Filomat 35:12 (2021), 3983–4004
https://doi.org/10.2298/FIL2112983W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cauchy Completion of Fuzzy Quasi-Uniform Spaces

Yongchao Wang[a], Yueli Yue[a]

[a]*School of Mathematical Sciences, Ocean University of China, Qingdao 266100, China*

Abstract. In this paper, we study the completion of fuzzy quasi-uniform spaces from a categorical point of view. Firstly, we introduce the concept of prorelations and describe fuzzy quasi-uniform spaces as enriched categories. Then we construct the Yoneda embedding in fuzzy quasi-uniform spaces through promodules, and prove the validness of Yoneda Lemma for right adjoint promodules. Finally, we study the Cauchy completion of fuzzy quasi-uniform spaces by the Yoneda embedding. We show that the inclusion functor from the category of T_0 separated complete fuzzy quasi-uniform spaces to the category of fuzzy quasi-uniform spaces has a left adjoint functor. The monad related to this adjunction is just the T_0 completion monad of fuzzy quasi-uniform spaces.

1. Introduction

Since Lawvere presented generalized metric spaces as enriched categories in [29], enriched categories have been proved to be a powerful tool for studying topological structures. For example, Zhang studied many valued topologies through the approach of category in [45]. Hofmann and Reis treated probabilistic (quasi-)metric spaces as enriched categories and studied these structures by enriched category in [16]. Chai also gave a research on probabilistic quasi-metric spaces from the enriched categorical point of view in [4]. Similar to the study of probabilistic quasi-metric spaces, He, Lai and Shen considered the categorical interpretation of fuzzy partial metric spaces in [14]. Following Lawvere and Bar's idea, Clementino, Hofmann and Tholen developed the theory of monoidal topology, and showed that many topological structures such as approach spaces, metric spaces, (quasi-)unform spaces and so on all can be viewed as lax algebras with respect to certain monads (see [5–8, 15, 17, 23]).

Uniformity plays an important role in the research and application of topology. The study of both classical (quasi-)uniform spaces and lattice-valued (quasi-)uniform spaces draws much attention in the research of topological structures (see [3, 9, 11, 12, 18–21, 25, 27, 28, 30, 32–35, 37, 38, 41–44, 46]). Due to the close relation between uniformities and metrics, this promotes the study of quasi-uniform structures by means of enriched categories. The first description of quasi-uniform spaces as enriched categories is attributed to Schmitt [36]. Then Clementino, Hofmann and Tholen used the theory of lax algebras and put the quasi-uniform spaces in the framework of monoidal topology in [7]. Furthermore Clementino

2020 *Mathematics Subject Classification*. Primary 54B30; Secondary 54A40,18D20
Keywords. Cauchy completion, promodule, Yoneda embedding, prorelation, fuzzy quasi-uniform space
Received: 08 September 2020; Revised: 09 June 2021; Accepted: 16 June 2021
Communicated by Ljubiša D.R. Kočinac
Corresponding author: Yueli Yue
Research supported by National Natural Science Foundation of China (No.11971448) and the Fundamental Research Funds for the Central Universities (No.3016000841964007)
Email addresses: ycwang566@163.com (Yongchao Wang), ylyue@ouc.edu.cn (Yueli Yue)

and Hofmann described quasi-uniform spaces as enriched categories, introduced Yoneda embedding in quasi-uniform spaces and studied the completion monad in [8].

There are many kinds of lattice-valued quasi-uniformities, such as Lowen and Höhle's quasi-uniformity [18, 32], Hutton's quasi-uniformity [21] and Shi's pointwise quasi-uniformity [37]. A natural question would be whether the lattice-valued quasi-uniform spaces could be viewed as enriched categories. In this paper, this question positively for fuzzy quasi-uniform space in the sense of Lowen and Höhle, and other kinds of fuzzy quasi-uniform spaces are left for future study.

Following the idea of Clementino and Hofmann in [8], we describe fuzzy quasi-uniform spaces as enriched categories by means of the concept of prorelation. Then we construct the Yoneda embedding in fuzzy quasi-uniform spaces from enriched category theory. And Yoneda Lemma is shown right on the condition that the closure operator, which is generated by its fuzzy quasi-uniformity.

As an application of Yoneda embedding, we focus on Cauchy completeness and completion of fuzzy quasi-uniform spaces. In [42], Yue and Fang have studied a kind of completeness of fuzzy quasi-uniform spaces based on pair ⊤-filters. We will continue the research of this kind of completeness in this paper, and show that it can be also easily generalized to saturated prefilter setting. Using right adjoint promodules, we can establish a pair of adjoint functors between the category of fuzzy quasi-uniform spaces with uniformly continuous maps and the category of fuzzy quasi-uniform spaces with right adjoint promodules. The monad related to this adjunction is just the T_0 completion monad of fuzzy quasi-uniform spaces. We also give a direct proof of the result which the category of T_0 separated complete fuzzy quasi-uniform spaces is a reflective full subcategory of the category of fuzzy quasi-uniform spaces.

2. Preliminaries

A commutative quantale is a pair $(\mathsf{Q}, \&)$, where Q is a complete lattice with the top element $\top (= \wedge\emptyset)$ and the bottom element $\bot (= \vee\emptyset)$, and $\&$ is a commutative semigroup operation on Q such that

$$\alpha \& \Big(\bigvee_{j\in J} \beta_j \Big) = \bigvee_{j\in J} \alpha \& \beta_j,$$

for all $\alpha \in \mathsf{Q}$ and $\{\beta_j \mid j \in J\} \subseteq \mathsf{Q}$. For a given commutative quantale $(\mathsf{Q}, \&)$, there exists a binary operation $\to: \mathsf{Q} \times \mathsf{Q} \to \mathsf{Q}$ defined by

$$\alpha \to \beta = \bigvee \{\gamma \in \mathsf{Q} \mid \alpha \& \gamma \leq \beta\},$$

called the implication (operation).

Let $f : X \to Y$ and $g : Y \to X$ be a pair of maps between ordered sets. We say that f is left adjoint to g (or g is right adjoint to f) and write $f \dashv g$ if

$$f(x) \leq y \Leftrightarrow x \leq g(y)$$

for all $x \in X$ and $y \in Y$. The pair (f, g) is said to be an adjunction. For example, if $f : X \to Y$ is an order isomorphism, then it is both left and right adjoint to its inverse f^{-1}.

A commutative quantale $(\mathsf{Q}, \&)$ is said to be unital if there exists an element $k \in \mathsf{Q}$ such that $k \& \alpha = \alpha \& k = \alpha$ for all $\alpha \in \mathsf{Q}$ (k is usually called the unit of Q). When the unit of Q is the top element $\top$, $(\mathsf{Q}, \&)$ is called integral.

A complete lattice Q is said to be meet continuous if for all $\alpha \in \mathsf{Q}$, $\alpha \wedge (\bigvee_{\beta\in\Gamma} \beta) = \bigvee_{\beta\in\Gamma} (\alpha \wedge \beta)$ for all directed subset $\Gamma \subseteq \mathsf{Q}$. In this paper, we always assume that $(\mathsf{Q}, \&)$ is an integral and commutative quantale, and Q is meet continuous.

An Q-subset on a set X is a map from X to Q, and the family of all Q-subsets on X will be denoted by Q^X, called the Q-power set of X. By $\bot_X$ and $\top_X$, we denote the constant Q-subsets on X taking the value $\bot$ and $\top$, respectively. Function $\alpha : X \to \mathsf{Q}$ where $\alpha(x) = \alpha$ for all $x \in X$. i,e., We don't distinguish between constant Q-set α and its value. For $U \subseteq X$, χ_U denotes the characteristic function of U, i.e., $\chi_U(x) = \top$ when $x \in U$ and $\chi_U(x) = \bot$ when $x \notin U$. χ_U is usually written by $\top_U$.

All algebraic operations on Q can be extended to Q^X pointwise. For example, $(A \vee B)(x) = A(x) \vee B(x)$, $(A \wedge B)(x) = A(x) \wedge B(x)$ for $A, B \in Q^X$ and $x \in X$. For a map $f : X \to Y$, we can define $f^{\to} : Q^X \to Q^Y$ and $f^{\leftarrow} : Q^Y \to Q^X$ by $f^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$ and $f^{\leftarrow}(B)(x) = B(f(x))$, respectively. Then $f^{\to}$ is the left adjoint of $f^{\leftarrow}$. From the definition of $f^{\to}$, we know $f^{\to}(\top_U) = \top_{f(U)}$.

A Q-relation $r : X \nrightarrow Y$ from X to Y is map $r : X \times Y \to Q$. The composition $s \circ r : X \nrightarrow Z$ of Q-relations $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ is defined by

$$s \circ r(x, z) = \bigvee_{y \in Y} r(x, y) \& s(y, z).$$

The identity on X for this composition is the Q-relation $1_X : X \nrightarrow X$ which sends (x, y) to $\top$ when $x = y$ and to $\bot$ otherwise. The category of sets and Q-relations is denoted by Q-**Rel** and the set of all Q-relations from X to Y is denoted by Q-**Rel**(X, Y). The theory of category can be found in [1]. For Q-relation $r : X \nrightarrow Y$, there is an opposite Q-relation $r^{\circ} : Y \nrightarrow X$ given by $r^{\circ}(y, x) = r(x, y)$ for all $x \in X$ and $y \in Y$. In fact, a map $f : X \to Y$ can be seen as a Q-relation $f : X \nrightarrow Y$:

$$f(x, y) = \begin{cases} \top, & y = f(x), \\ \bot, & \text{others.} \end{cases}$$

and its dual Q-relation $f^{\circ} : Y \nrightarrow X$ induced by $f : X \to Y$ is as follows:

$$f^{\circ}(y, x) = \begin{cases} \top, & y = f(x), \\ \bot, & \text{others.} \end{cases}$$

Now we recall some basic concepts about Q-ordered sets. The theory of Q-ordered sets can be found in many places, for instance [17, 39, 45].

A Q-order on X is a Q-relation $r : X \nrightarrow X$ such that (1) $\top \leq r(x, x)$ for all $x \in X$ and (2) $r(y, z) \& r(x, y) \leq r(x, z)$ for all $x, y, z \in X$. A set X equipped with a Q-order relation is called a Q-ordered set. Usually we simply "X is a Q-ordered set (X, r)" and write $X(x, y)$ for $r(x, y)$. When it is necessary to specify the Q-order we write (X, r). We say a map $f : X \to Y$ preserves Q-order (or Q-order preserving) if $X(x, y) \leq Y(f(x), f(y))$ for all $x, y \in X$. It is trivial to see that $(Q, \to)$ is a Q-ordered set.

Given a Q-ordered set X, define $x \leq y \Leftrightarrow X(x, y) = \top$. Then $\leq$ is a reflexive and transitive relation, hence a preorder on X. This preorder is called the underlying order of X. X is antisymmetric if $X(x, y) \& X(y, x) = \top \Rightarrow x = y$. A Q-ordered set X is said to be separated if the underlying order on X is antisymmetric.

For any Q-ordered set X and Y, let $[X, Y]$ denote the set of all Q-order-preserving maps from X to Y. For all $f, g \in [X, Y]$, let

$$[X, Y](f, g) = \bigwedge_{x \in X} Y(f(x), g(x)).$$

Then $[X, Y]$ becomes a Q-ordered set. Specially, for Q-subsets $A, B : X \to Q$, let $S_X(A, B) = \bigwedge_{x \in X} A(x) \to B(x)$. Then (Q^X, S_X) is a separated Q-ordered set. $S_X(A, B)$ can be interpreted as the degree to which A is a subset of B. It is sometimes called the fuzzy inclusion order in [2]. For convenience, S_X is simplified by S in this paper. For Q-relations, in this paper, we often use the following result: $S(r_1, r_2) \leq S(s \circ r_1, s \circ r_2)$ and $S(r_1, r_2) \leq S(r_1 \circ t, r_2 \circ t)$ for $r_1, r_2 : X \nrightarrow Y$, $s : Y \nrightarrow Z$ and $t : W \nrightarrow X$.

Besides Q-order preserving maps, there is another important morphisms between Q-ordered sets, namely Q-distributors. A Q-distributor $\phi : X \mapsto Y$ is a Q-relation $\phi : X \nrightarrow Y$ such that $\phi \circ X \leq \phi$ and $Y \circ \phi \leq \phi$. Each Q-order preserving map $f : X \to Y$ can give rise a Q-distributor $f_* : X \mapsto Y$ defined by $f_*(x, y) = Y(f(x), y)$ for all $x \in X$ and $y \in Y$. f_* has a right adjoint $f^* : Y \mapsto X$ defined by $f_*(y, x) = Y(y, f(x))$ for $x \in X$ and $y \in Y$, here the adjunction $f_* \dashv f^*$ means $f_* \circ f^* \leq Y$ and $f^* \circ f_* \geq X$. An important connection between Q-distributor and Q-order preserving maps is given by the fact that $\phi : X \nrightarrow Y$ is a Q-distributor precisely when $\phi : X \times Y \to Q$ is a Q-order preserving map between $X^{op} \otimes Y \to Q$.

Given a Q-ordered set X, the Yoneda embedding is the map $\mathbf{y}_X : X \to [X^{op}, Q]$ given by $\mathbf{y}_X(x)(y) = X(y, x)$. The Yoneda Lemma is as follows: $[X^{op}, Q](\mathbf{y}_X(x), \phi) = \phi(x)$ for all $x \in X$ and $\phi \in [X^{op}, Q]$. From [16], we know that another way to read the Yoneda Lemma goes as it follows: for any module $\phi : X \nrightarrow 1$, seen also as an element of $[X^{op}, \mathsf{Q}]$, one has $\phi^* \circ (\mathbf{y}_X)_* = \phi$. In this paper, for fuzzy quasi-uniform spaces, we will establish the Yoneda Lemma in the later form.

As filter plays a pivotal role in defining the notion of classical quasi-uniformity, in lattice-valued setting prefilter, $\top$-filter and Q-filter are the three most important lattice-valued filters and the relationships among them can be found in [9, 10, 22, 24]. Now we briefly describe the concepts of saturated prefilter and fuzzy quasi-uniformity.

Definition 2.1. (Lowen [31, 32]) Let X be a nonempty set. A nonempty subset $\mathcal{F} \subseteq \mathsf{Q}^X$ is called a prefilter on X if it satisfies the following properties:
(F1) $\top_X \in \mathcal{F}$;
(F2) If $A \in \mathcal{F}$ and $A \leq B$, then $B \in \mathcal{F}$;
(F3) $A \wedge B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$.

Definition 2.2. (Höhle [18]) Let $\mathcal{F}$ be a prefilter on X. Then
(1) $\mathcal{F}$ is called saturated if it satisfies the following (S):
(S) If $B \in \mathsf{Q}^X$ such that $\bigvee_{A\in\mathcal{F}} S(A, B) = \top$, then $B \in \mathcal{F}$.
(2) $\mathcal{F}$ is called a $\top$-filter if $\mathcal{F}$ is a saturated prefilter and fulfills $\bigvee_{x\in X} A(x) = \top$ for all $A \in \mathcal{F}$.

For each $x \in X$, $[x]_\top = \{A \in \mathsf{Q}^X \mid A(x) = \top\}$ is both a saturated prefilter and a $\top$-filter.

Remark 2.3. In fact, the saturation of a prefilter originally introduced by Lowen [33]. Höhle [18] introduced so called κ-condition [18]. Later, it is J. Gutiérrez García [9, 13] who proved that the κ-condition and saturation of prefilters are equavalent. So in the above definition of saturated prefilter, we use the κ-condition introduced by Höhle directly.

Definition 2.4. (Höhle [18]) A nonempty subset $\mathcal{B} \subseteq \mathsf{Q}^X$ is called a base of one saturated prefilter on X if it satisfies the following condition:
(B) $\bigvee_{B\in\mathcal{B}} S(B, C \wedge D) = \top$ for all $C, D \in \mathcal{B}$.

Every base $\mathcal{B}$ can generate a saturated prefilter $\mathcal{F}_\mathcal{B}$ given by

$$\mathcal{F}_\mathcal{B} = \{A \in \mathsf{Q}^X \mid \bigvee_{B\in\mathcal{B}} S(B, A) = \top\}.$$

From the definition of base of saturated prefilter, we know that if there exists $C \in \mathcal{B}$ such that $C \leq A \wedge B$ for all $A, B \in \mathcal{B}$, then $\mathcal{B}$ must be a base of one saturated prefilter. Especially, if $\mathcal{B}$ is closed for finite meet, then $\mathcal{B}$ is a base.

Definition 2.5. (Lowen [32, 34]for $\mathsf{Q} = [0, 1]$ and Höhle [18]) A nonempty subset $\mathcal{U} \subseteq \mathsf{Q}^{X\times X}$ is called a fuzzy quasi-uniformity on X if $\mathcal{U}$ is a saturated prefilter on $X \times X$ and satisfies the following conditions:
(U0) $U \in \mathcal{U}$ implies $U(x, x) = \top$ for all $x \in X$;
(UC) $U \in \mathcal{U}$ implies $\bigvee_{V\in\mathcal{U}} S(V \circ V, U) = \top$.
The pair $(X, \mathcal{U})$ is called a fuzzy quasi-uniform space.

A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is called uniformly continuous if $(f \times f)^\leftarrow(V) \in \mathcal{U}$ for all $V \in \mathcal{V}$, where $(f \times f)^\leftarrow(V)(x_1, x_2) = V(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.Note that $f^\circ \circ V \circ f(x_1, x_2) = V(f(x_1), f(x_2))$, so the equality $(f \times f)^\leftarrow(V) = f^\circ \circ V \circ f$ is hold. Let Q-**FQunif** denote the category of fuzzy quasi-uniform spaces and uniformly continuous maps.

In [8], Clementino and Hofmann introduced the concept of "prorelation" based on classical filter, and they viewed the quasi-uniform spaces as lax proalgebras. In [42], Yue and Fang generalized the concept of prorelation to $\top$-filter setting. By adopting their ideas, we can give the corresponding concepts according to saturated prefilter similarly.

Definition 2.6. Let $\Phi \subseteq$Q-**Rel**(X, Y). If Φ is a saturated prefilter on $X \times Y$, then Φ is called a prorelation from X to Y, denoted by $\Phi : X \multimap\!\!\!\rightarrow Y$.

Remark 2.7. (1) Any Q-relation $r : X \nrightarrow Y$ can be seen as a saturated prefilter by its upper set $\uparrow r = \{s : X \nrightarrow Y \mid s \geq r\}$. We usually simply $\uparrow r$ by $r : X \multimap\!\!\!\rightarrow Y$.

(2) Let Ψ and Φ be two saturated prefilters. Define the composition $\Phi \circ \Psi$ as follows:

$$\Phi \circ \Psi := \{W \mid \bigvee_{\phi \in \Phi} \bigvee_{\psi \in \Psi} S(\phi \circ \psi, W) = \top\}.$$

i.e., $\{\phi \circ \psi \mid \phi \in \Phi, \psi \in \Psi\}$ is the base of $\Phi \circ \Psi$. If $\mathcal{B}_1$ is a base of Φ and $\mathcal{B}_2$ is a base of Ψ, then we know

$$\Phi \circ \Psi = \{W \mid \bigvee_{\phi \in \mathcal{B}_1} \bigvee_{\psi \in \mathcal{B}_2} S(\phi \circ \psi, W) = \top\}.$$

It is routine to check that $\mathcal{W} \circ (\Phi \circ \Psi) = (\mathcal{W} \circ \Phi) \circ \Psi$ for $\Psi : X \multimap\!\!\!\rightarrow Y$, $\Phi : Y \multimap\!\!\!\rightarrow Z$ and $\mathcal{W} : Z \multimap\!\!\!\rightarrow W$. For the identity Q-relation $1_X : X \nrightarrow X$, $1_X : X \multimap\!\!\!\rightarrow X$ is the identity of the composition of prorelations. Hence sets and prorelations form a category, denote it by PQ-**Rel**.

Let PQ-**Rel**(X, Y) denote all the prorelations from X to Y. The Q-order Υ on PQ-**Rel**(X, Y) by letting:

$$\forall \Phi, \Psi \in P\text{Q-}\mathbf{Rel}(X, Y), \;\; \Upsilon(\Phi, \Psi) = \bigwedge_{\psi \in \Psi} \bigvee_{\phi \in \Phi} S(\phi, \psi).$$

It is obvious that the underlying order of Υ is as follows:

$$\forall \Phi, \Psi \in P\text{Q-}\mathbf{Rel}(X, Y), \;\; \Phi \leq \Psi \iff \Psi \subseteq \Phi.$$

Hence, if $\mathcal{B}_1$ is a base of Φ and $\mathcal{B}_2$ is a base of Ψ, then $\Phi \leq \Psi$ when $\mathcal{B}_2 \subseteq \mathcal{B}_1$. Note that the above definition make sense whenever Φ and Ψ are sets of Q-relations.

According to the above discussion, a fuzzy quasi-uniformity $\mathcal{U}$ on X can be seen as a prorelation $\mathcal{U} : X \multimap\!\!\!\rightarrow X$ satisfies the following condition:

$$1_X \leq \mathcal{U}, \;\; \mathcal{U} \circ \mathcal{U} \leq \mathcal{U}.$$

Similarly, a uniformly continuous map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ can be seen as a map $f : X \to Y$ such that

$$\mathcal{U} \leq f^{\circ} \circ \mathcal{V} \circ f \quad \text{or equivalently} \quad f \circ \mathcal{U} \leq \mathcal{V} \circ f.$$

Now we have described fuzzy quasi-uniform space into the form of enriched category. Hence, we can use the categorical method to study fuzzy quasi-uniform spaces.

Definition 2.8. A prorelation $\Phi : (X, \mathcal{U}) \multimap\!\!\!\rightarrow (Y, \mathcal{V})$ is said to be a promodule if it satisfies

$$\Phi \circ \mathcal{U} \leq \Phi, \;\; \mathcal{V} \circ \Phi \leq \Phi.$$

For each fuzzy quasi-uniform space $(X, \mathcal{U})$, $\mathcal{U} : (X, \mathcal{U}) \multimap\!\!\!\rightarrow (X, \mathcal{U})$ itself is a promodule. For promudule $\Phi : (X, \mathcal{U}) \multimap\!\!\!\rightarrow (Y, \mathcal{V})$, since $\Phi \leq \Phi \circ \mathcal{U}$ is always true, then $\Phi \circ \mathcal{U} = \Phi$ holds. Similarly, $\mathcal{V} \circ \Phi = \Phi$. It is easy to see that the composition of promodules is still a promodule and $\mathcal{U}$ acts as the identity of the composition. Let PQ-**Mod** denote the category of fuzzy quasi-uniform spaces and promodules.

Definition 2.9. For two promodules $\Phi : (X, \mathcal{U}) \multimap\!\!\!\rightarrow (Y, \mathcal{V})$, $\Psi : (Y, \mathcal{V}) \multimap\!\!\!\rightarrow (X, \mathcal{U})$, if $\Psi \circ \Phi \geq \mathcal{U}$ and $\Phi \circ \Psi \leq \mathcal{V}$ hold, then Φ is called the left adjoint of Ψ or Ψ is called the right adjoint of Φ, denoted by $\Phi \dashv \Psi$.

Each given uniformly continuous map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ can determine a pair of promodules $f_* : (X, \mathcal{U}) \multimap (Y, \mathcal{V})$ and $f^* : (Y, \mathcal{V}) \multimap (X, \mathcal{U})$ as follows:

$$X \overset{f}{\multimap} Y \overset{\mathcal{V}}{\multimap} Y \quad f_* = \mathcal{V} \circ f$$

$$Y \overset{\mathcal{V}}{\multimap} Y \overset{f^\circ}{\multimap} X \quad f^* = f^\circ \circ \mathcal{V}$$

Remark 2.10. In fact, the above $\mathcal{V} \circ f$ should be $\mathcal{V} \circ (\uparrow f)$. The readers can easily check that $\{V \circ f \mid V \in \mathcal{V}\}$ and $\{V \circ g \mid V \in \mathcal{V},\ g \geq f\}$ generate the same prorelation. Hence we use the form $\mathcal{V} \circ f$. Similarly, we use $f^\circ \circ \mathcal{V}$ instead of $(\uparrow f^\circ) \circ \mathcal{V}$.

For $1_X : (X, \mathcal{U}) \to (X, \mathcal{U})$, clearly, $(1_X)_* = (1_X)^* = \mathcal{U}$. Given two uniformly continuous maps $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$, $g : (Y, \mathcal{V}) \to (Z, \mathcal{W})$, it has $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^* = f^* \circ g^*$. These operations define two functors:

$$(-)_* : \mathsf{Q}\text{-}\mathbf{FQunif} \to P\mathsf{Q}\text{-}\mathbf{Mod}$$

and

$$(-)^* : \mathsf{Q}\text{-}\mathbf{FQunif} \to P\mathsf{Q}\text{-}\mathbf{Mod}^{op}.$$

Proposition 2.11. *Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly continuous map. Then $f_* \dashv f^*$.*

Proof. It is straightforward to check $\mathcal{U} \leq f^* \circ f_*$ and $f_* \circ f^* \leq \mathcal{V}$. □

In particular, let $(X, \mathcal{U})$ be a fuzzy quasi-uniform space and $\mathcal{P} : * \twoheadrightarrow *$ be the unique fuzzy quasi-uniform structure on the singleton $\{*\}$ (in fact, $\mathcal{P} = \{W \in \mathsf{Q}^{* \times *} \mid W(*, *) = \top\}$), denoted by $1 = (\{*\}, \mathcal{P})$. The uniformly continuous map $x : 1 \to X (* \mapsto x, x \in X)$, defines two adjoint promodules $x_* \dashv x^* : X \multimap 1$, where $x_* = \mathcal{U} \circ x$ and $x^* = x^\circ \circ \mathcal{U}$.

Definition 2.12. ([8]) Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly continuous map. Then:
(1) f is said to be fully faithful if $f^* \circ f_* = \mathcal{U}$;
(2) f is said to be fully dense if $f_* \circ f^* = \mathcal{V}$.

Proposition 2.13. *Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly continuous map. Then:*
(1) f is fully faithful if and only if $f^\circ \circ \mathcal{V} \circ f \leq \mathcal{U}$.
(2) f is fully dense if and only if $\mathcal{V} \leq \mathcal{V} \circ f \circ f^\circ \circ \mathcal{V}$.

Proof. (1) f is fully faithful if and only if $\mathcal{U} = f^* \circ f_* = f^\circ \circ \mathcal{V} \circ f$. Since f is a uniformly continuous map, it always has $\mathcal{U} \leq f^\circ \circ \mathcal{V} \circ f$. So (1) is obvious.
(2) f is fully dense if and only if $\mathcal{V} = f_* \circ f^* = \mathcal{V} \circ f \circ f^\circ \circ \mathcal{V}$. Since $f_* \dashv f^*$, it always has $f_* \circ f^* \leq \mathcal{V}$, so (2) is obvious. □

3. Yoneda embedding in fuzzy quasi-uniform spaces

When studying the completion of fuzzy uniform spaces, one can construct the completion $(\check{X}, \check{\mathcal{U}})$ of $(X, \mathcal{U})$ in the following way (see [34, 42]):

$$\check{X} = \{\mathcal{F} \mid \mathcal{F} \text{ is a minimal Cauchy } \top\text{-filter}\}$$

and $\{\check{U} \mid U \in \mathcal{U}\}$ is the base of $\check{\mathcal{U}}$, where

$$\check{U} : \check{X} \times \check{X} \to \mathsf{Q} \text{ is given by } \check{U}(\mathcal{F}, \mathcal{G}) = \bigvee_{F \in \mathcal{F}} \bigvee_{G \in \mathcal{G}} S(F \times G, U).$$

From Lemma 4.2 of [42], we know that there is a close relation between Cauchy pair ⊤-filters and adjoint promodules. Hence, when using promodules as the basic tool to define the completion, we may consider the following construction for the right adjoint promudules:

$$\check{U}(\Psi_1, \Psi_2) = \bigvee_{\phi_2 \in \Psi_2^{\vdash}} \bigvee_{\psi_1 \in \Psi_1} S(\phi_2 \circ \psi_1, U),$$

where $\Psi_2^{\vdash}$ is the left adjoint of Ψ_2. The value of $\check{U}(\Psi_1, \Psi_2)$ measures the degree to which $\Psi_2^{\vdash} \circ \Psi_1$ is smaller than U. Since $\Psi_2^{\vdash} \circ \Psi_1 \le U$ is equivalent to $\Psi_1 \le \Psi_2 \circ U$, we can generalize $\check{U}$ from right adjoint promodules to promodules as follows $\check{U}(\Psi_1, \Psi_2) = \bigwedge_{\psi_2 \in \Psi_2} \bigvee_{\psi_1 \in \Phi_1} S(\psi_1, \psi_2 \circ U)$ by using $\Psi_1 \le \Psi_2 \circ U$.

From the above motivation, now we can describe a Yoneda embedding in fuzzy quasi-uniform spaces. For a given fuzzy quasi-uniform space $X = (X, \mathcal{U})$, we consider the following set:

$$PX = \{\Psi : X \nrightarrow 1 \mid \Psi \text{ is a promodule}\}$$

For a Q-relation $U : X \nrightarrow X$, it is natural to lift U to a Q-relation on PX:

$$\widetilde{U}(\Phi, \Psi) = \Upsilon\,(\Phi, \Psi \circ U).$$

where $\Psi \circ U = \{\psi \circ U \mid \psi \in \Psi\}$ and then we equip PX with $\widetilde{\mathcal{U}}$:

$$\widetilde{\mathcal{U}} = \{\mathcal{H} \mid \bigvee_{U \in \mathcal{U}} S(\widetilde{U}, \mathcal{H}) = \top\},$$

Here we first need to check that $\widetilde{\mathcal{U}}$ is a fuzzy quasi-uniformity on PX.

Lemma 3.1. $(PX, \widetilde{\mathcal{U}})$ *is a fuzzy quasi-uniform space.*

Proof. Step1: we want to prove that $\{\widetilde{U} \mid U \in \mathcal{U}\}$ is a saturated prefilter base.

It is easy to check $U \le V \Rightarrow \widetilde{U} \le \widetilde{V}$ for all $U, V \in \mathcal{U}$. Now we can assert $S(U, V) \le S(\widetilde{U}, \widetilde{V})$. In fact,

$$\begin{aligned} S(\widetilde{U}, \widetilde{V}) &= \bigwedge_{\Phi, \Psi \in PX} \widetilde{U}(\Phi, \Psi) \to \widetilde{V}(\Phi, \Psi) \\ &= \bigwedge_{\Phi, \Psi \in PX} \Upsilon(\Phi, \Psi \circ U) \to \Upsilon(\Phi, \Psi \circ V) \\ &\ge \bigwedge_{\Psi \in PX} \Upsilon(\Psi \circ U, \Psi \circ V) \\ &\ge S(U, V). \end{aligned}$$

(B): For $C, D \in \mathcal{U}$, since $\mathcal{U}$ is a fuzzy quasi-uniformity, it follows that

$$\bigvee_{B \in \mathcal{U}} S(\widetilde{B}, \widetilde{C} \wedge \widetilde{D}) \ge \bigvee_{B \in \mathcal{U}} S(\widetilde{B}, \widetilde{C \wedge D}) \ge \bigvee_{B \in \mathcal{U}} S(B, C \wedge D) = \top.$$

Step 2: we check that $\widetilde{\mathcal{U}}$ fulfills (U0) and (UC).

(U0): It is easy to check $\widetilde{U}(\Psi, \Psi) = \top$ for all $U \in \mathcal{U}$. Hence, for each $\mathcal{H} \in \widetilde{\mathcal{U}}$, we have

$$\top = \bigvee_{U \in \mathcal{U}} S(\widetilde{U}, \mathcal{H}) \le \bigvee_{U \in \mathcal{U}} \widetilde{U}(\Psi, \Psi) \to \mathcal{H}(\Psi, \Psi) = \mathcal{H}(\Psi, \Psi).$$

(UC): We first prove $\widetilde{V} \circ \widetilde{U} \leq \widetilde{V \circ U}$. For $\Psi_1, \Psi_2 \in PX$,

$$\begin{aligned}
\widetilde{V} \circ \widetilde{U}(\Psi_1, \Psi_2) &= \bigvee_{\Phi \in PX} \widetilde{U}(\Psi_1, \Phi) \& \widetilde{V}(\Phi, \Psi_2) \\
&= \bigvee_{\Phi \in PX} \Upsilon(\Psi_1, \Phi \circ U) \& \Upsilon(\Phi, \Psi_2 \circ V) \\
&\leq \bigvee_{\Phi \in PX} \Upsilon(\Psi_1, \Phi \circ U) \& \Upsilon(\Phi \circ U, \Psi_2 \circ V \circ U) \\
&\leq \Upsilon(\Psi_1, \Psi_2 \circ V \circ U) \\
&= \widetilde{V \circ U}(\Psi_1, \Psi_2).
\end{aligned}$$

Then for each $\mathcal{H} \in \widetilde{\mathcal{U}}$, we have

$$\begin{aligned}
\top &= \bigvee_{U \in \mathcal{U}} S(\widetilde{U}, \mathcal{H}) \\
&= \bigvee_{U \in \mathcal{U}} (\bigvee_{V \in \mathcal{U}} S(V \circ V, U)) \& S(\widetilde{U}, \mathcal{H}) \\
&\leq \bigvee_{U \in \mathcal{U}} \bigvee_{V \in \mathcal{U}} S(\widetilde{V \circ V}, \widetilde{U}) \& S(\widetilde{U}, \mathcal{H}) \\
&\leq \bigvee_{U \in \mathcal{U}} \bigvee_{V \in \mathcal{U}} S(\widetilde{V \circ V}, \mathcal{H}) \\
&\leq \bigvee_{V \in \mathcal{U}} S(\widetilde{V} \circ \widetilde{V}, \mathcal{H}) \\
&\leq \bigvee_{\mathcal{A} \in \widetilde{\mathcal{U}}} S(\mathcal{A} \circ \mathcal{A}, \mathcal{H}).
\end{aligned}$$

In conclusion, $\widetilde{\mathcal{U}}$ is a fuzzy quasi-uniformity on PX. □

Lemma 3.2. *Let $(X, \mathcal{U})$ be a fuzzy quasi-uniform space, $\Phi, \Psi \in PX$ and $\mathcal{B}, \mathcal{D}$ be the bases of Φ, Ψ respectively. Then for every $U \in \mathcal{U}$,*

$$\widetilde{U}(\Phi, \Psi) = \bigwedge_{\psi \in \Psi} \bigvee_{\phi \in \Phi} S(\phi, \psi \circ U) = \bigwedge_{\psi \in \mathcal{D}} \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U).$$

Proof. Firstly, we check $\bigvee_{\phi \in \Phi} S(\phi, \psi \circ U) = \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U)$. It is obvious that $\bigvee_{\phi \in \Phi} S(\phi, \psi \circ U) \geq \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U)$, and $\bigvee_{\phi \in \Phi} S(\phi, \psi \circ U) \leq \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U)$ is obtained by

$$\bigvee_{\phi \in \Phi} S(\phi, \psi \circ U) = \bigvee_{\phi \in \Phi} (\bigvee_{B \in \mathcal{B}} S(B, \phi)) \& S(\phi, \psi \circ U) \leq \bigvee_{B \in \mathcal{B}} S(B, \psi \circ U).$$

Secondly, we prove $\bigwedge_{\psi \in \Psi} \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U) = \bigwedge_{\psi \in \mathcal{D}} \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U)$. On one hand, $\bigwedge_{\psi \in \Psi} \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ$

$U) \leq \bigwedge_{\psi \in \mathcal{D}} \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U)$ is obvious. On the other hand, we have

$$\begin{aligned}
\bigwedge_{\psi \in \Psi} \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U) &= \bigwedge_{\psi \in \Psi} [\top \to \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U)] \\
&= \bigwedge_{\psi \in \Psi} [\bigvee_{\psi' \in \mathcal{D}} S(\psi', \psi) \to \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U)] \\
&= \bigwedge_{\psi' \in \mathcal{D}} \bigwedge_{\psi \in \Psi} [S(\psi', \psi) \to \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi \circ U)] \\
&\geq \bigwedge_{\psi' \in \mathcal{D}} \bigwedge_{\psi \in \Psi} \bigvee_{\phi \in \mathcal{B}} [S(\psi', \psi) \to S(\phi, \psi \circ U)] \\
&\geq \bigwedge_{\psi' \in \mathcal{D}} \bigwedge_{\psi \in \Psi} \bigvee_{\phi \in \mathcal{B}} [S(\psi' \circ U, \psi \circ U) \to S(\phi, \psi \circ U)] \\
&\geq \bigwedge_{\psi' \in \mathcal{D}} \bigvee_{\phi \in \mathcal{B}} S(\phi, \psi' \circ U),
\end{aligned}$$

as desired. □

In the following part, we construct the Yoneda embedding in fuzzy quasi-uniform spaces with the help of PX.

Proposition 3.3. (Yoneda embedding) *Let $(X, \mathcal{U})$ be a fuzzy quasi-uniform space. For each $x \in X$, the assignment $x \mapsto x^*$ defines a map $\mathfrak{y}_X : X \to PX$. Then*

(1) $\mathfrak{y}_X : (X, \mathcal{U}) \to (PX, \widetilde{\mathcal{U}})$ is uniformly continuous;

(2) $\mathfrak{y}_X : (X, \mathcal{U}) \to (PX, \widetilde{\mathcal{U}})$ is fully faithful.

Proof. (1) We want to show $\mathcal{U} \leq \mathfrak{y}_X^\circ \circ \widetilde{\mathcal{U}} \circ \mathfrak{y}_X$, it suffices to check $\{\mathfrak{y}_X^\circ \circ \widetilde{U} \circ \mathfrak{y}_X \mid U \in \mathcal{U}\} \subseteq \mathcal{U}$. For each $U \in \mathcal{U}$, by Lemma 3.2, we have

$$\forall x, y \in X,\ \widetilde{U}(x^*, y^*) = \bigwedge_{C \in \mathcal{U}} \bigvee_{D \in \mathcal{U}} S(x^\circ \circ D, y^\circ \circ C \circ U) \geq \bigvee_{D \in \mathcal{U}} S(x^\circ \circ D, y^\circ \circ U).$$

Then

$$\begin{aligned}
&\bigvee_{V \in \mathcal{U}} S(V, \mathfrak{y}_X^\circ \circ \widetilde{U} \circ \mathfrak{y}_X) \\
&= \bigvee_{V \in \mathcal{U}} \bigwedge_{x,y \in X} V(x, y) \to \widetilde{U}(x^*, y^*) \\
&\geq \bigvee_{V \in \mathcal{U}} \bigwedge_{x,y \in X} V(x, y) \to \bigvee_{D \in \mathcal{U}} S(x^\circ \circ D, y^\circ \circ U) \\
&\geq \bigvee_{V \in \mathcal{U}} \bigwedge_{x,y \in X} V(x, y) \to S(x^\circ \circ V, y^\circ \circ U) \\
&= \bigvee_{V \in \mathcal{U}} \bigwedge_{x,y,z \in X} V(x, y) \& V(z, x) \to U(z, y) \\
&= \bigvee_{V \in \mathcal{U}} \bigwedge_{y,z \in X} V \circ V(z, y) \to U(z, y) \\
&= \bigvee_{V \in \mathcal{U}} S(V \circ V, U) \\
&= \top.
\end{aligned}$$

So we obtain $\bigvee_{V\in\mathcal{U}} S(V, \mathfrak{y}_X^\circ \circ \widetilde{U} \circ \mathfrak{y}_X) = \top$ for all $U \in \mathcal{U}$, this implies $\mathfrak{y}_X^\circ \circ \widetilde{U} \circ \mathfrak{y}_X \in \mathcal{U}$ by the condition (S).

(2) According to Proposition 2.13, we need to prove $\mathfrak{y}_X^\circ \circ \widetilde{\mathcal{U}} \circ \mathfrak{y}_X \leq \mathcal{U}$, which is equivalent to

$$U \in \mathcal{U} \Longrightarrow U \in \mathfrak{y}_X^\circ \circ \widetilde{\mathcal{U}} \circ \mathfrak{y}_X = \{W \mid \bigvee_{V\in\mathcal{U}} S(\mathfrak{y}_X^\circ \circ \widetilde{V} \circ \mathfrak{y}_X, W) = \top\}.$$

For $V \in \mathcal{U}$ and $x, y \in X$, we have

$$\begin{aligned}
\widetilde{V}(x^*, y^*) &= \bigwedge_{W\in\mathcal{U}} \bigvee_{D\in\mathcal{U}} S(x^\circ \circ D, y^\circ \circ W \circ V) \\
&\leq \bigvee_{D\in\mathcal{U}} S(x^\circ \circ D, y^\circ \circ V \circ V) \\
&= \bigvee_{D\in\mathcal{U}} \bigwedge_{z\in X} D(z, x) \to V \circ V(z, y) \\
&\leq \bigvee_{D\in\mathcal{U}} D(x, x) \to V \circ V(x, y) \\
&= V \circ V(x, y).
\end{aligned}$$

So we obtain $\mathfrak{y}_X^\circ \circ \widetilde{V} \circ \mathfrak{y}_X \leq V \circ V$, then

$$\bigvee_{V\in\mathcal{U}} S(\mathfrak{y}_X^\circ \circ \widetilde{V} \circ \mathfrak{y}_X, U) \geq \bigvee_{V\in\mathcal{U}} S(V \circ V, U) = \top.$$

□

From [18], for each fuzzy quasi-uniform space $(X, \mathcal{U})$, we can define $\overline{(-)} : \mathsf{Q}^X \to \mathsf{Q}^X$, $A \mapsto \overline{A}$ by

$$\forall x \in X, \quad \overline{A}(x) = \bigwedge_{U\in\mathcal{U}} \bigvee_{y\in X} A(y) \& U(x, y) \& U(y, x).$$

Remark 3.4. In classical quasi-uniform space, the above operator will be a topological closure operator. But in general lattice-valued setting, $\overline{(-)} : \mathsf{Q}^X \to \mathsf{Q}^X$ is not necessary a topological closure operator.Q is also joint continuous, i.e., $\vee$ is distributive over directed meets, we can assert that $\overline{(-)} : \mathsf{Q}^X \to \mathsf{Q}^X$ must be a topological closure operator.

Theorem 3.5. (Yoneda Lemma) *Let $(X, \mathcal{U})$ be a fuzzy quasi-unform space. For each $\Psi \in PX$, then*

(1) $\Psi \geq \Psi^* \circ (\mathfrak{y}_X)_*$;

(2) if $\overline{\mathfrak{y}_X^{\rightarrow}(\top_X)}(\Psi) = \top$, then $\Psi \leq \Psi^* \circ (\mathfrak{y}_X)_*$.

Proof. (1) Since $\Psi^* \circ (\mathfrak{y}_X)_* = \Psi^\circ \circ \widetilde{\mathcal{U}} \circ \widetilde{\mathcal{U}} \circ \mathfrak{y}_X = \Psi^\circ \circ \widetilde{\mathcal{U}} \circ \mathfrak{y}_X$, we need to check $\Psi \subseteq \Psi^\circ \circ \widetilde{\mathcal{U}} \circ \mathfrak{y}_X$. That is to say $\bigvee_{U\in\mathcal{U}} S(\Psi^\circ \circ \widetilde{U} \circ \mathfrak{y}_X, \psi) = \top$ for all $\psi \in \Psi$. Since Ψ is a promodule, it follows that $\Psi \circ \mathcal{U} \leq \Psi$. Hence, $\forall \psi \in \Psi, \bigvee_{\phi\in\Psi} \bigvee_{U\in\mathcal{U}} S(\phi \circ U, \psi) = \top$. For each $\phi \in \Psi, x \in X$,

$$\begin{aligned}
\widetilde{U}(x^*, \Psi) &= \bigwedge_{\psi\in\Psi} \bigvee_{V\in\mathcal{U}} S(x^\circ \circ V, \psi \circ U) \\
&\leq \bigvee_{V\in\mathcal{U}} S(x^\circ \circ V, \phi \circ U) \\
&= \bigvee_{V\in\mathcal{U}} \bigwedge_{z\in X} V(z, x) \to \phi \circ U(z) \\
&\leq \bigvee_{V\in\mathcal{U}} V(x, x) \to \phi \circ U(x) \\
&= \phi \circ U(x).
\end{aligned}$$

Therefore,

$$\top=\bigvee_{U\in\mathcal{U}}\bigvee_{\phi\in\Psi}S(\phi\circ U,\psi)\leq\bigvee_{U\in\mathcal{U}}S(\Psi^{\circ}\circ\widetilde{U}\circ\mathfrak{y}_X,\psi).$$

(2) Since $\overline{\mathfrak{y}_X^{\rightarrow}(\top_X)}(\Psi)=\top$, it follows that

$$\begin{aligned}\top&=\overline{\mathfrak{y}_X^{\rightarrow}(\top_X)}(\Psi)\\&=\bigwedge_{U\in\mathcal{U}}\bigvee_{\lambda\in PX}\mathfrak{y}_X^{\rightarrow}(\top_X)(\lambda)\&\widetilde{U}(\Psi,\lambda)\&\widetilde{U}(\lambda,\Psi)\\&=\bigwedge_{U\in\mathcal{U}}\bigvee_{\lambda\in PX}\bigvee_{\mathfrak{y}_X(x)=\lambda}\top_X(x)\&\widetilde{U}(\Psi,\lambda)\&\widetilde{U}(\lambda,\Psi)\\&=\bigwedge_{U\in\mathcal{U}}\bigvee_{x\in X}\widetilde{U}(\Psi,\mathfrak{y}_X(x))\&\widetilde{U}(\mathfrak{y}_X(x),\Psi)\\&=\bigwedge_{U\in\mathcal{U}}\bigvee_{x\in X}\widetilde{U}(\Psi,x^*)\&\widetilde{U}(x^*,\Psi).\end{aligned}$$

That is to say $\bigvee_{x\in X}\widetilde{U}(\Psi,x^*)\&\widetilde{U}(x^*,\Psi)=\top$ for all $U\in\mathcal{U}$. Now let $U\in\mathcal{U}$, then $\bigvee_{V\in\mathcal{U}}S(V\circ V,U)=\top$. Since $\mathfrak{y}_X$ is uniformly continuous, we have $\mathfrak{y}_X^{\circ}\circ\widetilde{V}\circ\mathfrak{y}_X\in\mathcal{U}$ for $V\in\mathcal{U}$. Then $\bigvee_{U_1\in\mathcal{U}}S(U_1,\mathfrak{y}_X^{\circ}\circ\widetilde{V}\circ\mathfrak{y}_X)=\top$. Moreover, $\bigvee_{U_2\in\mathcal{U}}S(U_2\circ U_2,U_1)=\top$. So we have

$$\begin{aligned}\top&=\bigvee_{V\in\mathcal{U}}[\bigvee_{U_1\in\mathcal{U}}(\bigvee_{U_2\in\mathcal{U}}S(U_2\circ U_2,U_1))\&S(U_1,\mathfrak{y}_X^{\circ}\circ\widetilde{V}\circ\mathfrak{y}_X)]\&S(V\circ V,U)\\&=\bigvee_{V\in\mathcal{U}}\bigvee_{U_1\in\mathcal{U}}\bigvee_{U_2\in\mathcal{U}}S(U_2\circ U_2,U_1)\&S(U_1,\mathfrak{y}_X^{\circ}\circ\widetilde{V}\circ\mathfrak{y}_X)\&S(V\circ V,U)\\&\leq\bigvee_{V\in\mathcal{U}}\bigvee_{U_2\in\mathcal{U}}S(U_2\circ U_2,\mathfrak{y}_X^{\circ}\circ\widetilde{V}\circ\mathfrak{y}_X)\&S(V\circ V,U)\\&\leq\bigvee_{V\in\mathcal{U}}\bigvee_{U_2\in\mathcal{U}}S(U_2\circ U_2,\mathfrak{y}_X^{\circ}\circ\widetilde{V}\circ\mathfrak{y}_X)\&S(\widetilde{V\circ V},\widetilde{U})\\&\leq\bigvee_{V\in\mathcal{U}}\bigvee_{U_2\in\mathcal{U}}S(U_2\circ U_2,\mathfrak{y}_X^{\circ}\circ\widetilde{V}\circ\mathfrak{y}_X)\&S(\widetilde{V}\circ\widetilde{V},\widetilde{U})\\&\leq\bigvee_{V\in\mathcal{U}}\bigvee_{U_2\in\mathcal{U}}S((U_2\wedge V)\circ(U_2\wedge V),\mathfrak{y}_X^{\circ}\circ\widetilde{V}\circ\mathfrak{y}_X)\&S((\widetilde{U_2\wedge V})\circ\widetilde{V},\widetilde{U})\\&\leq\bigvee_{V\in\mathcal{U}}\bigvee_{U_0\in\mathcal{U}}S(U_0\circ U_0,\mathfrak{y}_X^{\circ}\circ\widetilde{V}\circ\mathfrak{y}_X)\&S(\widetilde{U_0}\circ\widetilde{V},\widetilde{U})\\&=\bigvee_{V\in\mathcal{U}}\bigvee_{U_0\in\mathcal{U}}[\bigwedge_{x,y\in X}U_0\circ U_0(x,y)\rightarrow\widetilde{V}(x^*,y^*)]\&[\bigwedge_{\Phi,\Psi'\in PX}\widetilde{U_0}\circ\widetilde{V}(\Phi,\Psi')\rightarrow\widetilde{U}(\Phi,\Psi')]\\&\leq\bigvee_{V\in\mathcal{U}}\bigvee_{U_0\in\mathcal{U}}\bigwedge_{x,y\in X}\bigwedge_{\Phi,\Psi'\in PX}[U_0\circ U_0(x,y)\rightarrow\widetilde{V}(x^*,y^*)]\&[\widetilde{U_0}\circ\widetilde{V}(\Phi,\Psi')\rightarrow\widetilde{U}(\Phi,\Psi')]\\&\leq\bigvee_{V\in\mathcal{U}}\bigvee_{U_0\in\mathcal{U}}\bigwedge_{x,y\in X}[U_0\circ U_0(x,y)\rightarrow\widetilde{V}(x^*,y^*)]\&[\widetilde{U_0}\circ\widetilde{V}(x^*,\Psi)\rightarrow\widetilde{U}(x^*,\Psi)]\\&\leq\bigvee_{V\in\mathcal{U}}\bigvee_{U_0\in\mathcal{U}}\bigwedge_{x,y\in X}[U_0\circ U_0(x,y)\rightarrow\widetilde{V}(x^*,y^*)]\&[\widetilde{V}(x^*,y^*)\&\widetilde{U_0}(y^*,\Psi)\rightarrow\widetilde{U}(x^*,\Psi)]\\&\leq\bigvee_{U_0\in\mathcal{U}}\bigwedge_{x,y\in X}U_0\circ U_0(x,y)\&\widetilde{U_0}(y^*,\Psi)\rightarrow\widetilde{U}(x^*,\Psi)\\&=\bigvee_{U_0\in\mathcal{U}}\bigwedge_{x\in X}(\bigvee_{y\in X}U_0\circ U_0(x,y)\&\widetilde{U_0}(y^*,\Psi)\rightarrow\widetilde{U}(x^*,\Psi)).\end{aligned}$$

According to the above formula, then we have

$$
\begin{aligned}
&\bigvee_{\psi\in\Psi} S(\psi, \Psi^{\circ}\circ \widetilde{U}\circ \mathfrak{y}_X)\\
&= \top \to \bigvee_{\psi\in\Psi} S(\psi, \Psi^{\circ}\circ \widetilde{U}\circ \mathfrak{y}_X)\\
&= \bigvee_{U_0\in\mathcal{U}} \bigwedge_{x\in X}(\bigvee_{y\in X} U_0\circ U_0(x,y)\&\widetilde{U_0}(y^*,\Psi)\to \widetilde{U}(x^*,\Psi))\to \bigvee_{\psi\in\Psi} S(\psi, \Psi^{\circ}\circ \widetilde{U}\circ \mathfrak{y}_X)\\
&= \bigwedge_{U_0\in\mathcal{U}} [\bigwedge_{x\in X}(\bigvee_{y\in X} U_0\circ U_0(x,y)\&\widetilde{U_0}(y^*,\Psi)\to \widetilde{U}(x^*,\Psi))\to \bigvee_{\psi\in\Psi} S(\psi, \Psi^{\circ}\circ \widetilde{U}\circ \mathfrak{y}_X)]\\
&\geq \bigwedge_{U_0\in\mathcal{U}} \bigvee_{\psi\in\Psi}[\bigwedge_{x\in X}(\bigvee_{y\in X} U_0\circ U_0(x,y)\&\widetilde{U_0}(y^*,\Psi)\to \widetilde{U}(x^*,\Psi))]\to [\bigwedge_{x\in X}\psi(x)\to \widetilde{U}(x^*,\Psi)]\\
&\geq \bigwedge_{U_0\in\mathcal{U}} \bigvee_{\psi\in\Psi}\bigwedge_{x\in X}[\bigvee_{y\in X} U_0\circ U_0(x,y)\&\widetilde{U_0}(y^*,\Psi)\to \widetilde{U}(x^*,\Psi)]\to [\psi(x)\to \widetilde{U}(x^*,\Psi)]
\end{aligned}
$$

$$
\begin{aligned}
&\geq \bigwedge_{U_0\in\mathcal{U}} \bigvee_{\psi\in\Psi}\bigwedge_{x\in X}[\psi(x)\to \bigvee_{y\in X} U_0\circ U_0(x,y)\&\widetilde{U_0}(y^*,\Psi)]\\
&\geq \bigwedge_{U_0\in\mathcal{U}} \bigvee_{\psi\in\Psi}\bigwedge_{x\in X}\bigvee_{y\in X}\psi(x)\to (U_0\circ U_0(x,y)\&\widetilde{U_0}(y^*,\Psi))\\
&\geq \bigwedge_{U_0\in\mathcal{U}} \bigvee_{\psi\in\Psi}\bigwedge_{x\in X}\bigvee_{y\in X}(\psi(x)\to U_0\circ U_0(x,y))\&\widetilde{U_0}(y^*,\Psi)\\
&\geq \bigwedge_{U_0\in\mathcal{U}} \bigvee_{y\in X}\bigvee_{\psi\in\Psi}\bigwedge_{x\in X}(\psi(x)\to U_0\circ U_0(x,y))\&\widetilde{U_0}(y^*,\Psi).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\top &= \bigwedge_{U_0\in\mathcal{U}} \bigvee_{y\in X}\widetilde{U_0}(\Psi,y^*)\&\widetilde{U_0}(y^*,\Psi)\\
&= \bigwedge_{U_0\in\mathcal{U}} \bigvee_{y\in X}[\bigwedge_{V\in\mathcal{U}}\bigvee_{\psi\in\Psi} S(\psi, y^{\circ}\circ V\circ U_0)\&\widetilde{U_0}(y^*,\Psi)]\\
&\leq \bigwedge_{U_0\in\mathcal{U}} \bigvee_{y\in X}[\bigvee_{\psi\in\Psi} S(\psi, y^{\circ}\circ U_0\circ U_0)\&\widetilde{U_0}(y^*,\Psi)]\\
&= \bigwedge_{U_0\in\mathcal{U}} \bigvee_{y\in X}\bigvee_{\psi\in\Psi}(\bigwedge_{x\in X}\psi(x)\to U_0\circ U_0(x,y))\&\widetilde{U_0}(y^*,\Psi)\\
&\leq \bigwedge_{U_0\in\mathcal{U}} \bigvee_{y\in X}\bigvee_{\psi\in\Psi}\bigwedge_{x\in X}\psi(x)\to U_0\circ U_0(x,y)\&\widetilde{U_0}(y^*,\Psi).
\end{aligned}
$$

So we obtain $\bigvee_{\psi\in\Psi} S(\psi, \Psi^{\circ}\circ \widetilde{U}\circ \mathfrak{y}_X) = \top$ for each $U\in\mathcal{U}$, which implies $\{\Psi^{\circ}\circ\widetilde{U}\circ\mathfrak{y}_X \mid U\in\mathcal{U}\}\subseteq\Psi$. Therefore, $\Psi\leq \Psi^{\circ}\circ\widetilde{\mathcal{U}}\circ\mathfrak{y}_X$. □

Proposition 3.6. *Let $(X,\mathcal{U})$ be a fuzzy quasi-unform space and $\Psi\in PX$. Then Ψ is a right adjoint if and only if* $\overline{\mathfrak{y}_X^{\to}(\top_X)}(\Psi) = \top$.

Proof. Sufficiency: If $\overline{\mathfrak{y}_X^{\to}(\top_X)}(\Psi) = \top$, then $\Psi = \Psi^*\circ(\mathfrak{y}_X)_*$ by Theorem 3.5. Let $\Phi = (\mathfrak{y}_X)^*\circ\Psi_*$. We can assert

$(\Phi : 1 \to X) \dashv (\Psi : X \to 1)$. In fact, on one hand

$$\begin{aligned}\Phi \circ \Psi &= (\mathfrak{y}_X)^* \circ \Psi_* \circ \Psi^* \circ (\mathfrak{y}_X)_* \\ &\leq (\mathfrak{y}_X)^* \circ (\mathfrak{y}_X)_* \\ &= \mathcal{U} \ \text{(since } \mathfrak{y}_X \text{ is fully faithful).}\end{aligned}$$

On the other hand, note that

$$\begin{aligned}\Psi \circ \Phi &= \Psi^* \circ (\mathfrak{y}_X)_* \circ (\mathfrak{y}_X)^* \circ \Psi_* \\ &= \Psi^\circ \circ \widetilde{\mathcal{U}} \circ \widetilde{\mathcal{U}} \circ \mathfrak{y}_X \circ (\mathfrak{y}_X)^\circ \circ \widetilde{\mathcal{U}} \circ \widetilde{\mathcal{U}} \circ \Psi \\ &= \Psi^\circ \circ \widetilde{\mathcal{U}} \circ \mathfrak{y}_X \circ \mathfrak{y}_X^\circ \circ \widetilde{\mathcal{U}} \circ \Psi.\end{aligned}$$

For any $W, V \in \mathcal{U}$, let $U = W \wedge V$. Then we have $U \in \mathcal{U}$ and

$$\begin{aligned}\Psi^\circ \circ \widetilde{W} \circ \mathfrak{y}_X \circ \mathfrak{y}_X^\circ \circ \widetilde{V} \circ \Psi &= \bigvee_{x \in X} \widetilde{V}(\Psi, x^*) \& \widetilde{W}(x^*, \Psi) \\ &\geq \bigvee_{x \in X} \widetilde{U}(\Psi, x^*) \& \widetilde{U}(x^*, \Psi) \\ &= \top \ \text{(by } \overline{\mathfrak{y}_X(\top_X)}(\Psi) = \top).\end{aligned}$$

Then $\Psi \circ \Phi \geq 1$, as desired.

Necessity: Suppose Ψ is a right adjoint and Φ is the left adjoint to Ψ. We want to show that for each $U \in \mathcal{U}$, it has

$$\bigvee_{x \in X} \widetilde{U}(\Psi, x^*) \& \widetilde{U}(x^*, \Psi) = \top.$$

Firstly, on account of $\Phi \dashv \Psi$, we have $\Phi \circ \Psi \leq \mathcal{U}$ and $\Psi \circ \Phi \geq 1$. Then $\bigvee_{\phi \in \Phi} \bigvee_{\psi \in \Psi} S(\phi \circ \psi, U) = \top$ by $\Phi \circ \Psi \leq \mathcal{U}$, i.e.,

$$\begin{aligned}\top &= \bigvee_{\phi \in \Phi} \bigvee_{\psi \in \Psi} \bigwedge_{x,z \in X} \psi(z) \& \phi(x) \to U(z, x) \\ &= \bigvee_{\phi \in \Phi} \bigvee_{\psi \in \Psi} \bigwedge_{x,z \in X} \phi(x) \to (\psi(z) \to U(z, x)) \\ &= \bigvee_{\phi \in \Phi} \bigvee_{\psi \in \Psi} \bigwedge_{x \in X} \phi(x) \to [\bigwedge_{z \in X} (\psi(z) \to U(z, x))] \\ &= \bigvee_{\phi \in \Phi} \bigvee_{\psi \in \Psi} S(\phi, \bigwedge_{z \in X} (\psi(z) \to U(z, -))) \\ &\leq \bigvee_{\phi \in \Phi} S(\phi, \bigvee_{\psi \in \Psi} \bigwedge_{z \in X} (\psi(z) \to U(z, -))).\end{aligned}$$

Hence $\phi^\flat \in \Phi$, where $\phi^\flat = \bigvee_{\psi\in\Psi}\bigwedge_{z\in X}(\psi(z) \to U(z,-))$. And then we have

$$\begin{aligned}
\widetilde{U}(\Psi, x^*) &= \bigwedge_{V\in\mathcal{U}}\bigvee_{\psi_1\in\Psi} S(\psi_1, x^\circ \circ V \circ U)\\
&= \bigwedge_{V\in\mathcal{U}}\bigvee_{\psi_1\in\Psi}\bigwedge_{z\in X} \psi_1(z) \to V\circ U(z,x)\\
&= \bigwedge_{V\in\mathcal{U}}\bigvee_{\psi_1\in\Psi}\bigwedge_{z\in X}[\psi_1(z) \to \bigvee_{y\in X} U(z,y)\&V(y,x)]\\
&\geq \bigvee_{\psi_1\in\Psi}\bigwedge_{z\in X}(\psi_1(z) \to U(z,x))\\
&= \phi^\flat(x).
\end{aligned}$$

Since $\mathcal{U}$ is a fuzzy quasi-uniformity, it has $\bigvee_{V\in\mathcal{U}} S(V\circ V, U) = \top$. Furthermore, by $\bigvee_{\phi'\in\Phi}\bigvee_{\psi'\in\Psi} S(\phi'\circ\psi', V) = \top$, we have

$$\begin{aligned}
\top &= \bigvee_{V\in\mathcal{U}}(\bigvee_{\phi'\in\Phi}\bigvee_{\psi'\in\Psi} S(\phi'\circ\psi', V))\&S(V\circ V, U)\\
&= \bigvee_{V\in\mathcal{U}}\bigvee_{\phi'\in\Phi}\bigvee_{\psi'\in\Psi} S(\phi'\circ\psi', V)\&S(V\circ V, U)\\
&\leq \bigvee_{V\in\mathcal{U}}\bigvee_{\phi'\in\Phi}\bigvee_{\psi'\in\Psi} S(\phi'\circ\psi'\circ V, V\circ V)\&S(V\circ V, U)\\
&\leq \bigvee_{V\in\mathcal{U}}\bigvee_{\phi'\in\Phi}\bigvee_{\psi'\in\Psi} S(\phi'\circ\psi'\circ V, U)
\end{aligned}$$

$$\begin{aligned}
&= \bigvee_{V\in\mathcal{U}}\bigvee_{\phi'\in\Phi}\bigvee_{\psi'\in\Psi}\bigwedge_{y,z\in X} \phi'\circ\psi'\circ V(y,z) \to U(y,z)\\
&= \bigvee_{V\in\mathcal{U}}\bigvee_{\phi'\in\Phi}\bigvee_{\psi'\in\Psi}\bigwedge_{y,z\in X}(\bigvee_{x\in X} V(y,x)\&\psi'(x)\&\phi'(z)) \to U(y,z)\\
&= \bigvee_{V\in\mathcal{U}}\bigvee_{\phi'\in\Phi}\bigvee_{\psi'\in\Psi}\bigwedge_{x,y,z\in X} \psi'(x) \to (V(y,x)\&\phi'(z) \to U(y,z))\\
&= \bigvee_{V\in\mathcal{U}}\bigvee_{\phi'\in\Phi}\bigvee_{\psi'\in\Psi}\bigwedge_{x\in X}[\psi'(x) \to \bigwedge_{y,z\in X}(V(y,x)\&\phi'(z) \to U(y,z))]\\
&\leq \bigvee_{\psi'\in\Psi}\bigwedge_{x\in X}[\psi'(x) \to \bigvee_{V\in\mathcal{U}}\bigvee_{\phi'\in\Phi}\bigwedge_{y,z\in X}(V(y,x)\&\phi'(z) \to U(y,z))]\\
&= \bigvee_{\psi'\in\Psi} S(\psi', \bigvee_{V\in\mathcal{U}}\bigvee_{\phi'\in\Phi}\bigwedge_{y,z\in X}(V(y,-)\&\phi'(z) \to U(y,z))).
\end{aligned}$$

We can obtain $\psi^\flat \in \Psi$, where $\psi^\flat = \bigvee_{V\in\mathcal{U}} \bigvee_{\phi'\in\Phi} \bigwedge_{y,z\in X} V(y,-)\&\phi'(z) \to U(y,z)$. Moreover,

$$\begin{aligned}
\widetilde{U}(x^*,\Psi) &\geq \widetilde{U}(\Psi\circ\Phi\circ x^*,\Psi) \ \ (\text{since } \Psi\circ\Phi\circ x^* \geq x^*)\\
&= \bigwedge_{\psi_2\in\Psi} \bigvee_{\phi_1\in\Psi\circ\Phi\circ x^*} S(\phi_1,\psi_2\circ U)\\
&= \bigwedge_{\psi_2\in\Psi} \bigvee_{\psi'\in\Psi} \bigvee_{\phi'\in\Phi} \bigvee_{V\in\mathcal{U}} S(\psi'\circ\phi'\circ x^\circ\circ V,\psi_2\circ U))\\
&\geq \bigwedge_{\psi_2\in\Psi} \bigvee_{\phi'\in\Phi} \bigvee_{V\in\mathcal{U}} S(\psi_2\circ\phi'\circ x^\circ\circ V,\psi_2\circ U)\\
&\geq \bigvee_{\phi'\in\Phi} \bigvee_{V\in\mathcal{U}} S(\phi'\circ x^\circ\circ V,U)\\
&= \bigvee_{\phi'\in\Phi} \bigvee_{V\in\mathcal{U}} \bigwedge_{y,z\in X} V(y,x)\&\phi'(z)\to U(y,z)\\
&= \psi^\flat(x).
\end{aligned}$$

Finally, we know $\psi^\flat\circ\phi^\flat \geq 1$ from $\Psi\circ\Phi \geq 1$. This is to say $\bigvee_{x\in X}\phi^\flat(x)\&\psi^\flat(x) = \top$. And with the above calculation, it follows that

$$\bigvee_{x\in X} \widetilde{U}(\Psi,x^*)\&\widetilde{U}(x^*,\Psi) \geq \bigvee_{x\in X} \phi^\flat(x)\&\psi^\flat(x) = \top.$$

From the arbitrariness of $U\in\mathcal{U}$, we know $\overline{\mathfrak{y}_X^{\rightarrow}(\top_X)}(\Psi) = \top$. $\square$

For a fuzzy quasi-uniform space $(X,\mathcal{U})$, let RX denote the subset of PX consisting of the right adjoint promodules $\Psi : X \circ\!\!\rightarrow 1$. Then RX is also a fuzzy quasi-uniform space (with fuzzy quasi-uniformity inherited from PX) and the Yoneda embedding $\mathfrak{y}_X : X \to PX$ factors through RX. For convenience, for every fuzzy quasi-uniform space $(X,\mathcal{U})$, write

$$\mathfrak{r}_X : X \to RX$$

for the map obtained by restricting the codomain of $\mathfrak{y}_X : X \to PX$.

Corollary 3.7. *For each $\Psi\in RX$, then $\Psi = \Psi^*\circ(\mathfrak{r}_X)_*$.*

In general, the Yoneda embedding $\mathfrak{y}_X$ is not fully dense, but $\mathfrak{r}_X : X \to RX$ fulfills.

Lemma 3.8. *A map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is fully dense if and only if $\overline{f^{\rightarrow}(\top_X)}(y) = \top$ for all $y\in Y$.*

Proof. Necessity: Let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be fully dense. By Proposition 2.13, we have $\mathcal{V} \leq \mathcal{V}\circ f\circ f^\circ\circ\mathcal{V}$. Since $1_Y \leq \mathcal{V}$, then we have $1_Y \leq V\circ f\circ f^\circ\circ V$ for all $V\in\mathcal{V}$. Then for each $y\in Y$,

$$\top = 1_Y(y,y) \leq \bigwedge_{V\in\mathcal{V}} V\circ f\circ f^\circ\circ V(y,y) = \bigwedge_{V\in\mathcal{V}}\bigvee_{x\in X} V(y,f(x))\&V(f(x),y) = \overline{f^{\rightarrow}(\top_X)}(y).$$

Sufficiency: Suppose $\overline{f^{\rightarrow}(\top_X)}(y) = \top$ for all $y\in Y$. We need to show $\mathcal{V} \leq \mathcal{V}\circ f\circ f^\circ\circ\mathcal{V}$ or $\{V\circ f\circ f^\circ\circ V \mid V\in\mathcal{V}\} \subseteq \mathcal{V}$. Let $V\in\mathcal{V}$. Then $\top = \bigvee_{W_1\in\mathcal{V}} S(W_1\circ W_1, V)$ and $\top = \bigvee_{W_2\in\mathcal{V}} S(W_2,V)$. By $\top = \overline{f^{\rightarrow}(\top_X)}(y)$ for all $y \in Y$, we know $1_Y \leq W \circ f \circ f^\circ \circ W$ for all $W \in \mathcal{V}$. Since $\mathcal{V}$ is a fuzzy quasi-uniformity, we have

$$\begin{aligned}
\top &= \bigvee_{W_1 \in \mathcal{V}} S(W_1 \circ W_1, V) \\
&= \bigvee_{W_1 \in \mathcal{V}} S(W_1 \circ W_1, V) \& \top \\
&= \bigvee_{W_1 \in \mathcal{V}} S(W_1 \circ W_1, V) \& \bigvee_{W_2 \in \mathcal{V}} S(W_2, V) \\
&= \bigvee_{W_1 \in \mathcal{V}} \bigvee_{W_2 \in \mathcal{V}} S(W_1 \circ W_2, V) \& S(W_2, V) \\
&\leq \bigvee_{W \in \mathcal{V}} S(W \circ W, V) \& S(W, V) \text{ ($\mathcal{V}$ is closed for finite meets)} \\
&\leq \bigvee_{W \in \mathcal{V}} S(W \circ W \circ f \circ f^\circ \circ W, V \circ f \circ f^\circ \circ W) \& S(V \circ f \circ f^\circ \circ W, V \circ f \circ f^\circ \circ V) \\
&\leq \bigvee_{W \in \mathcal{V}} S(W \circ W \circ f \circ f^\circ \circ W, V \circ f \circ f^\circ \circ V) \\
&\leq \bigvee_{W \in \mathcal{V}} S(W, V \circ f \circ f^\circ \circ V).
\end{aligned}$$

Therefore, $V \circ f \circ f^\circ \circ V \in \mathcal{V}$ as desired. □

Theorem 3.9. *The map* $\mathrm{r}_X : X \to RX$ *is fully dense.*

Proof. It is obvious by Proposition 3.6 and Lemma 3.8. □

4. Completeness and completion of fuzzy quasi-uniform spaces

In this section, we study the applications of Yoneda embedding and we focus on the completion of fuzzy quasi-uniform spaces. In [42], Yue and Fang used pair $\top$-filters to study Cauchy completeness of fuzzy quasi-uniform spaces following the idea of Lindgren and Fletcher in [30]. For two $\top$-filters $\mathbb{F}$ and $\mathbb{G}$, $(\mathbb{F}, \mathbb{G})$ is called a pair $\top$-filter in [42] if $\bigvee_{x \in X} F(x) \& G(x) = \top$ for all $F \in \mathbb{F}, G \in \mathbb{G}$. Similarly, for two saturated prefilter $(\mathcal{F}, \mathcal{G})$, we can also define pair saturated prefilter as follows: $(\mathcal{F}, \mathcal{G})$ is called a pair saturated prefilter if $\bigvee_{x \in X} F(x) \& G(x) = \top$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. In this way, $\mathcal{F}$ and $\mathcal{G}$ must be $\top$-filters since $\top = \bigvee_{x \in X} F(x) \& G(x) \leq \bigvee_{x \in X} F(x)$ and $\top = \bigvee_{x \in X} F(x) \& G(x) \leq \bigvee_{x \in X} G(x)$ for $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Hence the results about completeness in [42] based on $\top$-filters are also valid for saturated prefilters. For convenience, we list the concepts and results as follows (Definition 4.1–Theorem 4.5).

Definition 4.1. Let $(X, \mathcal{U})$ be a fuzzy quasi-uniform space.

(1) A pair saturated prefilter $(\mathcal{F}, \mathcal{G})$ on $(X, \mathcal{U})$ is called a Cauchy pair saturated prefilter if $\bigvee_{F \in \mathcal{F}} \bigvee_{G \in \mathcal{G}} S(F \times G, U) = \top$ for $U \in \mathcal{U}$, where $F \times G : X \times X \to \mathsf{Q}$ is defined by $F \times G(x, y) = F(x) \& G(y)$ for all $x, y \in X$.

(2) A pair saturated prefilter $(\mathcal{F}, \mathcal{G})$ on $(X, \mathcal{U})$ converges to $x_0 \in X$ if $(\mathcal{L}_{x_0}, \mathcal{R}_{x_0}) \subseteq (\mathcal{F}, \mathcal{G})$, where $\mathcal{L}_{x_0}$ and $\mathcal{R}_{x_0}$ are the saturated prefilters generated by the bases $L_{x_0} = \{U(-, x_0) \mid U \in \mathcal{U}\}$ and $R_{x_0} = \{U(x_0, -) \mid U \in \mathcal{U}\}$, respectively.

Lemma 4.2. *Let* $(X, \mathcal{U})$ *be a fuzzy quasi-uniform space.* $\Phi \dashv \Psi : X \mathrel{\circ\!\!\to} 1$ *is a pair adjoint promodule if and only if* $(\mathcal{F}_\Psi, \mathcal{G}_\Phi)$ *is a minimal Cauchy saturated prefilter on* $(X, \mathcal{U})$*, where* $\mathcal{F}_\Psi$ *and* $\mathcal{G}_\Phi$ *are the saturated prefilters* $\{\psi(-, *) \mid \psi \in \Psi\}$ *and* $\{\phi(*, -) \mid \phi \in \Phi\}$.

In fact, the condition $\Psi \circ \Phi \geq 1$ guarantees that $(\mathcal{F}_\Psi, \mathcal{G}_\Phi)$ is a pair saturated prefilter on $(X, \mathcal{U})$, $\Phi \circ \Psi \leq \mathcal{U}$ ensures that it is a Cauchy saturated prefilter, while the condition of Φ, Ψ are promodules assures that it is minimal.

Definition 4.3. A fuzzy quasi-uniform space $(X, \mathcal{U})$ is called Cauchy complete if each Cauchy pair saturated prefilter on $(X, \mathcal{U})$ converges.

Since we have described fuzzy quasi-unform spaces as enriched categories. Therefore, we can study the completeness of fuzzy quasi-unform spaces by using pair adjoint promodules.

Definition 4.4. A fuzzy quasi-unform space $(X, \mathcal{U})$ is said to be Lawvere complete if for each adjoint promodule $\Phi \dashv \Psi : X \multimap 1$, there exists $x \in X$ such that $\Phi = x_*, \Psi = x^*$.

It is easy to see $(X, \mathcal{U})$ is Lawvere complete if and only if $\mathfrak{r}_X(X) = RX$.

Theorem 4.5. *$(X, \mathcal{U})$ is Cauchy complete if and only if $(X, \mathcal{U})$ is Lawvere complete.*

Now we begin to study the completion of fuzzy quasi-uniform spaces. First, we recall the definition of T_0 separability of fuzzy quasi-uniform space ([40] Theorem 5.1). $(X, \mathcal{U})$ is said to be T_0 separated if there is some $U \in \mathcal{U}$ such that $U(x, y) < \top$ or $U(y, x) < \top$ for all $x \neq y$.

Proposition 4.6. *For a fuzzy quasi-uniform space $(X, \mathcal{U})$, the following statements are equivalent:*
(1) $(X, \mathcal{U})$ is T_0 separated;
(2) $\mathfrak{y}_X : X \to PX$ is injective;
(3) For any uniformly continuous maps $f, g : Y \to X$, if $f^* = g^*$, then $f = g$;
(4) For any uniformly continuous maps $f, g : Y \to X$, if $f_* = g_*$, then $f = g$.

Proof. Firstly, we show (1) $\Leftrightarrow$ (2).
(1) $\Rightarrow$ (2) It suffices to show that $x \neq y \Rightarrow \mathfrak{y}_X(x) \neq \mathfrak{y}_X(y)$. Since

$$\begin{aligned}\mathfrak{y}_X(x) \neq \mathfrak{y}_X(y) &\Leftrightarrow x^* \neq y^* \\ &\Leftrightarrow x^\circ \circ \mathcal{U} \neq y^\circ \circ \mathcal{U} \\ &\Leftrightarrow \{x^\circ \circ\ U \mid U \in \mathcal{U}\} \nsubseteq y^\circ \circ \mathcal{U} \text{ or } \{y^\circ \circ\ U \mid U \in \mathcal{U}\} \nsubseteq x^\circ \circ \mathcal{U} \\ &\Leftrightarrow \exists U \in \mathcal{U}, x^\circ \circ\ U \notin y^\circ \circ \mathcal{U} \text{ or } y^\circ \circ\ U \notin x^\circ \circ \mathcal{U}\end{aligned}$$

Let $x \neq y$. Since $(X, \mathcal{U})$ is T_0 separated, there exists $U \in \mathcal{U}$ such that $U(x, y) < \top$ or $U(y, x) < \top$. Without loss of generality, assume that $U(y, x) < \top$. Then we have $x^\circ \circ U \notin y^\circ \circ \mathcal{U}$. In fact, if $x^\circ \circ U \in y^\circ \circ \mathcal{U}$, then

$$\top = \bigvee_{V \in \mathcal{U}} S(y^\circ \circ V, x^\circ \circ U) = \bigvee_{V \in \mathcal{U}} \bigwedge_{z \in X} V(z, y) \to U(z, x) \leq \bigvee_{V \in \mathcal{U}} V(y, y) \to U(y, x) = U(y, x).$$

This contradicts to the hypothesis. So we obtain $x^* \neq y^*$, as desired.
(2) $\Rightarrow$ (1) Let $x \neq y$. We assume that there exists $U_0 \in \mathcal{U}$ such that $x^\circ \circ U_0 \notin y^\circ \circ \mathcal{U}$. Then we have

$$\bigvee_{V \in \mathcal{U}} S(y^\circ \circ V, x^\circ \circ U_0) < \top,$$

i.e.,

$$\bigvee_{V \in \mathcal{U}} \bigwedge_{z \in X} V(z, y) \to U_0(z, x) < \top.$$

Next we can assert that there exists some $V_0 \in \mathcal{U}$ such that $V_0(y, x) < \top$. If $V(y, x) = \top$ for all $V \in \mathcal{U}$, then we have

$$\begin{aligned}\top &= \bigvee_{V\in\mathcal{U}} S(V\circ V, U_0)\\ &\leq \bigvee_{V\in\mathcal{U}}\bigwedge_{z\in X} V\circ V(z,x)\to U_0(z,x)\\ &= \bigvee_{V\in\mathcal{U}}\bigwedge_{z\in X}(\bigvee_{a\in X} V(z,a)\& V(a,x)\to U_0(z,x))\\ &\leq \bigvee_{V\in\mathcal{U}}\bigwedge_{z\in X}(V(z,y)\& V(y,x)\to U_0(z,x))\\ &= \bigvee_{V\in\mathcal{U}}\bigwedge_{z\in X} V(z,y)\to U_0(z,x)(\text{ since } V(y,x)=\top).\end{aligned}$$

This contradicts to the hypothesis. Therefore, there exists $V_0 \in \mathcal{U}$ such that $V_0(y,x) < \top$.

Secondly, we prove (2) ⇔ (3).

(2) ⇒ (3) Suppose that $f, g : Y \to X$ are uniformly continuous maps and satisfy $f^* = g^*$. We need to check that $\forall y \in Y, f(y) = g(y)$. Since $f^* = g^*$, then $y^* \circ f^* = y^* \circ g^*$, i.e. $f(y)^* = g(y)^*$. Then $f(y) = g(y)$ on account of the injectivity of $\mathfrak{y}_X$.

(3) ⇒ (2) Since for any $x \in X$, $x : 1 \to X$ is a special uniformly continuous map, this proof is obvious.

Finally, since a promodule has at most one left (right, resp.) adjoint, then $f^* = g^* \Leftrightarrow f_* = g_*$. So (3) ⇔ (4) is obvious. □

Let $(X, \mathcal{U})$ be a fuzzy quasi-uniform space. If $A \in \mathsf{Q}^X$ satisfies $\overline{A} = \top_X$ or $\overline{A}(x) = \top$ for all $x \in X$, then A is called dense in $(X, \mathcal{U})$. In particular, $Z \subseteq X$ is dense in $(X, \mathcal{U})$ if $\overline{\top_Z}(x) = \top$ for all $x \in X$.

A T_0 separated and Cauchy complete fuzzy quasi-uniform space $(X^\heartsuit, \mathcal{U}^\heartsuit)$ is called the T_0 completion of $(X, \mathcal{U})$ if there exists a uniformly continuous map $c_X : (X, \mathcal{U}) \to (X^\heartsuit, \mathcal{U}^\heartsuit)$ satisfies the following properties:

(I) $c_X^\rightarrow(\top_X)$ or $\top_{c_X(X)}$ is dense in Y, i.e., $\overline{c_X^\rightarrow(\top_X)} = \top_Y$;

(II) whenever $(Y, \mathcal{V})$ is T_0 separated and Cauchy complete, and $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a uniformly continuous mapping, then there is a unique uniformly continuous map $f^\triangle : (X^\heartsuit, \mathcal{U}^\heartsuit) \to (Y, \mathcal{V})$ such that $f^\triangle \cdot c_X = f$.

Remark 4.7. By Lemma 3.8 and above definition of dense, we can see that a map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is fully dense if and only if $f_X^\rightarrow(\top_X)$ is dense in Y.

Now we consider the subspace RX of PX, where its fuzzy quasi-uniformity is the restriction of $\widetilde{\mathcal{U}}$. For convenience, we still use $\widetilde{\mathcal{U}}$ instead of $\widetilde{\mathcal{U}}|RX$. In the following, we will show that $(RX, \widetilde{\mathcal{U}})$ is just the T_0 completion of $(X, \mathcal{U})$.

Lemma 4.8. *$(RX, \widetilde{\mathcal{U}})$ is T_0 separated.*

Proof. Let $\Psi_1, \Psi_2 \in RX$ with $\Psi_1 \neq \Psi_2$. From Corollary 3.7, we know $\Psi_1 = \Psi_1^* \circ (\mathfrak{r}_X)_*$ and $\Psi_2 = \Psi_2^* \circ (\mathfrak{r}_X)_*$. Then $\Psi_1^* \neq \Psi_2^*$ by $\Psi_1 \neq \Psi_2$. Therefore, $(RX, \widetilde{\mathcal{U}})$ is T_0 separated from Proposition 4.6. □

Lemma 4.9. *$(RX, \widetilde{\mathcal{U}})$ is Cauchy complete.*

Proof. It suffices to check that for each adjoint promodule $\Phi \dashv \Psi : RX \circ\!\!\!\to 1$, there exists $\psi \in RX$ such that $\Psi = \psi^*$ and $\Phi = \psi_*$. Let

$$\psi = \Psi \circ (\mathfrak{r}_X)_* : X \overset{(\mathfrak{r}_X)_*}{\circ\!\!\!\longrightarrow} RX \overset{\Psi}{\circ\!\!\!\longrightarrow} 1 \ \text{ and } \ \phi = (\mathfrak{r}_X)^* \circ \Phi : 1 \overset{\Phi}{\circ\!\!\!\longrightarrow} RX \overset{(\mathfrak{r}_X)^*}{\circ\!\!\!\longrightarrow} X.$$

It is easy to see $\phi \dashv \psi$, then $\psi \in RX$. Next we prove $\Psi = \psi^*$. Since $\psi \in RX$, it follows that $\psi = \psi^* \circ (\mathfrak{r}_X)_*$ from Corollary 3.7. Hence $\Psi \circ (\mathfrak{r}_X)_* = \psi^* \circ (\mathfrak{r}_X)_*$. Since $\mathfrak{r}_X$ is fully dense, we have $\Psi = \psi^*$. Then by $\Phi \dashv \psi^*, \psi_* \dashv \psi^*$ and the uniqueness of left adjoint, which implies $\Phi = \psi_*$. □

If $(X,\mathcal{U})$ is a T_0 separated and Cauchy complete fuzzy quasi-uniform space, then $(X,\mathcal{U}) \cong (RX,\widetilde{\mathcal{U}})$. In fact, since $\mathrm{r}_X(X) = RX$, it follows that r_X is surjective. We already know that r_X is uniformly continuous and injective. To show $(X,\mathcal{U}) \cong (RX,\widetilde{\mathcal{U}})$, We still need to show $\mathrm{r}_X^{-1} : (RX,\widetilde{\mathcal{U}}) \to (X,\mathcal{U})$ is uniformly continuous. On account of $(\mathrm{r}_X^{-1})^\circ \circ U \circ \mathrm{r}_X^{-1}(x^*, y^*) = U(x, y)$ and $\widetilde{U}(x^*, y^*) \leq U \circ U(x, y)$ for $U \in \mathcal{U}$ and $x, y \in X$, we have

$$\bigvee_{V\in\mathcal{U}} S(\widetilde{V}, (\mathrm{r}_X^{-1})^\circ \circ U \circ \mathrm{r}_X^{-1}) \geq \bigvee_{V\in\mathcal{U}} S(V \circ V, U) = \top.$$

Hence $(\mathrm{r}_X^{-1})^\circ \circ U \circ \mathrm{r}_X^{-1} \in \widetilde{\mathcal{U}}$. Therefore, r_X^{-1} is uniformly continuous. Especially, if $(X,\mathcal{U})$ is a T_0 separated and Cauchy complete fuzzy quasi-uniform space, $(X,\mathcal{U})$ itself is just the T_0 completion of $(X,\mathcal{U})$.

For a right adjoint promodule $\Phi : (X,\mathcal{U}) \circ\!\!\to (Y,\mathcal{V})$, its left adjoint is denoted by $\widehat{\Phi}$. Then $\Phi \circ \widehat{\Phi} \geq \mathcal{V}$ and $\widehat{\Phi} \circ \Phi \leq \mathcal{U}$. We equip RX with the structure $\mathring{\mathcal{U}} = \{\mathcal{A} \mid \bigvee_{U\in\mathcal{U}} S(\mathring{U}, \mathcal{A}) = \top\}$, where $\mathring{U}(\Psi_1, \Psi_2) = \bigvee_{\widehat{\psi_2}\in\widehat{\Psi_2}} \bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2} \circ \psi_1, U)$ for all $\Psi_1, \Psi_2 \in RX$.

Lemma 4.10. $\widetilde{\mathcal{U}} = \mathring{\mathcal{U}}$.

Proof. We first check $\mathring{U} \leq \widetilde{U}$ for all $U \in \mathcal{U}$. It suffices to prove $\mathring{U}(\Psi_1, \Psi_2) \leq \widetilde{U}(\Psi_1, \Psi_2)$ for all $\Psi_1, \Psi_2 \in RX$. For each $\widehat{\psi_2} \in \widehat{\Psi_2}$ and $\psi_2 \in \Psi_2$, then we have $\psi_2 \circ \widehat{\psi_2} \geq 1$ by $\Psi_2 \circ \widehat{\Psi_2} \geq 1$. Then

$$\begin{aligned}\bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2} \circ \psi_1, U) &\leq \bigvee_{\psi_1\in\Psi_1} S(\psi_2 \circ \widehat{\psi_2} \circ \psi_1, \psi_2 \circ U)\\ &\leq \bigvee_{\psi_1\in\Psi_1} S(\psi_1, \psi_2 \circ U).\end{aligned}$$

We have

$$\mathring{U}(\Psi_1, \Psi_2) = \bigvee_{\widehat{\psi_2}\in\widehat{\Psi_2}} \bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2} \circ \psi_1, U) \leq \bigwedge_{\psi_2\in\Psi_2} \bigvee_{\psi_1\in\Psi_1} S(\psi_1, \psi_2 \circ U) = \widetilde{U}(\Psi_1, \Psi_2).$$

Therefore, $\{\widetilde{U} \mid U \in \mathcal{U}\} \subseteq \mathring{\mathcal{U}}$. This is to say $\mathring{\mathcal{U}} \leq \widetilde{\mathcal{U}}$. Now, we prove the opposite direction. We first check that $\widetilde{U} \leq U \mathring{\circ} U$. Since $\widehat{\Psi_2} \circ \Psi_2 \leq \mathcal{U}$, it follows that $\bigvee_{\widehat{\psi_2}\in\widehat{\Psi_2}} \bigvee_{\psi_2\in\Psi_2} S(\widehat{\psi_2} \circ \psi_2, U) = \top$. Then we have

$$\begin{aligned}U \mathring{\circ} U(\Psi_1, \Psi_2) &= \bigvee_{\widehat{\psi_2}\in\widehat{\Psi_2}} \bigvee_{\psi_2\in\Psi_2} S(\widehat{\psi_2} \circ \psi_2, U) \to U \mathring{\circ} U(\Psi_1, \Psi_2)\\ &= \bigvee_{\widehat{\psi_2}\in\widehat{\Psi_2}} \bigvee_{\psi_2\in\Psi_2} S(\widehat{\psi_2} \circ \psi_2, U) \to \bigvee_{\widehat{\psi_2}\in\widehat{\Psi_2}} \bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2} \circ \psi_1, U \circ U)\\ &\geq \bigwedge_{\widehat{\psi_2}\in\widehat{\Psi_2}} \Big(\bigvee_{\psi_2\in\Psi_2} S(\widehat{\psi_2} \circ \psi_2, U) \to \bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2} \circ \psi_1, U \circ U)\Big)\\ &\geq \bigwedge_{\widehat{\psi_2}\in\widehat{\Psi_2}} \bigwedge_{\psi_2\in\Psi_2} \bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2} \circ \psi_2, U) \to S(\widehat{\psi_2} \circ \psi_1, U \circ U)\\ &\geq \bigwedge_{\widehat{\psi_2}\in\widehat{\Psi_2}} \bigwedge_{\psi_2\in\Psi_2} \bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2} \circ \psi_2 \circ U, U \circ U) \to S(\widehat{\psi_2} \circ \psi_1, U \circ U)\\ &\geq \bigwedge_{\widehat{\psi_2}\in\widehat{\Psi_2}} \bigwedge_{\psi_2\in\Psi_2} \bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2} \circ \psi_1, \widehat{\psi_2} \circ \psi_2 \circ U)\\ &\geq \bigwedge_{\psi_2\in\Psi_2} \bigvee_{\psi_1\in\Psi_1} S(\psi_1, \psi_2 \circ U)\\ &= \widetilde{U}(\Psi_1, \Psi_2).\end{aligned}$$

And it is obvious that $S(U,V) \leq S(\mathring{U},\mathring{V})$, then

$$\top = \bigvee_{V\in\mathcal{U}} S(V\circ V,U) \leq \bigvee_{V\in\mathcal{U}} S(V\mathbin{\mathring{\circ}} V,\mathring{U}) \leq \bigvee_{V\in\mathcal{U}} S(\widetilde{V},\mathring{U}).$$

Therefore, $\{\mathring{U} \mid U\in\mathcal{U}\}\subseteq\widetilde{\mathcal{U}}$. So $\widetilde{\mathcal{U}}\leq\mathring{\mathcal{U}}$ $\square$

Proposition 4.11. *For each right adjoint $\Phi:(X,\mathcal{U}){\circ\!\!\!\rightarrow}(Y,\mathcal{V})$, the map*

$$R\Phi:(RY,\widetilde{\mathcal{V}})\to(RX,\widetilde{\mathcal{U}}),\quad \Psi\mapsto\Psi\circ\Phi$$

is uniformly continuous.

Proof. From Lemma 4.10, we need to check that $\mathring{\mathcal{V}}\leq R\Phi^{\circ}\circ\mathring{\mathcal{U}}\circ R\Phi$. It suffices to prove the base $\{R\Phi^{\circ}\circ\mathring{U}\circ R\Phi \mid U\in\mathcal{U}\}\subseteq\mathring{\mathcal{V}}$. Since $\widehat{\Phi}\circ\Phi\leq\mathcal{U}$ and $\mathcal{V}\circ\Phi\leq\Phi$, we have

$$\begin{aligned}
\top &= \bigvee_{\widehat{\phi}\in\widehat{\Phi}}\bigvee_{\phi\in\Phi}\Big(\bigvee_{V\in\mathcal{V}}\bigvee_{\phi'\in\Phi} S(V\circ\phi',\phi)\Big)\& S(\widehat{\phi}\circ\phi,U)\\
&= \bigvee_{\widehat{\phi}\in\widehat{\Phi}}\bigvee_{\phi,\phi'\in\Phi}\bigvee_{V\in\mathcal{V}} S(V\circ\phi',\phi)\& S(\widehat{\phi}\circ\phi,U)\\
&\leq \bigvee_{\widehat{\phi}\in\widehat{\Phi}}\bigvee_{\phi,\phi'\in\Phi}\bigvee_{V\in\mathcal{V}} S(\widehat{\phi}\circ V\circ\phi',\widehat{\phi}\circ\phi)\& S(\widehat{\phi}\circ\phi,U)\\
&\leq \bigvee_{\widehat{\phi}\in\widehat{\Phi}}\bigvee_{\phi'\in\Phi}\bigvee_{V\in\mathcal{V}} S(\widehat{\phi}\circ V\circ\phi',U).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\bigvee_{V\in\mathcal{V}} S(\mathring{V},R\Phi^{\circ}\circ\mathring{U}\circ R\Phi)\\
&= \bigvee_{V\in\mathcal{V}}\bigwedge_{\Psi_1,\Psi_2\in RY}\mathring{V}(\Psi_1,\Psi_2)\to\mathring{U}(\Psi_1\circ\Phi,\Psi_2\circ\Phi)\\
&= \bigvee_{V\in\mathcal{V}}\bigwedge_{\Psi_1,\Psi_2}\Big[\bigvee_{\widehat{\psi_2}\in\widehat{\Psi_2}}\bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2}\circ\psi_1,V)\to\bigvee_{\widehat{\alpha}\in\widehat{\Psi_2\circ\Phi}}\bigvee_{\beta\in\Psi_1\circ\Phi} S(\widehat{\alpha}\circ\beta,U)\Big]\\
&= \bigvee_{V\in\mathcal{V}}\bigwedge_{\Psi_1,\Psi_2}\Big[\bigvee_{\widehat{\psi_2}\in\widehat{\Psi_2}}\bigvee_{\psi_1\in\Psi_1} S(\widehat{\psi_2}\circ\psi_1,V)\to\bigvee_{\widehat{\phi}\in\widehat{\Phi}}\bigvee_{\widehat{\psi_2}\in\widehat{\Psi_2}}\bigvee_{\psi_1\in\Psi_1}\bigvee_{\phi'\in\Phi} S(\widehat{\phi}\circ\widehat{\psi_2}\circ\psi_1\circ\phi',U)\Big]\\
&\geq \bigvee_{V\in\mathcal{V}}\bigwedge_{\Psi_1,\Psi_2}\bigwedge_{\widehat{\psi_2}\in\widehat{\Psi_2}}\bigwedge_{\psi_1\in\Psi_1}\Big[S(\widehat{\psi_2}\circ\psi_1,V)\to\bigvee_{\widehat{\phi}\in\widehat{\Phi}}\bigvee_{\phi'\in\Phi} S(\widehat{\phi}\circ\widehat{\psi_2}\circ\psi_1\circ\phi',U)\Big]\\
&\geq \bigvee_{V\in\mathcal{V}}\bigwedge_{\Psi_1,\Psi_2}\bigwedge_{\widehat{\psi_2}\in\widehat{\Psi_2}}\bigwedge_{\psi_1\in\Psi_1}\bigvee_{\widehat{\phi}\in\widehat{\Phi}}\bigvee_{\phi'\in\Phi} S(\widehat{\psi_2}\circ\psi_1,V)\to S(\widehat{\phi}\circ\widehat{\psi_2}\circ\psi_1\circ\phi',U)\\
&\geq \bigvee_{V\in\mathcal{V}}\bigwedge_{\Psi_1,\Psi_2}\bigwedge_{\widehat{\psi_2}\in\widehat{\Psi_2}}\bigwedge_{\psi_1\in\Psi_1}\bigvee_{\widehat{\phi}\in\widehat{\Phi}}\bigvee_{\phi'\in\Phi} S(\widehat{\phi}\circ\widehat{\psi_2}\circ\psi_1\circ\phi',\widehat{\phi}\circ V\circ\phi')\to S(\widehat{\phi}\circ\widehat{\psi_2}\circ\psi_1\circ\phi',U)\\
&\geq \bigvee_{V\in\mathcal{V}}\bigvee_{\widehat{\phi}\in\widehat{\Phi}}\bigvee_{\phi'\in\Phi} S(\widehat{\phi}\circ V\circ\phi',U)\\
&= \top.
\end{aligned}$$

By the above formula, we have $R\Phi^{\circ}\circ\mathring{U}\circ R\Phi\in\mathring{\mathcal{V}}$ for all $U\in\mathcal{U}$. $\square$

Let Q-**CSepFQuinf** denote the category of T_0 separated and complete fuzzy quasi-uniform spaces. From Lemma 4.8, Lemma 4.9 and Proposition 4.11, we can define the functor F : Q-**FQunif** → Q-**CSepFQuinf** as follows:

- For $(X, \mathcal{U}) \in$ Q-**FQunif**, $F(X, \mathcal{U}) = (RX, \widetilde{\mathcal{U}})$;
- For uniformly continuous map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$, $F(f) : (RX, \widetilde{\mathcal{U}}) \to (RY, \widetilde{\mathcal{V}})$ is given by $F(f) = R(f^*) = (-) \circ f^*$.

Theorem 4.12. F : *Q*-**FQunif** → *Q*-**CSepFQuinf** *defined above is the left adjoint of the inclusion functor* i : *Q*-**CSepFQuinf** → *Q*-**FQunif**.

Proof. It is easy to see that family $\{\mathfrak{r}_X : X \to i \circ F(X) = RX\}_X$ is a natural transformation since

$$F(f) \circ \mathfrak{r}_X(x) = F(f)(x^*) = x^* \circ f^* = f(x)^* = \mathfrak{r}_Y(f(x))$$

for each uniformly continuous map $f : X \to Y$ and $x \in X$. To show $F \dashv i$, it suffices to show there exists a unique $h : (RX, \widetilde{\mathcal{U}}) \to (Y, \mathcal{V})$ such that $f = h \circ \mathfrak{r}_X$ for each $(Y, \mathcal{V}) \in$ Q-**CSepFQuinf** and each uniformly continuous map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$.

Step1 Existence: Since $(Y, \mathcal{V})$ is T_0 separated and Cauchy complete, we have $(Y, \mathcal{V}) \cong (RY, \widetilde{\mathcal{V}})$ and $\mathfrak{r}_Y$ is the isomorphic morphism. Let $h = \mathfrak{r}_Y^{-1} \circ F(f)$. Then h is the desired map.

Step2 Uniqueness: Suppose $k : (RX, \widetilde{\mathcal{U}}) \to (Y, \mathcal{V})$ is another uniformly continuous map such that $f = k \circ \mathfrak{r}_X$. We need to show $h = k$. If $h \neq k$, then there exists $\Psi_0 \in RX$ such that $h(\Psi_0) \neq k(\Psi_0)$. Since $(Y, \mathcal{V})$ is T_0 separated, we suppose there exists $V \in \mathcal{V}$ such that $V(h(\Psi_0), k(\Psi_0)) < \top$ (the case $V(k(\Psi_0), h(\Psi_0)) < \top$ is similar). By $\bigvee_{W \in \mathcal{V}} S(W \circ W, V) = \top$, we know there exists $W_0 \in \mathcal{V}$ such that $W_0 \circ W_0(h(\Psi_0), k(\Psi_0)) < \top$. Since h and k are both uniformly continuous, it follows that $h^\circ \circ W_0 \circ h \in \widetilde{\mathcal{U}}$ and $k^\circ \circ W_0 \circ k \in \widetilde{\mathcal{U}}$. Hence $\bigvee_{U_1 \in \mathcal{U}} S(\widetilde{U_1}, h^\circ \circ W_0 \circ h) = \top$ and $\bigvee_{U_2 \in \mathcal{U}} S(\widetilde{U_2}, k^\circ \circ W_0 \circ k) = \top$. Therefore,

$$\bigvee_{U_1 \in \mathcal{U}} \bigvee_{U_2 \in \mathcal{U}} S(\widetilde{U_1}, h^\circ \circ W_0 \circ h) \& S(\widetilde{U_2}, k^\circ \circ W_0 \circ k) = \top.$$

And then

$$\bigvee_{U \in \mathcal{U}} S(\widetilde{U}, h^\circ \circ W_0 \circ h) \& S(\widetilde{U}, k^\circ \circ W_0 \circ k) = \top.$$

By $\Psi_0 \in RX$, we also know that $\bigvee_{x \in X} \widetilde{U}(\Psi_0, x^*) \& \widetilde{U}(x^*, \Psi_0) = \top$. So

$$\begin{aligned}
\top &= \bigvee_{U \in \mathcal{U}} S(\widetilde{U}, h^\circ \circ W_0 \circ h) \& S(\widetilde{U}, k^\circ \circ W_0 \circ k) \\
&= \bigvee_{U \in \mathcal{U}} \top \& S(\widetilde{U}, h^\circ \circ W_0 \circ h) \& S(\widetilde{U}, k^\circ \circ W_0 \circ k) \\
&= \bigvee_{U \in \mathcal{U}} (\bigvee_{x \in X} \widetilde{U}(\Psi_0, x^*) \& \widetilde{U}(x^*, \Psi_0)) \& S(\widetilde{U}, h^\circ \circ W_0 \circ h) \& S(\widetilde{U}, k^\circ \circ W_0 \circ k) \\
&= \bigvee_{U \in \mathcal{U}} \bigvee_{x \in X} \widetilde{U}(\Psi_0, x^*) \& \widetilde{U}(x^*, \Psi_0) \& S(\widetilde{U}, h^\circ \circ W_0 \circ h) \& S(\widetilde{U}, k^\circ \circ W_0 \circ k) \\
&\leq \bigvee_{x \in X} h^\circ \circ W_0 \circ h(\Psi_0, x^*) \& k^\circ \circ W_0 \circ k(x^*, \Psi_0) \\
&= \bigvee_{x \in X} W_0(h(\Psi_0), f(x)) \& W_0(k(x^*), k(\Psi_0)) \\
&= \bigvee_{x \in X} W_0(h(\Psi_0), f(x)) \& W_0(f(x), k(\Psi_0)) \\
&\leq \bigvee_{y \in Y} W_0(h(\Psi_0), y) \& W_0(y, k(\Psi_0)) \\
&= W_0 \circ W_0(h(\Psi_0), k(\Psi_0)).
\end{aligned}$$

This contradict to $W_0 \circ W_0(h(\Psi_0), k(\Psi_0)) < \top$. Therefore, $k = h$. □

Remark 4.13. From the proof of Theorem 4.12, we know $(RX, \widetilde{\mathcal{U}})$ is exactly the T_0 completion of $(X, \mathcal{U})$. The adjunction $F \dashv i$ induces a monad $\mathbb{C} = (C, \mathfrak{r}, \mu)$ on Q-**FQunif** according to the following information:

- The functor C : Q-**FQunif** → Q-**FQunif** sends a fuzzy quasi-uniform space $(X, \mathcal{U})$ to $(RX, \widetilde{\mathcal{U}})$.
- The unit $\mathfrak{r}_X : (X, \mathcal{U}) \to C(X, \mathcal{U}) = (RX, \widetilde{\mathcal{U}})$ is the Yoneda embedding.
- The multiplication $\mu_X : C(C(X, \mathcal{U})) \to C(X, \mathcal{U})$ is the the inverse map of $\mathfrak{r}_{RX}$.

The readers can easily show that the algebra with respect to $\mathbb{C} = (C, \mathfrak{r}, \mu)$ is exactly the T_0 separated and Cauchy complete fuzzy quasi-uniform space.

5. Conclusions

In this paper, we describe fuzzy quasi-uniform spaces in the sense of Lowen and Höhle as enriched categories. We construct the Yoneda embedding in fuzzy quasi-uniform spaces through promodules. As an application of Yoneda embedding, we study the completeness and completion of fuzzy quasi-uniform spaces. When $\mathsf{Q} = [0, 1]$ and & is a continuous t-norm, from Corollary 4.5 in [44], we know "$\bigvee_{V \in \mathcal{U}} S(V \circ V, U) = 1$ for all $U \in \mathcal{U}$" can be replaced by "$U \in \mathcal{U} \Rightarrow \exists V \in \mathcal{U}$, s.t. $V \circ V \leq U$". In this case, the proofs of this paper can be simplified.

Since there are many kinds of lattice-valued quasi-uniform spaces, we want to know whether other lattice-valued quasi-uniform spaces can be viewed as enriched categories. The relationship on completeness between fuzzy quasi-uniform spaces and fuzzy quasi-metric spaces from a categorical point of view may be also an interesting question, we leave them for the future study.

Acknowledgement

The authors thank the referees gratefully for their valuable comments and suggestions which help improve the paper significantly.

References

[1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, J. Wiley & Sons, New York, 1990.
[2] R. Bělohlávek, Fuzzy Relation Systems, Foundation and Principles, Klumer Academic/Plenum Publishers, New York, Boston, Dordrecht, London, Moscow, 2002.
[3] D. Bushaw, A stability criterion for general systems, Math. Systems Theory 1 (1967) 79–88.
[4] Y. Chai, A note on the probabilistic quasi-metric spaces, J.Sichuan Univ. (Natur. Sci. Ed.) 46 (2009) 543–547.
[5] M.M. Clementino, D. Hofmann, Lawvere completeness in topology, Appl. Categ. Structures 17 (2009) 175–210.
[6] M.M. Clementino, D. Hofmann, Topological features of lax algebras, Appl. Categ. Structures 11 (2003) 267–286.
[7] M.M. Clementino, D. Hofmann, W. Tholen, One setting for all: metric, topology, uniformity, approach structure, Appl. Categ. Structures 12 (2004) 127–154.
[8] M.M. Clementino, D. Hofmann, On the completion monad via the Yoneda embedding in quasi-uniform spaces, Topol. Appl. 158 (2011) 2423–2430.
[9] J. Gutiérrez García, A unified approach to the concept of a fuzzy L-unifom space, Ph.D. Thesis, 2000.
[10] J. Gutiérrez García, On stratified L-valued filters induced by ⊤-filters, Fuzzy Sets Syst. 157 (2006) 813–819.
[11] J. Gutiérrez García, M. A. De Prada Vicente, S. Romaguera, Completeness of Hutton [0,1]-quasi-uniform spaces, Fuzzy Sets Syst. 158 (2007) 1791–1802.
[12] J. Gutiérrez García, J. Rodriguez-Lopez, S. Romaguera, On fuzzy uniformities induced by a fuzzy metric space, Fuzzy Sets Syst. 330 (2018) 52–78.
[13] J. Gutiérrez García, Charactistic values of ⊤-filters, Fuzzy Sets Syst. 156(1) (2005) 55–67.
[14] J. He, H. Lai, L. Shen, Towards probabilistic partial metric spaces: Diagonals between distance distributions, Fuzzy Sets Syst. 370 (2019), 99–119.
[15] D. Hofmann, W. Tholen, Lawvere completion and separation via closure, Appl. Categ. Structures 18 (2010) 259–287.
[16] D. Hofmann, C. D. Reis, Probabilistic metric spaces as enriched categories, Fuzzy Sets Syst. 210 (2013) 1–21.
[17] D. Hofmann, G.J. Seal, W. Tholen, Monoidal Topology, a Categorical Approach to Order, Metric, and Topology, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2014.
[18] U. Höhle, Probabilistic topologies induced by L-fuzzy uniformities, Manuscripta Math 38 (1982) 289–323.

[19] U. Höhle, Probabilistic metrization of fuzzy uniformities, Fuzzy Sets and Systems 8 (1982) 63–69.
[20] U. Höhle, Lineare probabilistisch topologische Raüme, Habilitations Thesis (Unpublished).
[21] B. Hutton, Uniformities on fuzzy topological spaces, J. Math. Anal. Appl. 58 (1977) 557–571.
[22] H. Lai, D. Zhang, Fuzzy topological spaces with conical neighborhood systems, Fuzzy Sets Syst. 330 (2018) 87–104.
[23] H. Lai, W. Tholen, Quantale-valued topological spaces via closure and convergence, Topology Appl. 230 (2017) 599–620.
[24] H. Lai, D. Zhang, G. Zhang, The saturated prefilter monad, Topology Appl. (2020) https://doi.org/10.1016/j.topol.2020.107525.
[25] A.K. Katsaras, On fuzzy uniform spaces, J. Math. Anal. Appl. 101 (1984) 97–113.
[26] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Mathematical Soceity Lecture Notes Series 64, Cambridge University Press,1982.
[27] D.C. Kent, On the scale of a uniform space, Invent. Math. 4 (1967) 159–164.
[28] H.P. A. K*ü*nzi, Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology, in: C. E. Aull, R. Lowen (Eds.), Handbook of the History of General Topology, vol.3, Kluwer, Dordrecht, 2001, pp.853–968.
[29] F.W. Lawvere, Metric spaces, generalized logic, and closed categories, Rend. Sem. Mat. Fis. Milano 43 (1973) 135–166.
[30] W. F. Lindgren, P. Fletcher, A construction of the pair completion of a quasi-uniform space, Can. Math. Bull. 21 (1978) 53–59.
[31] R. Lowen, Convergence in fuzzy topological spaces, Gen. Topol. Appl. 10 (1979), 147–160.
[32] R. Lowen, Fuzzy uniform spaces, J. Math. Anal. Appl. 82 (1981) 370–385.
[33] R. Lowen, Fuzzy neighbourhood spaces, Fuzzy Sets Syst. 7 (1982) 165–189.
[34] R. Lowen, P. Wuyts. Completeness, compactness and precompactness in fuzzy uniform spaces: part I and part II, J. Math. Anal. Appl. 90 (1982) 563–581; 92 (1983), 342–371.
[35] S. Salbany, The completion monad and its algebra, Comment. Math. Univ. Carol. 23 (1982) 301–311.
[36] V. Schmitt, Enriched categories and quasi-uniform spaces, Elect. Notes Theoret. Comput. Sci. 73 (2004) 165–205.
[37] F.G. Shi, J. Zhang, C.Y. Zheng, L-proximities and totally bounded pointwise L-uniformities, Fuzzy Sets Syst. 133 (2003) 321–331.
[38] J.L. Sieber, W.J. Pervin, Completeness in quasi-uniform spaces, Math. Ann. 158 (1965) 79–81.
[39] K.R. Wagner, Liminf convergence in Ω-categories, Theoretical Computer Sci. 184 (1997) 61–104.
[40] P. Wuyts, R. Lowen, Separation axioms for fuzzy topological spaces, fuzzy neighborhood spaces and fuzzy uniform spaces, J. Math. Anal Appl. 93 (1983) 21–41.
[41] Y. Yue, J. Fang, Uniformities in fuzzy metric spaces, Iran. J. Fuzzy Syst. 12 (2015) 43–57.
[42] Y. Yue, J. Fang, Completeness in probabilistic quasi-uniform spaces, Fuzzy Sets Syst. 370 (2019) 34–62.
[43] Y. Yue, J. Fang, The $\top$-filter monad and its applications, Fuzzy Sets Syst. 382 (2020) 79–97.
[44] D. Zhang, A comparison of various uniformities in fuzzy topology, Fuzzy Sets Syst. 140 (2003) 399–422.
[45] D. Zhang, An enriched category approach to many valued topology, Fuzzy Sets Syst. 158 (2007) 349–366.
[46] D. Zhang, Uniform environments as a general framework for metrics and uniformities, Fuzzy Sets Syst. 159 (2008) 559–572.