lambda\mu \neq 0$, $GF = 0$, and $FG = 0$, from (9), we have $D_1 = 0$ and $D_3 = 0$. Therefore

$$D = \left[\begin{array}{cc|cc} 0 & D_2 & 0 & D_4 \end{array}\right]\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}. \tag{10}$$

Combining (7) and (10) with the fourth equation $D\Lambda C = 0$ of (4) yields

$$\begin{aligned} D\Lambda C &= \widetilde{D}\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}\begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}\widetilde{C} \\ &= \begin{bmatrix} 0 & D_2 & 0 & D_4 \end{bmatrix}\left[\begin{array}{cc|cc} \lambda I_{t_1} & 0 & 0 & 0 \\ 0 & \lambda I_{k-t_1} & 0 & 0 \\ \hline 0 & 0 & \mu I_{t_2} & 0 \\ 0 & 0 & 0 & \mu I_{m-k-t_2} \end{array}\right]\begin{bmatrix} 0 \\ C_2 \\ 0 \\ C_4 \end{bmatrix} \\ &= \lambda D_2 C_2 + \mu D_4 C_4 \\ &= 0. \end{aligned}$$

In summary, we have proved the following theorem.

Theorem 3.1. *Suppose that A satisfies Assumption 2.1 with $\lambda\mu \neq 0$ and $\lambda^2 - \lambda\mu + \mu^2 = 0$. Then all solutions of the Yang-Baxter-like matrix equation $AXA = XAX$ have the form*

$$X = S\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I_{n-m} \end{bmatrix}\left[\begin{array}{cc|cc|c} \lambda I_{t_1} & 0 & F & 0 & 0 \\ 0 & 0_{k-t_1} & 0 & 0 & C_2 \\ \hline G & 0 & \mu I_{t_2} & 0 & 0 \\ 0 & 0 & 0 & 0_{m-k-t_2} & C_4 \\ \hline 0 & D_2 & 0 & D_4 & W \end{array}\right]\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & I_{n-m} \end{bmatrix}S^{-1},$$

in which, $P \in \mathbb{C}^{k\times k}$, $Q \in \mathbb{C}^{(m-k)\times(m-k)}$ are any invertible matrices, $0 \leq t_1 \leq k$, $0 \leq t_2 \leq m-k$, F is an arbitrary $t_1 \times t_2$ matrix, $G = (I - F^\dagger F)M(I - FF^\dagger)$, M is an arbitrary $t_2 \times t_1$ matrix, $C_2 \in \mathbb{C}^{(k-t_1)\times(n-m)}$, $C_4 \in \mathbb{C}^{(m-k-t_2)\times(n-m)}$, $D_2 \in \mathbb{C}^{(n-m)\times(k-t_1)}$, $D_4 \in \mathbb{C}^{(n-m)\times(m-k-t_2)}$, $\lambda D_2 C_2 = -\mu D_4 C_4$, and W is an arbitrary $(n-m)\times(n-m)$ matrix.

Corollary 3.2. *Under the assumption of Theorem 3.1, if $t_1 = k$, $t_2 = m - k$, then all solutions of the Yang-Baxter-like matrix equation $AXA = XAX$ have the form*

$$X = S\begin{bmatrix} \lambda I_k & PFQ^{-1} & 0 \\ QGP^{-1} & \mu I_{m-k} & 0 \\ 0 & 0 & W \end{bmatrix} S^{-1},$$

in which, $P \in \mathbb{C}^{k\times k}$, $Q \in \mathbb{C}^{(m-k)\times(m-k)}$ are any invertible matrices, F is an arbitrary $k \times (m-k)$ matrix, $G = (I - F^{\dagger}F)M(I - FF^{\dagger})$, M is an arbitrary $(m-k)\times k$ matrix, and W is an arbitrary $(n-m)\times(n-m)$ matrix.

4. All solutions for $AXA = XAX$ in the case $\lambda^2 - \lambda\mu + \mu^2 \neq 0$

If $\lambda\mu \neq 0$ and $\lambda^2 - \lambda\mu + \mu^2 \neq 0$, by Theorem 4.6 in [22], all solutions K of the first equation $K\Lambda K = \Lambda K \Lambda$ of (4) are

$$K = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \left[\begin{array}{ccc|ccc} \widehat{\lambda} I_r & 0 & 0 & F & 0 & 0 \\ 0 & \lambda I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{k-r-v} & 0 & 0 & 0 \\ \hline G & 0 & 0 & \widehat{\mu} I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{m-k-r-\tau} \end{array}\right] \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix},$$

in which, $P \in \mathbb{C}^{k\times k}$, $Q \in \mathbb{C}^{(m-k)\times(m-k)}$ are any invertible matrices, $0 \leq r \leq \min\{k, m-k\}$, $0 \leq v \leq k - r$, $0 \leq \tau \leq m - k - r$, $\widehat{\lambda} = \frac{\mu^2}{\mu-\lambda}$, $\widehat{\mu} = \frac{\lambda^2}{\lambda-\mu}$, F is an arbitrary $r \times r$ invertible matrix, and $G = \frac{-\lambda\mu(\lambda^2-\lambda\mu+\mu^2)}{(\lambda-\mu)^2}F^{-1}$. Let

$$\widehat{C} = \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix},$$

where $C_1 \in \mathbb{C}^{r\times(n-m)}$, $C_2 \in \mathbb{C}^{v\times(n-m)}$, $C_3 \in \mathbb{C}^{(k-r-v)\times(n-m)}$, $C_4 \in \mathbb{C}^{r\times(n-m)}$, $C_5 \in \mathbb{C}^{\tau\times(n-m)}$ and $C_6 \in \mathbb{C}^{(m-k-r-\tau)\times(n-m)}$. According to the second equation $K\Lambda C = 0$ of (4) and

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}\begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix} = \begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix}\begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix},$$

we have

$$\left[\begin{array}{ccc|ccc} \lambda\widehat{\lambda} I_r & 0 & 0 & \mu F & 0 & 0 \\ 0 & \lambda^2 I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{k-r-v} & 0 & 0 & 0 \\ \hline \lambda G & 0 & 0 & \mu\widehat{\mu} I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^2 I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{m-k-r-\tau} \end{array}\right] \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} = 0.$$

Thus

$$\begin{cases} \lambda\widehat{\lambda} C_1 + \mu F C_4 = 0, \\ \lambda G C_1 + \mu\widehat{\mu} C_4 = 0, \\ \lambda^2 C_2 = 0, \\ \mu^2 C_5 = 0. \end{cases} \tag{11}$$

Since $\lambda\mu \neq 0$, from the last two equations of (11), we get $C_2 = 0$ and $C_5 = 0$. Because $\lambda\mu \neq 0$, $\lambda^2 - \lambda\mu + \mu^2 \neq 0$, and $G = \frac{-\lambda\mu(\lambda^2-\lambda\mu+\mu^2)}{(\lambda-\mu)^2}F^{-1}$, from the first two equations of (11), we have $C_1 = 0$ and $C_4 = 0$. Thus

$$C = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ C_3 \\ \hline 0 \\ 0 \\ C_6 \end{bmatrix}. \tag{12}$$

Let

$$\widehat{D} = D \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \end{bmatrix},$$

where $D_1 \in \mathbb{C}^{(n-m)\times r}$, $D_2 \in \mathbb{C}^{(n-m)\times v}$, $D_3 \in \mathbb{C}^{(n-m)\times(k-r-v)}$, $D_4 \in \mathbb{C}^{(n-m)\times r}$, $D_5 \in \mathbb{C}^{(n-m)\times\tau}$, and $D_6 \in \mathbb{C}^{(n-m)\times(m-k-r-\tau)}$. According to the third equation $D\Lambda K = 0$ of (4) and

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix} = \begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix},$$

we have

$$\begin{bmatrix} D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \end{bmatrix} \left[\begin{array}{ccc|ccc} \lambda\widehat{\lambda} I_r & 0 & 0 & \lambda F & 0 & 0 \\ 0 & \lambda^2 I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{k-r-v} & 0 & 0 & 0 \\ \hline \mu G & 0 & 0 & \mu\widehat{\mu} I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^2 I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{m-k-r-\tau} \end{array}\right] = 0.$$

Thus

$$\begin{cases} \lambda\widehat{\lambda} D_1 + \mu D_4 G = 0, \\ \lambda D_1 F + \mu\widehat{\mu} D_4 = 0 \\ \quad \lambda^2 D_2 = 0, \\ \quad \mu^2 D_5 = 0. \end{cases} \tag{13}$$

Since $\lambda\mu \neq 0$, from the last two equations of (13), we get $D_2 = 0$ and $D_5 = 0$. Because $\lambda\mu \neq 0$, $\lambda^2 - \lambda\mu + \mu^2 \neq 0$, and $G = \frac{-\lambda\mu(\lambda^2-\lambda\mu+\mu^2)}{(\lambda-\mu)^2}F^{-1}$, from the first two equations of (13), we have $D_1 = 0$ and $D_4 = 0$. Therefore

$$D = \left[\begin{array}{ccc|ccc} 0 & 0 & D_3 & 0 & 0 & D_6 \end{array}\right] \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}. \tag{14}$$

Combining (12) and (14) with the fourth equation $D\Lambda C = 0$ of (4) yields

$$\begin{aligned} D\Lambda C &= \widehat{D} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} \lambda I_k & 0 \\ 0 & \mu I_{m-k} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \widehat{C} \\ &= \begin{bmatrix} 0 & 0 & D_3 & 0 & 0 & D_6 \end{bmatrix} \left[\begin{array}{ccc|ccc} \lambda I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda I_v & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda I_{k-r-v} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mu I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu I_\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu I_{m-k-r-\tau} \end{array}\right] \begin{bmatrix} 0 \\ 0 \\ C_3 \\ 0 \\ 0 \\ C_6 \end{bmatrix} \\ &= \lambda D_3 C_3 + \mu D_6 C_6 \\ &= 0. \end{aligned}$$

In summary, we have proved the following theorem.

Theorem 4.1. *Suppose that A satisfies Assumption 2.1 with $\lambda\mu \neq 0$ and $\lambda^2 - \lambda\mu + \mu^2 \neq 0$. Then all solutions of the Yang-Baxter-like matrix equation $AXA = XAX$ have the form*

$$X = S\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I_{n-m} \end{bmatrix}\left[\begin{array}{ccc|ccc|c} \widehat{\lambda}I_r & 0 & 0 & F & 0 & 0 & 0 \\ 0 & \lambda I_v & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{k-r-v} & 0 & 0 & 0 & C_3 \\ \hline G & 0 & 0 & \widehat{\mu}I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu I_\tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{m-k-r-\tau} & C_6 \\ \hline 0 & 0 & D_3 & 0 & 0 & D_6 & W \end{array}\right]\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & I_{n-m} \end{bmatrix}S^{-1},$$

in which, $P \in \mathbb{C}^{k\times k}$, $Q \in \mathbb{C}^{(m-k)\times(m-k)}$ are any invertible matrices, $0 \leq r \leq \min\{k, m-k\}$, $0 \leq v \leq k-r$, $0 \leq \tau \leq m-k-r$, $\widehat{\lambda} = \frac{\mu^2}{\mu-\lambda}$, $\widehat{\mu} = \frac{\lambda^2}{\lambda-\mu}$, F is an arbitrary $r\times r$ invertible matrix, $G = \frac{-\lambda\mu(\lambda^2-\lambda\mu+\mu^2)}{(\lambda-\mu)^2}F^{-1}$, $C_3 \in \mathbb{C}^{(k-r-v)\times(n-m)}$, $C_6 \in \mathbb{C}^{(m-k-r-\tau)\times(n-m)}$, $D_3 \in \mathbb{C}^{(n-m)\times(k-r-v)}$, $D_6 \in \mathbb{C}^{(n-m)\times(m-k-r-\tau)}$, $\lambda D_3C_3 = -\mu D_6C_6$, and W is an arbitrary $(n-m)\times(n-m)$ matrix.

Corollary 4.2. *Under the assumption of Theorem 4.1, if $r + v = k$, $r + \tau = m - k$, then all solutions of the Yang-Baxter-like matrix equation $AXA = XAX$ have the form*

$$X = S\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I_{n-m} \end{bmatrix}\left[\begin{array}{cc|cc|c} \widehat{\lambda}I_r & 0 & F & 0 & 0 \\ 0 & \lambda I_v & 0 & 0 & 0 \\ \hline G & 0 & \widehat{\mu}I_r & 0 & 0 \\ 0 & 0 & 0 & \mu I_\tau & 0 \\ \hline 0 & 0 & 0 & 0 & W \end{array}\right]\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & I_{n-m} \end{bmatrix}S^{-1},$$

in which, $P \in \mathbb{C}^{k\times k}$, $Q \in \mathbb{C}^{(m-k)\times(m-k)}$ are any invertible matrices, $0 \leq r \leq \min\{k, m-k\}$, $0 \leq v \leq k-r$, $0 \leq \tau \leq m-k-r$, $\widehat{\lambda} = \frac{\mu^2}{\mu-\lambda}$, $\widehat{\mu} = \frac{\lambda^2}{\lambda-\mu}$, F is an arbitrary $r \times r$ invertible matrix, $G = \frac{-\lambda\mu(\lambda^2-\lambda\mu+\mu^2)}{(\lambda-\mu)^2}F^{-1}$, and W is an arbitrary $(n-m)\times(n-m)$ matrix.

5. Application

In this section we apply Theorem 4.1 to the case that the minimal polynomial of A is $g(x) = x^3 - x$. In this case, $A^3 = A$. M. Saeed Ibrahim Adam et al. have considered this case in [21]. It is well-known that the zeros of the minimal polynomial of any square matrix give all the eigenvalues of the matrix and an eigenvalue λ is semisimple, that is the algebraic multiplicity of λ equals its geometric multiplicity, if and only if the multiplicity of λ, as a zero of the minimal polynomial, is 1. Since $g(x) = x^3 - x = x(x+1)(x-1)$, A has three distinct eigenvalues -1, 0, and 1. Furthermore, each eigenvalue of A is semisimple, so A is diagonalizable. In other words, there is a nonsingular matrix S such that $AS = SJ$, where J is a diagonal matrix. Assume that the rank of A is m and the multiplicity of eigenvalue 1 is k. Then we can write the Jordan form of A as $J = diag\{I_k, -I_{m-k}, 0_{n-m}\}$. Since $1^2 - 1 \times (-1) + (-1)^2 = 3 \neq 0$, applying Theorem 4.1, we have the following theorem.

Theorem 5.1. *If $A^3 = A$. Assume that the rank of A is m and the multiplicity of eigenvalue 1 is k. Then all solutions of the Yang-Baxter-like matrix equation $AXA = XAX$ have the form*

$$X = S\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I_{n-m} \end{bmatrix}\left[\begin{array}{ccc|ccc|c} -\frac{1}{2}I_r & 0 & 0 & F & 0 & 0 & 0 \\ 0 & I_v & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{k-r-v} & 0 & 0 & 0 & C_3 \\ \hline \frac{3}{4}F^{-1} & 0 & 0 & \frac{1}{2}I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_\tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{m-k-r-\tau} & C_6 \\ \hline 0 & 0 & D_3 & 0 & 0 & D_6 & W \end{array}\right]\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & I_{n-m} \end{bmatrix}S^{-1},$$

in which, $P \in \mathbb{C}^{k\times k}$, $Q \in \mathbb{C}^{(m-k)\times(m-k)}$ are any invertible matrices, $0 \le r \le \min\{k, m-k\}$, $0 \le \upsilon \le k-r$, $0 \le \tau \le m-k-r$, F is an arbitrary $r \times r$ invertible matrix, $C_3 \in \mathbb{C}^{(k-r-\upsilon)\times(n-m)}$, $C_6 \in \mathbb{C}^{(m-k-r-\tau)\times(n-m)}$, $D_3 \in \mathbb{C}^{(n-m)\times(k-r-\upsilon)}$, $D_6 \in \mathbb{C}^{(n-m)\times(m-k-r-\tau)}$, $D_3C_3 = D_6C_6$, and W is an arbitrary $(n-m)\times(n-m)$ matrix.

The formula X in Theorem 5.1 are more general than the formula in Theorem 3.5 by M. Saeed Ibrahim Adam et al. [21].

6. Numerical examples

We present two numerical examples to illustrate our results.

Example 6.1. *Let*

$$A = \begin{bmatrix} 0 & -2 & 3 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -2 & 4 & -1 \\ -1 & -4 & 7 & -2 \end{bmatrix}.$$

Then $A^3 = A$. This is the example in [21]. There exists a nonsingular matrix

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix},$$

such that $A = SJS^{-1}$, $J = diag\{1, 1, -1, 0\}$. all solutions of (1) are $X = SYS^{-1}$. From Theorem 5.1, there are totally eight cases to be considered in finding the general solution Y.

Case I: $r = 1$, $\upsilon = 1$, $\tau = 0$.

$$Y = \begin{bmatrix} \frac{-0.5p_1p_4-p_2p_3}{p_1p_4-p_2p_3} & \frac{1.5p_1p_2}{p_1p_4-p_2p_3} & \frac{p_1f}{q} & 0 \\ \frac{-1.5p_3p_4}{p_1p_4-p_2p_3} & \frac{p_1p_4+0.5p_2p_3}{p_1p_4-p_2p_3} & \frac{p_3f}{q} & 0 \\ \frac{3qp_4}{4f(p_1p_4-p_2p_3)} & -\frac{3qp_2}{4f(p_1p_4-p_2p_3)} & 0.5 & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

for all p_1, p_2, p_3, p_4, f, q, $w \in \mathbb{C}$, $p_1p_4 - p_2p_3 \neq 0$, $f \neq 0$, and $q \neq 0$.

Case II: $r = 1$, $\upsilon = 0$, $\tau = 0$.

$$Y = \begin{bmatrix} \frac{-0.5p_1p_4}{p_1p_4-p_2p_3} & \frac{0.5p_1p_2}{p_1p_4-p_2p_3} & \frac{p_1f}{q} & 0 \\ \frac{-0.5p_3p_4}{p_1p_4-p_2p_3} & \frac{0.5p_2p_3}{p_1p_4-p_2p_3} & \frac{p_3f}{q} & 0 \\ \frac{3qp_4}{4f(p_1p_4-p_2p_3)} & -\frac{3qp_2}{4f(p_1p_4-p_2p_3)} & 0.5 & 0 \\ -d_3p_3 & d_3p_1 & 0 & w \end{bmatrix}, \begin{bmatrix} \frac{-0.5p_1p_4}{p_1p_4-p_2p_3} & \frac{0.5p_1p_2}{p_1p_4-p_2p_3} & \frac{p_1f}{q} & p_2c_3 \\ \frac{-0.5p_3p_4}{p_1p_4-p_2p_3} & \frac{0.5p_2p_3}{p_1p_4-p_2p_3} & \frac{p_3f}{q} & p_4c_3 \\ \frac{3qp_4}{4f(p_1p_4-p_2p_3)} & -\frac{3qp_2}{4f(p_1p_4-p_2p_3)} & 0.5 & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

for all p_1, p_2, p_3, p_4, f, q, w, c_3, $d_3 \in \mathbb{C}$, $p_1p_4 - p_2p_3 \neq 0$, $f \neq 0$, and $q \neq 0$.

Case III: $r = 0$, $\upsilon = 2$, $\tau = 1$.

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

for all $w \in \mathbb{C}$.

Case IV: $r = 0$, $v = 2$, $\tau = 0$.

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_6 & w \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & c_6 \\ 0 & 0 & 0 & w \end{bmatrix}$$

for all c_6, d_6, $w \in \mathbb{C}$.

Case V: $r = 0$, $v = 1$, $\tau = 1$.

$$Y = \begin{bmatrix} \frac{p_1p_4}{p_1p_4-p_2p_3} & \frac{-p_1p_2}{p_1p_4-p_2p_3} & 0 & 0 \\ \frac{p_3p_4}{p_1p_4-p_2p_3} & \frac{-p_2p_3}{p_1p_4-p_2p_3} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -d_3p_3 & d_3p_1 & 0 & w \end{bmatrix}, \begin{bmatrix} \frac{p_1p_4}{p_1p_4-p_2p_3} & \frac{-p_1p_2}{p_1p_4-p_2p_3} & 0 & p_2c_3 \\ \frac{p_3p_4}{p_1p_4-p_2p_3} & \frac{-p_2p_3}{p_1p_4-p_2p_3} & 0 & p_4c_3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

for all p_1, p_2, p_3, p_4, w, c_3, $d_3 \in \mathbb{C}$, *and* $p_1p_4 - p_2p_3 \neq 0$.

Case VI: $r = 0$, $v = 1$, $\tau = 0$.

$$Y = \begin{bmatrix} \frac{p_1p_4}{p_1p_4-p_2p_3} & \frac{-p_1p_2}{p_1p_4-p_2p_3} & 0 & p_2c_3 \\ \frac{p_3p_4}{p_1p_4-p_2p_3} & \frac{-p_2p_3}{p_1p_4-p_2p_3} & 0 & p_4c_3 \\ 0 & 0 & 0 & c_6 \\ \frac{-d_3p_3}{p_1p_4-p_2p_3} & \frac{d_3p_1}{p_1p_4-p_2p_3} & d_6 & w \end{bmatrix}$$

for all p_1, p_2, p_3, p_4, w, c_3, d_3, c_6, $d_6 \in \mathbb{C}$, $p_1p_4 - p_2p_3 \neq 0$, *and* $d_3c_3 = d_6c_6$.

Case VII: $r = 0$, $v = 0$, $\tau = 1$.

$$Y = \begin{bmatrix} 0 & 0 & 0 & c_{31} \\ 0 & 0 & 0 & c_{32} \\ 0 & 0 & -1 & 0 \\ d_{31} & d_{32} & 0 & w \end{bmatrix}$$

for all c_{31}, c_{32}, d_{31}, d_{32}, $w \in \mathbb{C}$, *and* $d_{31}c_{31} = -d_{32}c_{32}$.

Case VIII: $r = 0$, $v = 0$, $\tau = 0$.

$$Y = \begin{bmatrix} 0 & 0 & 0 & c_{31} \\ 0 & 0 & 0 & c_{32} \\ 0 & 0 & 0 & c_6 \\ d_{31} & d_{32} & d_6 & w \end{bmatrix}$$

for all c_{31}, c_{32}, d_{31}, d_{32}, c_6, d_6, $w \in \mathbb{C}$, *and* $d_{31}c_{31} + d_{32}c_{32} = d_6c_6$.

We find the explicit expressions of the solution X. Our results are more general than those obtained recently by M. Saeed Ibrahim Adam et al. [21].

Example 6.2. *Let* $A = J = diag(1 + i\sqrt{3}, 1 + i\sqrt{3}, 2, 0)$. *So the nonzero eigenvalues satisfy* $(1 + i\sqrt{3})^2 - (1 + i\sqrt{3}) \times 2 + 2^2 = 0$. *Then by Theorem 3.1, there are totally six cases to be considered in finding the general solution X.*

Case I: $t_1 = 2$, $t_2 = 1$.

$$X = \begin{bmatrix} 1+i\sqrt{3} & 0 & f_1 & 0 \\ 0 & 1+i\sqrt{3} & f_2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & w \end{bmatrix}, \begin{bmatrix} 1+i\sqrt{3} & 0 & 0 & 0 \\ 0 & 1+i\sqrt{3} & 0 & 0 \\ g_1 & g_2 & 2 & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

for all f_1, f_2, g_1, g_2, $w \in \mathbb{C}$.

Case II: $t_1 = 2, t_2 = 0.$

$$X = \begin{bmatrix} 1+i\sqrt{3} & 0 & 0 & 0 \\ 0 & 1+i\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & w \end{bmatrix}, \begin{bmatrix} 1+i\sqrt{3} & 0 & 0 & 0 \\ 0 & 1+i\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_4 & w \end{bmatrix}$$

for all $c_4,\ d_4,\ w \in \mathbb{C}$.

Case III: $t_1 = 1, t_2 = 1.$

$$X = \begin{bmatrix} \frac{(1+i\sqrt{3})p_1p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_1p_2}{p_1p_4-p_2p_3} & p_1f & p_2c_2 \\ \frac{(1+i\sqrt{3})p_3p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_2p_3}{p_1p_4-p_2p_3} & p_3f & p_4c_2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & w \end{bmatrix}, \begin{bmatrix} \frac{(1+i\sqrt{3})p_1p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_1p_2}{p_1p_4-p_2p_3} & p_1f & 0 \\ \frac{(1+i\sqrt{3})p_3p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_2p_3}{p_1p_4-p_2p_3} & p_3f & 0 \\ 0 & 0 & 2 & 0 \\ \frac{-d_2p_3}{p_1p_4-p_2p_3} & \frac{d_2p_1}{p_1p_4-p_2p_3} & 0 & w \end{bmatrix},$$

$$\begin{bmatrix} \frac{(1+i\sqrt{3})p_1p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_1p_2}{p_1p_4-p_2p_3} & 0 & p_2c_2 \\ \frac{(1+i\sqrt{3})p_3p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_2p_3}{p_1p_4-p_2p_3} & 0 & p_4c_2 \\ \frac{gp_4}{p_1p_4-p_2p_3} & \frac{-gp_2}{p_1p_4-p_2p_3} & 2 & 0 \\ 0 & 0 & 0 & w \end{bmatrix}, \begin{bmatrix} \frac{(1+i\sqrt{3})p_1p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_1p_2}{p_1p_4-p_2p_3} & 0 & 0 \\ \frac{(1+i\sqrt{3})p_3p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_2p_3}{p_1p_4-p_2p_3} & 0 & 0 \\ \frac{gp_4}{p_1p_4-p_2p_3} & \frac{-gp_2}{p_1p_4-p_2p_3} & 2 & 0 \\ \frac{-d_2p_3}{p_1p_4-p_2p_3} & \frac{d_2p_1}{p_1p_4-p_2p_3} & 0 & w \end{bmatrix}$$

for all $p_1,\ p_2,\ p_3,\ p_4,\ w,\ c_2,\ d_2,\ f,\ g \in \mathbb{C}$, *and* $p_1p_4 - p_2p_3 \neq 0$.

Case IV: $t_1 = 1, t_2 = 0.$

$$X = \begin{bmatrix} \frac{(1+i\sqrt{3})p_1p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_1p_2}{p_1p_4-p_2p_3} & 0 & p_2c_2 \\ \frac{(1+i\sqrt{3})p_3p_4}{p_1p_4-p_2p_3} & \frac{-(1+i\sqrt{3})p_2p_3}{p_1p_4-p_2p_3} & 0 & p_4c_2 \\ 0 & 0 & 0 & c_4 \\ \frac{-d_2p_3}{p_1p_4-p_2p_3} & \frac{d_2p_1}{p_1p_4-p_2p_3} & d_4 & w \end{bmatrix}$$

for all $p_1,\ p_2,\ p_3,\ p_4,\ w,\ c_2,\ d_2,\ c_4,\ d_4 \in \mathbb{C}$, *and* $p_1p_4 - p_2p_3 \neq 0$, *and* $(1 + i\sqrt{3})d_2c_2 = -2d_4c_4$.

Case V: $t_1 = 0, t_2 = 1.$

$$X = \begin{bmatrix} 0 & 0 & 0 & c_{21} \\ 0 & 0 & 0 & c_{22} \\ 0 & 0 & 2 & 0 \\ d_{21} & d_{22} & 0 & w \end{bmatrix}$$

for all $c_{21},\ c_{22},\ d_{21},\ d_{22},\ w \in \mathbb{C}$, *and* $d_{21}c_{21} = -d_{22}c_{22}$.

Case VI: $t_1 = 0, t_2 = 0.$

$$X = \begin{bmatrix} 0 & 0 & 0 & c_{21} \\ 0 & 0 & 0 & c_{22} \\ 0 & 0 & 0 & c_4 \\ d_{21} & d_{22} & d_4 & w \end{bmatrix}$$

for all $c_{21},\ c_{22},\ d_{21},\ d_{22},\ c_4, d_4,\ w \in \mathbb{C}$, *and* $(1 + i\sqrt{3})(d_{21}c_{21} + d_{22}c_{22}) = -2d_4c_4$.

7. Conclusions

When the given matrix A is diagonalizable matrix with three distinct eigenvalues 0, λ, and μ, we have derived all explicit expression for the solutions X of the Yang-Baxter-like matrix equation (1) under the conditions that $\lambda^2 - \lambda\mu + \mu^2 = 0$ and $\lambda^2 - \lambda\mu + \mu^2 \neq 0$, respectively. Our approach here is to use the Jordan decomposition of A to obtain a simplified Yang-Baxter-like matrix equation with A replaced by a simple block diagonal matrix, and then we solve a system of several matrix equations for the smaller sized solution blocks. The idea behind the technique can be generalized to consider more general cases. We demonstrate the application to the case that the minimal polynomial of A is $g(x) = x^3 - x$. Our results have extended the previous results of [17, 21]. Finding all the solutions of the Yang-Baxter-like matrix equation (1) for a general matrix A is a hard task, which is continuing research work in the future.

References

[1] B. M. McCoy, Advanced Statistical Mechanics, Oxford University Press, 2009. Published to Oxford Scholarship Online: February 2010.

[2] R. J. Baxter, Partition function of the eight-vertex lattice model, Annals of Physics 70 (1972) 193–228.

[3] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Physical Review Letters 19 (1967) 1312–1315.

[4] C. N. Yang, M. Ge, Braid Group, Knot Theory, and Statistical Mechanics, World Scientific, 1989.

[5] D. Shen, M. Wei, Z. Jia, On commuting solutions of the Yang-Baxter-like matrix equation, Journal of Mathematical Analysis and Applications 462 (2018) 665–696.

[6] J. Ding, N. H. Rhee, Spectral solutions of the Yang-Baxter matrix equation, Journal of Mathematical Analysis and Applications 402 (2013) 567–573.

[7] J. Ding, N. H. Rhee, Computing solutions of the Yang-Baxter-like matrix equation for diagonalisable matrices, East Asian Journal on Applied Mathematics 5 (2015) 75–84.

[8] J. Ding, C. Zhang, On the structure of the spectral solutions of the Yang-Baxter matrix equation, Applied Mathematics Letters 35 (2014) 86–89.

[9] J. Ding, C. Zhang, N. H. Rhee, Further solutions of a Yang-Baxter-like matrix equation, East Asian Journal on Applied Mathematics 3 (2013) 352–362.

[10] J. Ding, C. Zhang, N. H. Rhee, Commuting solutions of the Yang-Baxter matrix equation, Applied Mathematics Letters 44 (2015) 1–4.

[11] Q. Dong, J. Ding, Complete commuting solutions of the Yang-Baxter-like matrix equation for diagonalizable matrices, Computers & Mathematics with Applications 72 (2016) 194–201.

[12] Q. Dong, J. Ding, Q. Huang, Commuting solutions of a quadratic matrix equation for nilpotent matrices, Algebra Colloquium 25 (1) (2018) 31–44.

[13] H. Ren, X. Wang, and T. Wang, Commuting solutions of the Yang-Baxter-like matrix equation for a class of rank-two updated matrices, Computers & Mathematics with Applications 76 (2018) 1085–1098.

[14] H. Yin, X. Wang, X. Tang, L. Chen, On the commuting solutions to the Yang-Baxter-like matrix equation for identity matrix minus special rank-two matrices, Filomat 32 (13) 4591–4609.

[15] D. Zhou, G. Chen, G. Yu, J. Zhong, On the projection-based commuting solutions of the Yang-Baxter matrix equation, Applied Mathematics Letters, 79 (2018) 155–161.

[16] D. Zhou, J. Ding, Solving the Yang-Baxter-like matrix equation for nilpotent matrices of index three, International Journal of Computer Mathematics 295 (2) (2018) 303–315.

[17] D. Chen, Z. Chen, X. Yong, Explicit solutions of the Yang-Baxter-like matrix equation for a diagonalizable matrix with spectrum contained in $\{1, \alpha, 0\}$. Applied Mathematics and Computation 348 (2019) 523–530.

[18] Q. Huang, M. Saeed Ibrahim Adam, J. Ding, L. Zhu, All non-commuting solutions of the Yang-Baxter matrix equation for a class of diagonalizable matrices, Operators & Matrices 13 (1) (2019) 187–195.

[19] M. Saeed Ibrahim Adam, J. Ding, Q. Huang, Explicit solutions of the Yang-Baxter-like matrix equation for an idempotent matrix, Applied Mathematics Letters 63 (2017) 71–76.

[20] M. Saeed Ibrahim Adam, J.Ding, Q. Huang, L. Zhu, Solving a class of quadratic matrix equations. Applied Mathematics Letters 82 (2018) 58–63.

[21] M. Saeed Ibrahim Adam, J. Ding, Q. Huang, L. Zhu, All Solutions of the Yang-Baxter-like Matrix Equation when $A^3 = A$, Journal of Applied Analysis and Computation 9(3) (2019) 1022–1031.

[22] D. Shen, M. Wei, All solutions of the Yang-Baxter-like matrix equation for diagonalizable cofficient matrix with two different eigenvalues, Applied Mathematics Letters 101 (2020) 106048.

[23] H. Tian, All solutions of the Yang-Baxter-like matrix equation for rank-one matrices, Applied Mathematics Letters 51 (2016) 55–59.

[24] D. Zhou, G. Chen, J. Ding, Solving the Yang-Baxter-like matrix equation for rank-two matrices, Journal of Computational and Applied Mathematics 313 (2017) 142–151.

[25] D. Zhou, G. Chen, J. Ding, On the Yang-Baxter-like matrix equation for rank-two matrices, Open Mathematics 15 (2017) 340–353.

[26] D. Zhou, G. Chen, J. Ding, H. Tian, Solving the Yang-Baxter-like matrix equation with non-diagonalizable elementary matrices, Communications in Mathematical Sciences 17 (2) (2019) 393–411.

[27] D. Zhou, H. Vu, Some non-commuting solutions of the Yang-Baxter-like matrix equation, Open Mathematics 18 (2020) 948–969.

[28] D. Zhou, J. Ding, All Solutions of the Yang-Baxter-Like Matrix Equation for Nilpotent Matrices of Index Two, Complexity 2020 (2020) 2585602, 7 pp.

[29] B. D. Djordjević and N. Č. Dinčić, Classification and aproximation of solutions to Sylvester matrix equation, Filomat 33 (13) (2019) 4261–4280.