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Some New General Lower Bounds for Mixed Metric Dimension of Graphs

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Abstract. A vertex $w \in V$ resolves two elements $x, y \in V \cup E$ if $d(w, x) \neq d(w, y)$. The mixed resolving set is a set of vertices S, $S \subseteq V$ if any two elements of $E \cup V$ are resolved by some element of S. The minimum cardinality of a mixed resolving set is called the mixed metric dimension of a graph G. This paper introduces three new general lower bounds for the mixed metric dimension of a graph. The exact values of mixed metric dimension for torus graph are determined using one of these lower bounds. Finally, some illustrative examples of these new lower bounds and those known in the literature are presented on a set of some well-known graphs.

1. Introduction

Let $G = (V, E)$ be a connected simple graph. Distance between pairs of vertices is measured by the number of edges in a shortest $u - v$ path in G where $u, v \in V$, and denoted as $d_G(u, v)$. A vertex w *resolves* vertices u and v if $d_G(w, u) \neq d_G(w, v)$. Set of a graph $S \subseteq V$ is a *resolving set* if any pair of vertices from V are resolved by some element from S. For an ordered set S as $S = \{w_1, \ldots, w_k\}$ and arbitrary vertex $u \in V$ we can determine the vector of resolving coordinates denoted by $r_S(u) = (d_G(u, w_1), \ldots, d_G(u, w_k))$. In this context, S is a resolving set if every vertex u has the unique vector of resolving coordinates. A resolving set of minimum cardinality is called *the metric basis* and its cardinality *the metric dimension* of G. The metric dimension of G is denoted as $\beta(G)$. The term resolving set was introduced by Harrary and Melter [1], while Slater used the term *locating set* [2]. In the literature synonymous to these terms is also *metric generators*. In order to simplify the notation we replace $d_G(u, v)$ with $d(u, v)$.

As the metric dimension and resolving sets give some information about vertices of the graph, it is natural to ask if there is some parameter, or graph invariant, which deals in the same way with graph's edges. Answer to that question was given by Kelenc et al. in [3], where authors introduced the edge metric dimension of graphs. The distance between vertex and edge in a graph was defined as the minimum distance between given vertex and endpoints of a given edge. Formally, if $w \in V$ and $e = \{u, v\}$ then

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$$d(w, e) = \min(d(w, u), d(w, v)). \quad (1)$$

Now, the vertex w resolves two edges e_1 and e_2 if $d(w, e_1) \neq d(w, e_2)$. Similarly as for the metric dimension, the edge resolving set $S \subseteq V$ is defined as a set of vertices such that for any pair of edges from E, there is some element in S that resolves them. The minimum edge resolving set is the edge metric basis and its cardinality is called the edge metric dimension of a graph G and is denoted as $\beta_E(G)$.

Finally, as there is the metric and the edge metric dimension of a graph G, Kelenc et al. (2017) in [4] recently introduced a concept of mixed metric dimension and initiated the study of its mathematical properties. It is said that vertex w resolves two items $x, y \in V \cup E$ if $d(w, x) \neq d(w, y)$. The mixed resolving set $S \subseteq V$ is defined as a set such that for any pair of elements from $V \cup E$ there is some element in S that resolves them. Following the earlier definitions, the mixed metric basis and the mixed metric dimension are defined as the minimum mixed resolving set and the cardinality of such minimum resolving set, respectively. The mixed metric dimension of G is denoted as $\beta_M(G)$.

Example 1.1. *Graph G from Figure* 1 *has 5 vertices and 7 edges. The mixed metric dimension is equal 5 and its obtained by total enumeration, while metric dimension and edge metric dimension are equal 3 and 4, respectively. The mixed metric basis is* $\{v_1, v_2, v_3, v_4, v_5\}$, *i.e. all vertices are elements of the basis and deleting any element from basis, it will always exist two items with the same coordinates.*

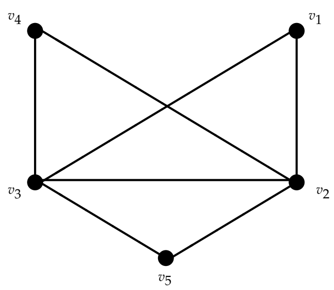

Figure 1: Small graph G with 5 vertices

1.1. Literature review

There are several proposed applications for the metric dimension in the literature. Originally, Slater considered unique recognition of intruders in the network, while others observed problems of navigating robots in networks [5], chemistry [6, 7], some applications in pattern recognition and image processing [8]. After initial works, some variations of this problem were introduced such as resolving dominating sets [9], independent resolving sets [10], strong metric dimension [11], local metric dimension [12] among others.

In the article [4] where the mixed metric dimension problem was introduced, the authors presented some facts considering structure of mixed resolving sets. In the cases were upper and lower bounds of mixed metric dimension could be easily obtained, the authors presented characterization of graphs whose mixed metric dimension reaches these bounds. Since lower and upper bounds are 2 and n, the authors have shown that for paths mixed metric dimension is 2, and mixed metric dimension is equal n if and only if every vertex has a maximal neighbor.

In order to better present mixed metric dimensions, the authors in [4] determined exact values for some classes of graphs, notably cycles, trees, complete bipartite graphs and grid graphs. Moreover, few general lower/upper bounds are presented. Finally, the authors proved that computing mixed metric dimension is NP-hard in general case.

An integer linear programming (ILP) formulation of the mixed metric problem is given in the paper [4]. ILP naturally produces a lower bound, notably the LP relaxation of the given problem.

Some other classes of graphs also attracted attention of researchers. Raza et al. (2019) in [13] gave the exact value of mixed metric dimension for three well-known classes of graphs: prism, antiprism and graph of convex polytope R_n. Milivojević Danas in [14], provided the exact results for two other important well-known classes of graphs: flower snarks and wheels.

1.2. Definitions and properties

Let us denote deg_v as degree of vertex $v \in V$ and $\delta(G)$ as the minimum degree of vertices in G, i.e. $\delta(G) = \min\{deg_v | v \in V\}$.

Torus graph $T_{m,n}$ can be defined as follows:

$$V(T_{m,n}) = \{(i, j) \,|\, 0 \le i \le m-1, 0 \le j \le n-1\}$$

and

$$E(T_{m,n}) = \{(i, j)(i+1, j) \,|\, 0 \le i \le m-2, 0 \le j \le n-1\} \bigcup$$
$$\{(i, j)(i, j+1) \,|\, 0 \le i \le m-1, 0 \le j \le n-2\} \bigcup$$
$$\{(m-1, j)(0, j) \,|\, 0 \le j \le n-1\} \bigcup \{(i, n-1)(i, 0) \,|\, 0 \le i \le m-1\}.$$

Torus graph can be also shown as Cartesian product of two cycles, i.e. $T_{m,n} = C_m \square C_n$.

Following propositions, theorems and their corollaries are used in rest of the paper to prove validity of new general lower bounds. We cite them as they were stated in articles [4, 15].

Proposition 1.2. ([4]) *For any graph G it holds*

$$\beta_M(G) \ge \max\{\beta(G), \beta_E(G)\}. \tag{2}$$

For two vertices $u, v \in G$, we say that they are *false twins* if they have the same open neighborhoods, i.e., $N(u) = N(v)$. The vertices u, v are called *true twins* if $N[u] = N[v]$. Also, vertex v is called an *extreme vertex* if $N(v)$ induces a complete graph. Further on, an arbitrary vertex $u \in N(v)$ is called a *maximal neighbor* of the vertex v if all neighbors of vertex v, including itself, are also in the closed neighborhood of u.

Proposition 1.3. ([4]) *If u, v are true twins in a graph G, then u, v belong to every mixed metric generator for G.*

Proposition 1.4. ([4]) *If u, v are false twins in a graph G and S is a mixed metric generator for G, then $\{u, v\} \cap S \ne \emptyset$.*

Proposition 1.5. ([4]) *If u is a simplicial (extreme) vertex in a graph G, then u belongs to every mixed metric generator for G.*

Corollary 1.6. ([4]) *If u is a vertex of degree 1 in a graph G, then u belongs to every mixed metric generator for G.*

Proposition 1.7. ([3]) *Let G be a connected graph and let $\triangle(G)$ be the maximum degree of G. Then*

$$\beta_E(G) \ge \lceil \log_2 \triangle(G) \rceil. \tag{3}$$

Theorem 1.8. ([15]) *Let G be a connected graph, then*

$$\beta_E(G) \ge 1 + \lceil \log_2 \delta(G) \rceil. \tag{4}$$

2. New general lower bounds

First, we introduce a lower bound for mixed metric dimension of any connected graph G, which slightly improves the lower bound for edge metric dimension from Theorem 1.8 given in [15].

Theorem 2.1. *Let G be a connected graph and let x be an arbitrary vertex from mixed resolving set S of G. Then,* $|S| \geq 1 + \lceil \log_2(1 + \deg_x) \rceil$.

Proof. Without loss of generality we can assume that $S = \{x, w_2, \ldots, w_p\}$. Vector of metric coordinates of vertex x with respect to S is $r(x, S) = (0, d_2, \ldots, d_p)$, where $d_i = d(w_i, x)$, for all $i, 2 \leq i \leq p$. Vertex x is incident to deg_x edges. Name them as $e_1, \ldots, e_{deg_x}$. For each position $i = 2, \ldots, p$ in the ordering of S and each index $j = 1, \ldots, deg_x$, edge e_j is incident to vertex x, so by the definition of $d(e_j, w_i)$ it directly implies $d(e_j, w_i) \in \{d_i - 1, d_i\}$, i.e. there can be only two different possible distances. Therefore vertex x and edges $e_1, \ldots, e_{deg_x}$ have at most 2^{p-1} different mixed metric representations with respect to S, implying $1 + deg_x \leq 2^{p-1}$ and $p \geq 1 + \log_2(1 + \deg_x)$ follows. Having in mind that $p = |S|$ is an integer, we have $|S| \geq 1 + \lceil \log_2(1 + \deg_x) \rceil$. □

Next corollary describes vertices which cannot be members of mixed resolving sets:

Corollary 2.2. *Let G be a connected graph and let v be an arbitrary vertex $v \in V$. If $deg_v > 2^{\beta_M(G)-1} - 1$, then v is not a member of any mixed resolving set S of cardinality $\beta_M(G)$ of G.*

Since $\delta(G)$ is the minimum degree of vertices in G we have another corollary of Theorem 2.1:

Corollary 2.3. *Let G be a connected graph, then*

$$\beta_M(G) \geq 1 + \lceil \log_2(\delta(G) + 1) \rceil. \tag{5}$$

Example 2.4. *For the graph given in Example 1.1 this lower bound is calculated as follows: Since the minimum degree of vertices $\delta(G) = 2$ and from the inequality (5), the lower bound is equal 3.*

For regular graphs, next corollary holds:

Corollary 2.5. *Let G be an r-regular graph, then*

$$\beta_M(G) \geq 1 + \lceil \log_2(r + 1) \rceil. \tag{6}$$

In the following consideration, we propose a relationship between the mixed metric dimension and the minimum hitting set problem. Let us first define the hitting set H. For a given set U and a collection $\mathcal{T}$ of subsets $S_1, \ldots, S_m$ of U such that their union is equal to U, the hitting set H is a set which has a nonempty intersection with each set from this collection, i.e. $(\forall i \in \{1, \ldots, m\})\, H \cap S_i \neq \emptyset$. Finding a hitting set of minimum cardinality is called the minimum hitting set problem (MHSP). It should be noted that minimum hitting set problem is equivalent to a famous *set covering problem*.

For an arbitrary edge uv we define sets $W_{uv} = \{w \in V | d(u, w) < d(v, w)\}$ and $W_{vu} = \{w \in V | d(u, w) > d(v, w)\}$. The relationship between these sets and mixed resolving sets are given in the following lemma.

Lemma 2.6. *Let G be a connected graph, $uv \in E$ an arbitrary edge and S a mixed resolving set, then*

a) $W_{vu} \cap S \neq \emptyset$;

b) $W_{uv} \cap S \neq \emptyset$.

Proof. *a*) Suppose the opposite, i.e. $(\exists uv \in E)$ so that $W_{vu} \cap S = \emptyset$, which means that for each vertex $w \in S$ holds $d(u, w) \leq d(v, w)$. According to equality (1), it is easy to see that mixed metric coordinate of u is same as the mixed metric coordinate of uv, i.e. $r(u, S) = r(uv, S)$, which means that S is not a mixed resolving set which is contradiction to the starting assumption.

b) The proof of this part of lemma is analogous to the proof of part *a*). □

To each edge $uv \in E$, we assign sets W_{uv} and W_{vu} and it is easy to see that there are $2m$ sets. Our idea is to find a minimal hitting set H^* for the family of sets $\{W_{vu}, W_{uv} | uv \in E\}$. The cardinality of minimum hitting set of this family of sets will be denoted as $MHSP(\{W_{uv}, W_{vu} | uv \in E\})$.

The following theorem proposes the new lower bound for mixed metric dimension based on the cardinality of above-mentioned minimum hitting set.

Theorem 2.7. *For any connected graph G, it holds*

$$\beta_M(G) \geq MHSP(\{W_{uv}, W_{vu} | uv \in E\}). \tag{7}$$

Proof. Let $\beta_M(G)$ be the mixed metric dimension of an arbitrary graph G. Then, there is a mixed resolving set S so that $|S| = \beta_M(G)$. From Lemma 2.6, for arbitrary edge uv, it follows that

$$W_{uv} \cap S \neq \emptyset \wedge W_{vu} \cap S \neq \emptyset. \tag{8}$$

This means that there is at least one element from S in each of these sets for every edge uv.

Let us now consider minimal hitting set problem over family of these sets W_{uv} and W_{vu} where $uv \in E$. Since a mixed resolving set S satisfies (8), S is a hitting set for a family of sets $\{W_{vu}, W_{uv} | uv \in E\}$. The cardinality of each hitting set is greater or equal to the cardinality of minimal hitting set, so it can be concluded that inequality (7) holds. □

Example 2.8. *The lower bound from Theorem* 2.7*, for the graph from the Figure* 1*, obtained by using total enumeration for hitting set problem is equal 5.*

Let us present another lower bound for mixed metric dimension based on the diameter of a graph, where the diameter of graph is defined as the maximum distance between the pair of vertices.

Theorem 2.9. *Let $G = (V, E)$ be a connected graph with mixed metric dimension $\beta_M(G)$ and let $D(G)$ be the diameter of G, then*

$$|V| + |E| \leq D(G)^{\beta_M(G)} + \beta_M(G)(\triangle(G) + 1). \tag{9}$$

Proof. We will consider all possible representations of metric coordinates for all items of the graph G. Since the diameter of graph is $D(G)$, then it is easy to see that each item of graph can have integer coordinates between 0 and $D(G)$. The set of all items can be separated into two disjunctive classes:

I) items whose metric coordinates do not have 0;

II) items whose metric coordinates have 0.

Each item from *I*) class, which does not have a coordinate equal 0, must have unique coordinates from one of $D(G)^{\beta_M(G)}$ possibilities. For items from *II*), i.e. with one coordinate equal to zero, it is easy to see that it will be the vertex which is an element of the basis, or an edge containing that vertex. Hence, for each element of the basis, there are at the most $\triangle(G) + 1$ possibilities, i.e. it must have a unique coordinate from one of $\beta_M(G)(\triangle(G) + 1)$ possibilities. Therefore, from the previous, it is easy to conclude that inequality (9) follows, thus completing the proof of theorem. □

Example 2.10. *The lower bound from Theorem* 2.9*, for the graph from the Figure* 1*, is obtained by calculating inequalities (9) and it is equal 2.*

3. Exact results on torus graph

In this section we will use previously introduced general lower bounds to obtain the exact values of mixed metric dimension of torus graph.

Theorem 3.1. *For $m, n \geq 3$ it holds $\beta_M(T_{m,n}) = 4$.*

Proof. We will prove that both upper and lower bounds for $\beta_M(T_{m,n})$ is equal to 4.

Step 1: *Upper bound is 4.*

There are four cases which are identified by parity of the torus dimensions:

Case 1. $m = 2k + 1, n = 2l + 1$

Let $S = \{(0,0), (0,l), (1,l+1), (k+1,l+1)\}$. Let us prove that S is mixed metric resolving set. The representation of coordinates of each vertex and each edge,with respect to S, is shown in Table 1 and Table 2.

Table 1: Metric coordinates of vertices of $T_{2k+1,2l+1}$

vetex	cond.	$r(v,S)$
$(0,0)$		$(0,l,l+1,l+k)$
$(i,0)$	$1 \leq i \leq k$	$(i,i+l,l+i-1,l+k-i+1)$
$(0,j)$	$1 \leq j \leq l$	$(j,l-j,l-j+2,l-j+k+1)$
(i,j)	$1 \leq i \leq k-1$, $1 \leq j \leq l$	$(j+i,i+l-j,l-j+i,l-j+k-i+2)$
$(0,j)$	$l+1 \leq j \leq n-1$	$(n-j,j-l,j-l,j-l+k-1)$
(i,j)	$l+1 \leq j \leq n-1$, $1 \leq i \leq k-1$	$(n-j+i,j-l+i,j-l+i-2,k-i+j-l)$
$(i,0)$	$k+2 \leq i \leq m-1$	$(m-i,m-i+l,m-i+l+1,l-k+i-1)$
(i,j)	$k+2 \leq i \leq m-1$, $1 \leq j \leq l$	$(m-i+j,m-i+l-j,m-i+l-j+2,i-k+l-j)$
(i,j)	$k+2 \leq i \leq m-1$, $l+1 \leq j \leq n-1$	$(m+n-i-j,j-l+m-i,$ $j-l-i+m,j-l+i-k-2)$
$(k+1,0)$		$(k,k+l,k+l,l)$
(k,j)	$1 \leq j \leq l$	$(k+j,k+l-j,k+l-j,l-j+2)$
$(k+1,j)$	$1 \leq j \leq l$	$(k+j,k+l-j,k+l-j+1,l-j+1)$
(k,j)	$l+1 \leq j \leq n-1$	$(k+n-j,j-l+k,k+j-l-2,j-l)$
$(k+1,j)$	$l+1 \leq j \leq n-1$	$(k+n-j,k+j-l,k+j-l-1,j-l-1)$

Table 2: Metric coordinates of edges of $T_{2k+1,2l+1}$

edge	cond.	$r(e,S)$
$(0,0)(1,0)$		$(0,l,l,k+l)$
$(0,0)(0,n-1)$		$(0,l,l,k+l-1)$
$(0,0)(m-1,0)$		$(0,l,l+1,l+k-1)$
$(0,j)(0,j+1)$	$0 \leq j \leq l-1$	$(j,l-j-1,l-j+1,k+l-j)$
$(i,0)(i+1,0)$	$1 \leq i \leq k-1$	$(i,l+i,l+i-1,k-i+l)$
$(i,j)(i,j+1)$	$1 \leq i \leq k$, $1 \leq j \leq l-1$	$(i+j,l-j-1+i,l+i-j-1,l-j+k-i+1)$
$(i,l)(i,l+1)$	$1 \leq i \leq k$	$(l+i,i,i-1,k-i+1)$
$(i,0)(i,1)$	$1 \leq i \leq k$	$(i,l+i-1,l+i-1,l+k-i+1)$
$(0,j)(1,j)$	$1 \leq j \leq l$	$(j,l-j,l-j+1,k+l-j+1)$
$(0,j)(0,j+1)$	$l+1 \leq j \leq n-2$	$(n-j-1,j-l,j-l,k+j-l-1)$
$(i,j)(i,j+1)$	$1 \leq i \leq k$, $l+1 \leq j \leq n-2$	$(n-j+i-1,j-l+i,$ $j-l+i-2,k+j-l-i)$
$(i,j)(i+1,j)$	$1 \leq i \leq k-1$, $1 \leq j \leq l$	$(i+j,l-j+i,l-j+i,l-j+k-i+1)$
$(i,0)(i,n-1)$	$1 \leq i \leq k$	$(i,l+i,l+i-2,l+k-i)$
$(i,j)(i+1,j)$	$1 \leq i \leq k-1$, $l+1 \leq j \leq n-1$	$(n-j+i,j-l+i,$ $j-l+i-2,k-i+j-l-1)$
$(0,j)(1,j)$	$l+1 \leq j \leq n-1$	$(n-j,j-l,j-l-1,k+j-l-1)$
$(i,j)(i+1,j)$	$k+1 \leq i \leq m-2$, $1 \leq j \leq l$	$(j+m-i-1,l-j+m-i-1,$ $m-i+l-j+1,l-j+i-k)$
$(i,j)(i,j+1)$	$k+2 \leq i \leq m-1$, $1 \leq j \leq l-1$	$(m-i+j,l-j-1+m-i,$ $l-j+m-i,l-j+i-k-1)$
$(i,0)(i+1,0)$	$k+1 \leq i \leq m-2$	$(m-i-1,m-i+l-1,m-i+l,i-k-1+l)$
$(i,0)(i,1)$	$k+2 \leq i \leq m-1$	$(m-i,m-i+l-1,m-i+l+1,l+i-k-1)$
$(k,j)(k+1,j)$	$1 \leq j \leq l$	$(k+j,k+l-j,k+l-j,l-j+1)$
$(0,j)(m-1,j)$	$1 \leq j \leq l$	$(j,l-j,l-j+2,k+l-j)$
$(k+1,j)(k+1,j+1)$	$1 \leq j \leq l-1$	$(k+j,k+l-j-1,k+l-j,l-j)$
$(i,j)(i+1,j)$	$k+1 \leq i \leq m-2$, $l+1 \leq j \leq n-1$	$(n-j+m-i-1,m-i+j-l-1,$ $m-i+j-l-1,i-k+j-l-2)$
$(i,j)(i,j+1)$	$k+2 \leq i \leq m-1$, $l+1 \leq j \leq n-2$	$(n-j-1+m-i,j-l+m-i,$ $j-l+m-i+1,j-l+i-k-2)$
$(i,l)(i,l+1)$	$k+2 \leq i \leq m-1$	$(m-i+l,m-i,m-i+1,i-k-1)$
$(k,j)(k+1,j)$	$l+1 \leq j \leq n-1$	$(n-j+k,j-l+k,k+j-l-2,j-l-1)$
$(i,0)(i,n-1)$	$k+2 \leq i \leq m-1$	$(m-i,m-i+l,m-i+l,l+i-k-2)$
$(0,j)(m-1,j)$	$l+1 \leq j \leq n-1$	$(n-j,j-l,j-l,k+j-l-2)$
$(0,l)(0,l+1)$		$(l,0,1,k)$
$(k,0)(k+1,0)$		$(k,k+l,k+l-1,l)$
$(k+1,0)(k+1,1)$		$(k,k+l-1,k+l,l)$
$(k+1,0)(k+1,n-1)$		$(k,k+l,k+l-1,l-1)$
$(k+1,j)(k+1,j+1)$	$l+1 \leq j \leq n-2$	$(n-j+k-1,j-l+k,j-l+k-1,j-l-1)$
$(k+1,l)(k+1,l+1)$		$(k+l,k,k,0)$

Since metric coordinates of all items are mutually different, S is a mixed resolving set. Therefore, $\beta_M(T_{2k+1,2l+1}) \leq 4$.

Case 2. $m = 2k + 1, n = 2l$

Let $S = \{(0, 0), (0, l), (1, 0), (k + 1, 1)\}$. Let us prove that S is mixed metric resolving set. The representation of coordinates of each vertex and each edge, with respect to S, is shown in Table 3 and Table 4.

Table 3: Metric coordinates of vertices of $T_{2k+1,2l}$

vetex	cond.	$r(v, S)$
$(0,0)$		$(0, l, 1, k+1)$
$(i,0)$	$1 \leq i \leq k$	$(i, i+l, i-1, k-i+2)$
$(0,j)$	$1 \leq j \leq l$	$(j, l-j, j+1, k+j-1)$
(i,j)	$1 \leq i \leq k$ $1 \leq j \leq l$	$(j+i, l-j+i, j+i-1, j+k-i)$
$(0,j)$	$l+1 \leq j \leq n-1$	$(n-j, j-l, n-j+1, n-j+k+1)$
(i,j)	$l+1 \leq j \leq n-1$ $1 \leq i \leq k$	$(n-j+i, j-l+i, n-j+i-1, n-j+k-i+2)$
$(i,0)$	$k+2 \leq i \leq m-1$	$(m-i, m-i+l, m-i+1, i-k)$
$(k+1,0)$		$(k, k+l, k, 1)$
(i,j)	$k+2 \leq i \leq m-1$ $1 \leq j \leq l$	$(m-i+j, m-i-j+l, m-i+j+1, i-k+j-2)$
$(k+1,j)$	$1 \leq j \leq l$	$(k+j, k+l-j, k+j, j-1)$
(i,j)	$k+2 \leq i \leq m-1$ $l+1 \leq j \leq n-1$	$(m-i+n-j, m-i+j-l,$ $m-i+1+n-j, n+i-k-j)$
$(k+1,j)$	$l+1 \leq j \leq n-1$	$(n-j+k, n-j+1, j-l+k, n-j+1)$

Table 4: Metric coordinates of edges of $T_{2k+1,2l}$

edge	cond.	$r(e, S)$
$(0,0)(0,1)$		$(0, l-1, 1, k)$
$(0,0)(1,0)$		$(0, l, 0, k+1)$
$(0,0)(0,n-1)$		$(0, l-1, 1, k+1)$
$(0,0)(m-1,0)$		$(0, l, 1, k)$
$(0,j)(0,j+1)$	$1 \leq j \leq l-1$	$(j, l-j-1, j+1, k+j-1)$
$(i,0)(i+1,0)$	$1 \leq i \leq k$	$(i, i+l, i-1, k-i+1)$
$(i,j)(i,j+1)$	$1 \leq i \leq k$ $1 \leq j \leq l-1$	$(i+j, i+l-j-1, j+i-1, k+j-i)$
$(i,l)(i,l+1)$	$1 \leq i \leq k$	$(l+i-1, i, l+i-2, k-i+l)$
$(i,0)(i,1)$	$1 \leq i \leq k$	$(i, l-1+i, i-1, k-i+1)$
$(0,j)(1,j)$	$1 \leq j \leq l$	$(j, l-j, j, k+j-1)$
$(0,j)(0,j+1)$	$l+1 \leq j \leq n-2$	$(n-j-1, j-l, n-j, k+n-j)$
$(i,j)(i,j+1)$	$1 \leq i \leq k$ $l+1 \leq j \leq n-2$	$(n-j-1+i, j-l+i, n-j-2+i, k+n-j-i+1)$
$(i,j)(i+1,j)$	$1 \leq i \leq k$ $1 \leq j \leq l$	$(i+j, i+l-j, j+i-1, k-i+j-1)$
$(i,0)(i,n-1)$	$1 \leq i \leq k$	$(i, i+l-1, i-1, k-i+2)$
$(i,j)(i+1,j)$	$1 \leq i \leq k$ $l+1 \leq j \leq n-1$	$(n-j+i, j-l+i,$ $n-j+i-1, n+k-i-j+1)$
$(0,j)(1,j)$	$l+1 \leq j \leq n-1$	$(n-j, j-l, n-j, n-j+k+1)$
$(i,j)(i+1,j)$	$k+1 \leq i \leq m-2$ $1 \leq j \leq l$	$(m-i+j-1, m-i+l-j-1,$ $m-i+j, i-k+j-2)$
$(i,j)(i,j+1)$	$k+2 \leq i \leq m-1$ $1 \leq j \leq l-1$	$(m-i+j, m-i+l-j-1,$ $m-i+1+j, i-k+j-2)$
$(k+1,j)(k+1,j+1)$	$1 \leq j \leq l-1$	$(k+j, k+l-j-1, k+j, j-1)$
$(i,0)(i+1,0)$	$k+2 \leq i \leq m-2$	$(m-i-1, m-i+l-1, m-i, i-k)$
$(i,0)(i,1)$	$k+2 \leq i \leq m-1$	$(m-i, m-i-1+l, m-i+1, i-k-1)$
$(k+1,0)(k+1,1)$		$(k, k+l-1, k, 0)$
$(0,j)(m-1,j)$	$1 \leq j \leq l$	$(j, l-j, j+1, k+j-2)$
$(i,j)(i+1,j)$	$k+1 \leq i \leq m-2$ $l+1 \leq j \leq n-1$	$(n-j+m-i-1, j-l+m-i-1,$ $m+n-j-i, n-j+i-k)$
$(i,j)(i,j+1)$	$k+2 \leq i \leq m-1$ $l+1 \leq j \leq n-2$	$(n-j+m-i-1, m+j-i-l,$ $n-j+m-i, n-j+i-k-1)$
$(k+1,j)(k+1,j+1)$	$l+1 \leq j \leq n-2$	$(n+k-j-1, k+j-l,$ $k+n-j-1, n-j)$
$(i,l)(i,l+1)$	$k+2 \leq i \leq m-1$	$(l+m-i-1, m-i, l+m-i, l+i-k-2)$
$(k+1,l)(k+1,l+1)$		$(l+k-1, k, l+k-1, l-1)$
$(i,0)(i,n-1)$	$k+2 \leq i \leq m-1$	$(m-i, m-i+l-1, m-i+1, i-k)$
$(k+1,0)(k+1,n-1)$		$(k, l+k-1, k, 1)$
$(0,j)(m-1,j)$	$l+1 \leq j \leq n-1$	$(n-j, j-l, n-j+1, k+n-j)$
$(0,l)(0,l+1)$		$(l-1, 0, l, k+l-1)$
$(k+1,0)(k+2,0)$		$(k-1, k+l-1, k, 1)$

Since metric coordinates of all items are mutually different, so S is a mixed resolving set. Therefore, $\beta_M(T_{2k+1,2l}) \leq 4$.

Case 3. $m = 2k, n = 2l + 1$

Let $S = \{(0, 0), (k, 0), (0, 1), (1, l + 1)\}$. Since $C_m \square C_n$ is the same as $C_n \square C_m$, the proof of this case is similar to the proof of Case 2.

Case 4. $m = 2k, n = 2l$

Let $S = \{(0, 0), (0, 1), (1, l), (k, 0)\}$. Let us prove that S is mixed metric resolving set. The representation of coordinates of each vertex and each edge, with respect to S, is shown in Table 5 and Table 6.

Table 5: Metric coordinates of vertices of $T_{2k,2l}$

vetex	cond.	$r(v,S)$
$(0,0)$		$(0,1,l+1,k)$
$(i,0)$	$1 \le i \le k$	$(i,i+1,l+i-1,k-i)$
$(0,j)$	$1 \le j \le l$	$(j,j-1,l-j+1,k+j)$
(i,j)	$1 \le i \le k$ $1 \le j \le l$	$(j+i,j-1+i,l-j+i-1,j+k-i)$
$(0,j)$	$l+1 \le j \le n-1$	$(n-j,n-j+1,j-l+1,n-j+k)$
(i,j)	$l+1 \le j \le n-1$ $1 \le i \le k$	$(n-j+i,n-j+i+1,j-l+i-1,n-j+k-i)$
$(i,0)$	$k+1 \le i \le m-1$	$(m-i,m-i+1,m-i+l+1,i-k)$
(i,j)	$k+1 \le i \le m-1$ $1 \le j \le l$	$(m-i+j,m-i+j-1,m-i+l-j+1,i-k+j)$
(i,j)	$k+1 \le i \le m-1$ $l+1 \le j \le n-1$	$(m+n-i-j,m+n-i-j+1,$ $m+j-l-i+1,n+i-k-j)$

Table 6: Metric coordinates of edges of $T_{2k,2l}$

edge	cond.	$r(e,S)$
$(0,0)(0,1)$		$(0,0,l,k)$
$(0,0)(1,0)$		$(0,1,l,k-1)$
$(0,0)(0,n-1)$		$(0,1,l,k)$
$(0,0)(m-1,0)$		$(0,1,l+1,k-1)$
$(0,j)(0,j+1)$	$1 \le j \le l-1$	$(j,j-1,l-j,k+j)$
$(i,0)(i+1,0)$	$1 \le i \le k-1$	$(i,i+1,l+i-1,k-i-1)$
$(i,j)(i,j+1)$	$1 \le i \le k$ $1 \le j \le l-1$	$(i+j,i+j-1,l-j+i-2,k+j-i)$
$(i,0)(i,1)$	$1 \le i \le k$	$(i,i,l+i-2,k-i)$
$(0,j)(1,j)$	$1 \le j \le l$	$(j,j-1,l-j,k+j-1)$
$(0,j)(0,j+1)$	$l+1 \le j \le n-2$	$(n-j-1,n-j,j-l+1,n-j-1+k)$
$(i,j)(i,j+1)$	$1 \le i \le k$ $l+1 \le j \le n-2$	$(n-j+i-1,n-j+i,$ $j-l+i-1,n-j+k-i-1)$
$(i,j)(i+1,j)$	$1 \le i \le k-1$ $1 \le j \le l$	$(i+j,i+j-1,l-j+i-1,j+k-i-1)$
$(i,l)(i,l+1)$	$1 \le i \le k$	$(l+i-1,l+i-1,i-1,l+k-i-1)$
$(i,0)(i,n-1)$	$1 \le i \le k$	$(i,i+1,l+i-2,k-i)$
$(i,j)(i+1,j)$	$1 \le i \le k-1$ $l+1 \le j \le n-1$	$(n-j+i,n-j+i+1,$ $j-l+i-1,n-j+k-i-1)$
$(0,j)(1,j)$	$l+1 \le j \le n-1$	$(n-j,n-j+1,j-l,n-j+k-1)$
$(i,j)(i+1,j)$	$k+1 \le i \le m-2$ $1 \le j \le l$	$(m-i+j-1,m-i+j-2,$ $m-i+l-j,i-k+j)$
$(i,j)(i,j+1)$	$k+1 \le i \le m-1$ $1 \le j \le l-1$	$(m-i+j,m-i+j-1,$ $m-i+l-j,j+i-k)$
$(i,0)(i+1,0)$	$k+1 \le i \le m-2$	$(m-i-1,m-i,m-i+l,i-k)$
$(i,0)(i,1)$	$k+1 \le i \le m-1$	$(m-i,m-i,m-i+l,i-k)$
$(k,j)(k+1,j)$	$1 \le j \le l$	$(k+j-1,k+j-2,$ $k+l-j-1,j)$
$(0,j)(m-1,j)$	$1 \le j \le l$	$(j,j-1,l-j+1,k+j-1)$
$(i,j)(i+1,j)$	$k+1 \le i \le m-2$ $l+1 \le j \le n-1$	$(n-j+m-i-1,n+m-j-i,$ $j-l+m-i,n-j+i-k)$
$(i,j)(i,j+1)$	$k+1 \le i \le m-1$ $l+1 \le j \le n-2$	$(n-j+m-i-1,n+m-j-i,$ $j-l+m-i+1,n-j+i-k-1)$
$(i,l)(i,l+1)$	$k+1 \le i \le m-1$	$(l+m-i-1,m+l-i-1,$ $m-i+1,l+i-k-1)$
$(k,j)(k+1,j)$	$l+1 \le j \le n-1$	$(n-j+k-1,n-j+k,$ $j-l+k-1,n-j)$
$(i,0)(i,n-1)$	$k+1 \le i \le m-1$	$(m-i,m-i+1,m-i+l,i-k)$
$(0,j)(m-1,j)$	$l+1 \le j \le n-1$	$(n-j,n-j+1,j-l+1,n-j+k-1)$
$(0,l)(0,l+1)$		$(l-1,l-1,1,l-1+k)$
$(k,0)(k+1,0)$		$(k-1,k,k-1+l,0)$

Since metric coordinates of all items are mutually different, S is a mixed resolving set. Therefore, $\beta_M(T_{2k,2l}) \le 4$.

Step 2: *Lower bound is 4.*
Torus graph is 4-regular graph, so by Corollary 2.5 follows $\beta_M(T_{m,n}) \ge 1 + \lceil log_2(r+1) \rceil = 1 + \lceil log_2 5 \rceil = 4$.
Therefore, from the previous two steps, it follows that $\beta_M(T_{m,n}) = 4$. □

4. Illustrative example of new and old lower bounds

In this section we illustrate the relationship between the lower bounds and the exact values of the mixed metric dimension for two sets of graphs. The lower bounds known in the literature [4, 15] and the new lower bounds proposed in this paper are determined for two sets of graphs: all connected graphs with 5 vertices and a set of 12 well-known graphs, which are listed in Table 8.

The set of all connected graphs with 5 vertices contains 21 different configurations. Graphical representation of all these graphs can be found at https://mathworld.wolfram.com/ConnectedGraph.html. The graph shown in Table 7 are ordered in the same way as their graphic representation on the provided link. Columns of Table 7 are organized as follows: In the first four columns, the ordinal number, number of edges $|E|$, metric dimension $\beta(G)$ and edge metric dimension $\beta_E(G)$ are shown respectively. The following three columns contain the lower bounds from literature: L1 and L2 denote lower bounds from Proposition 1.7 and Theorem 1.8. Each of Proposition 1.3, Proposition 1.4, Proposition 1.5 and Corollary 1.6 determines one lower bound. For the purpose of transparency of Table 7, we have decided to give a unified lower bound that encompasses all of them, denoted as L3. This lower bound cannot be obtained generally, while for each specific graph, all three lower bounds from propositions and Corollary 1.6 can be calculated separately and unified together.

Lower bound L4 is based on the LP relaxation of the mixed metric dimension problem. In the last three columns new lower bounds N1, N2 and N3 from Corollary 2.3, Theorem 2.7 and Theorem 2.9 are given respectively.

It should be noted that total enumeration is able to quickly compute metric dimension, edge metric dimension and mixed metric dimension for graphs up to 36 vertices, so it is used to obtain data for $\beta(G)$, $\beta_E(G)$ and $\beta_M(G)$ for all considered graphs. Data shown column labeled as L4, which represents a LP relaxation of the mixed metric dimension problem, can be quickly obtained by any linear programming software: CPLEX, Gurobi, GLPK, LP_solve, etc. Data displayed in column labeled as N2 is also computed by total enumeration.

Table 7: Direct comparison of lower bounds for connected graphs with 5 vertices

				LB from lit.				New LB			
Num	$\|E\|$	$\beta(G)$	$\beta_E(G)$	L1	L2	L3	L4	N1	N2	N3	$\beta_M(G)$
1.	4	3	3	2	1	**4**	**4**	2	**4**	2	4
2.	4	2	2	2	1	**3**	**3**	2	**3**	2	3
3.	5	2	3	2	1	**4**	**4**	2	**4**	2	4
4.	5	2	2	2	1	**3**	**3**	2	**3**	2	3
5.	5	2	2	2	1	2	**3**	2	2	2	3
6.	6	2	3	2	1	3	**4**	2	**4**	2	4
7.	6	3	3	2	2	2	3	3	2	2	4
8.	7	3	4	2	2	**5**	**5**	3	**5**	2	5
9.	4	1	1	1	1	**2**	**2**	**2**	**2**	**2**	2
10.	5	2	2	2	1	**3**	**3**	2	**3**	2	3
11.	6	2	3	2	2	**4**	**4**	3	**4**	2	4
12.	6	2	3	2	1	**4**	**4**	2	**4**	2	4
13.	7	3	3	2	1	**4**	**4**	2	**4**	2	4
14.	5	2	2	1	2	0	**3**	**3**	**3**	2	3
15.	6	2	2	2	2	1	**3**	**3**	**3**	2	3
16.	7	2	3	2	2	2	**4**	3	**4**	2	4
17.	8	3	4	2	2	**5**	**5**	3	**5**	2	5
18.	7	2	3	2	2	3	**4**	3	3	2	4
19.	8	2	**4**	2	3	2	**4**	3	**4**	2	4
20.	9	3	4	2	3	**5**	**5**	3	**5**	2	5
21.	10	4	4	2	3	**5**	**5**	4	**5**	3	5

As it can be seen from Table 7 new lower bounds obtain very good results as well as bounds L1, L2, L3 and L4 from the literature, since number of vertices is relatively small ($|V| = 5$). Therefore, additional calculations will be conducted on 12 well-known graphs. In Table 8 are shown graph characteristics for each graph, while the values of the various lower bounds are shown in Table 9. Columns in Table 9, nominated as L1, L2, L3, L4, N1, N2 and N3, have the same meaning as in Table 7. From Table 9 it can be seen that the new lower bounds are better than the ones from the literature since in 10 cases new lower bounds are better or equal to the already known, while in 7 cases it produced the best lower bounds until now.

However, only in two cases mixed metric dimension equals one of the presented lower bounds (1 from literature and 1 from the new ones). For all 12 graphs some of the 7 lower bounds gives relatively good approximation of the mixed metric dimension. All 7 lower bounds should be used in union since different lower bounds are applicable for different graphs and no one is uniquely dominant over the others. The important feature of presented lower bounds is that their calculation complexity is usually much smaller in comparison with standard/edge/mixed metric dimension problem complexity.

Table 8: Graph characteristic

Num	Name	$\|V\|$	$\|E\|$	$\beta(G)$	$\beta_E(G)$	Another notions
1.	Rook's graph	36	180	7	8	srg(36,10,4,2)
2.	9-triangular graph	36	252	6	32	Johnson graph; srg(36,14,7,4)
3.	Clebsch graph	16	40	4	9	srg(16,5,0,2)
4.	Generalized quadrangle	27	135	5	18	srg(27,10,1,5)
5.	Hypercube Q_5	32	80	4	4	5−cube graph
6.	Kneser (7,2)	21	105	5	12	srg(21,10,3,6)
7.	Mobius Kantor	16	24	4	4	Generalized Petersen $GP(8,3)$
8.	Paley graph	13	39	4	6	srg(13,6,2,3)
9.	Petersen graph	10	15	3	4	Generalized Petersen $GP(5,2)$
10.	Small graph 6 vert.	6	11	3	4	
11.	Hamming H(2,6)	36	180	7	8	$K_6 \square K_6$
12.	Hamming H(3,3)	27	81	4	5	$K_3 \square K_3 \square K_3$

Table 9: Direct comparison of lower bounds for some graphs

	LB from lit.							
Num	L1	L2	L3	L4	N1	N2	N3	$\beta_M(G)$
1.	4	5	0	6	5	6	8	9
2.	4	5	0	18	5	9	8	32
3.	3	4	0	4	4	5	5	9
4.	4	5	0	4	5	6	8	18
5.	3	**4**	0	2	**4**	2	3	4
6.	4	5	0	4	5	6	6	12
7.	2	3	0	2	3	3	3	4
8.	3	4	0	4	4	5	5	6
9.	2	3	0	4	3	4	4	6
10.	2	2	**5**	**5**	3	4	3	5
11.	4	5	0	6	5	6	8	9
12.	3	4	0	3	4	3	4	6

It should be noted that actually the best lower bound with respect to experimental results is the edge metric dimension. However, this fact has very limited practical impact, since both mixed and edge metric dimension problems are NP-hard. So, computationally, we can obtain edge metric dimension (as lower bound) only in cases when we can obtain exact value of mixed metric dimension. Only gain can be obtained in cases, when exact value, or tight lower bound, of edge metric dimension can be found in literature, but exact value of mixed metric dimension is unknown.

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